Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

35th Anniversary 1963 – 1998

1998 Solutions

Euclid Contest

(Grade 12)

for the

NATIONAL BANK OF CANADA

Awards

1. (a) If one root of $x^2 + 2x - c = 0$ is $x = 1$, what is the value of c?

Solution 1
If $x = 1$, by substituting, $c = 3$.

Solution 2
By division,

$$\begin{array}{r|l} & x+3 \\ x-1 & x^2+2x-c \\ & \underline{x^2-x} \\ & 3x-c \\ & \underline{3x-3} \\ & -c+3 \end{array}$$

If the remainder is zero, $-c + 3 = 0$

$c = 3$.

(b) If $2^{2x-4} = 8$, what is the value of x?

Solution
$2^{2x-4} = 2^3$
Therefore, $2x - 4 = 3$

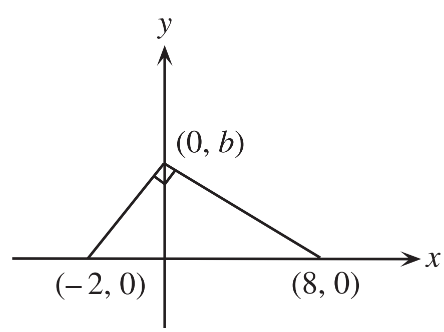
$x = \frac{7}{2}$.

(c) Two perpendicular lines with x-intercepts -2 and 8 intersect at $(0, b)$. Determine all values of b.

Solution 1
If the lines are perpendicular their slopes are negative reciprocals.

Thus, $\frac{b}{-8} \times \frac{b}{2} = -1$

$b^2 = 16$, $b = \pm 4$.



Solution 2
Using Pythagoras, $\left[(b-0)^2 + (0-8)^2\right] + \left[(b-0)^2 + (0+2)^2\right] = 10^2$

$2b^2 = 32$, $b = \pm 4$.

Solution 3
The vertices of the triangle represents three points on a circle with $(-2, 0)$ and $(8, 0)$ being the

coordinates of the end points of the diameter. This circle has centre $C(3, 0)$ and $r = 5$. The equation for this circle is $(x-3)^2 + y^2 = 25$ and if we want to find the y-intercepts we let $x = 0$ which gives $b = \pm 4$.

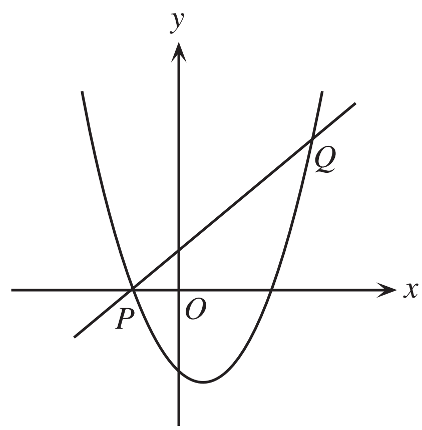
2. (a) The vertex of $y = (x-1)^2 + b$ has coordinates $(1, 3)$. What is the y-intercept of this parabola?

Solution
The vertex of parabola is $(1, b)$.
Therefore, $b = 3$.
The required equation is now $y = (x-1)^2 + 3$.
For the y-intercept, let $x = 0$.
Thus, $y_{\text{int}} = (0-1)^2 + 3 = 4$.

(b) What is the area of ΔABC with vertices $A(-3, 1)$, $B(5, 1)$ and $C(8, 7)$?

Solution
Drawing the diagram gives a triangle with a height of 6 and a base of 8 units.
The triangle has an area of 24 square units.

(c) In the diagram, the line $y = x + 1$ intersects the parabola $y = x^2 - 3x - 4$ at the points P and Q. Determine the coordinates of P and Q.



Solution
Consider the system of equations $y = x + 1$, $y = x^2 - 3x - 4$.
Comparison gives $x + 1 = x^2 - 3x - 4$

$$x^2 - 4x - 5 = 0$$
$$(x-5)(x+1) = 0.$$

Therefore $x = 5$ or $x = -1$.
If $x = 5$, $y = 6$ and if $x = -1$, $y = 0$.
The required coordinates are $P(-1, 0)$ and $Q(5, 6)$.

3. (a) The graph of $y = m^x$ passes through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

Solution
Since $(2,5)$ is on $y=m^x$, $5=m^2$.
Since $(5,n)$ is on $y=m^x$, $n=m^5$.
So $mn=m\left(m^5\right)=m^6=\left(m^2\right)^3=5^3=125$.

(b) Jane bought 100 shares of stock at \$10.00 per share. When the shares increased to a value of \$*N* each, she made a charitable donation of all the shares to the Euclid Foundation. She received a tax refund of 60% on the total value of her donation. However, she had to pay a tax of 20% on the increase in the value of the stock. Determine the value of *N* if the difference between her tax refund and the tax paid was \$1000.

Solution
Jane's charitable donation to the Euclid Foundation was $100N$ dollars.
Her tax refund was 60% of $100N$ or $60N$ dollars.
The increase in the value of her stock was $100(N-10)$ or $(100N-1000)$ dollars.
Jane's tax payment was 20% of $100N-1000$ or $20N-200$.
From the given, $60N-(20N-200)=1000$
Upon simplification, $40N=800$
$N=20$.
Therefore the value of *N* was 20.

4. (a) Consider the sequence $t_1=1$, $t_2=-1$ and $t_n=\left(\frac{n-3}{n-1}\right)t_{n-2}$ where $n\geq 3$. What is the value of t_{1998}?

Solution 1
Calculating some terms, $t_1=1$, $t_2=-1$, $t_3=0$, $t_4=\frac{-1}{3}$, $t_5=0$, $t_6=\frac{-1}{5}$ etc.
By pattern recognition, $t_{1998}=\frac{-1}{1997}$.

Solution 2
$$\begin{aligned}t_{1998}&=\frac{1995}{1997}t_{1996}=\frac{1995}{1997}\times\frac{1993}{1995}t_{1994}\\&=\frac{1995}{1997}\cdot\frac{1993}{1995}\cdot\frac{1991}{1993}\cdots\frac{3}{5}\cdot\frac{1}{3}t_2\\&=\frac{-1}{1997}\end{aligned}$$

(b) The *n*th term of an arithmetic sequence is given by $t_n=555-7n$.
If $S_n=t_1+t_2+...+t_n$, determine the smallest value of *n* for which $S_n<0$.

Solution 1

This is an arithmetic sequence in which $a = 548$ and $d = -7$.

Therefore, $S_n = \frac{n}{2}[2(548) + (n-1)(-7)] = \frac{n}{2}[-7n + 1103]$.

We now want $\frac{n}{2}(-7n + 1103) < 0$.

Since $n > 0$, $-7n + 1103 < 0$

$$n > 157\tfrac{4}{7}.$$

Therefore the smallest value of n is 158.

Solution 2

For this series we want, $\sum_{k=1}^{n} t_k < 0$, or $\sum_{k=1}^{n} (555 - 7k) < 0$.

Rewriting, $555n - 7\dfrac{(n)(n+1)}{2} < 0$

$$1110n - 7n^2 - 7n < 0$$

$$7n^2 - 1103n > 0$$

$$\text{or, } n > \frac{1103}{7}.$$

The smallest value of n is 158.

Solution 3

We generate the series as 548, 541, 534, ..., 2, -5, ..., -544, -551.

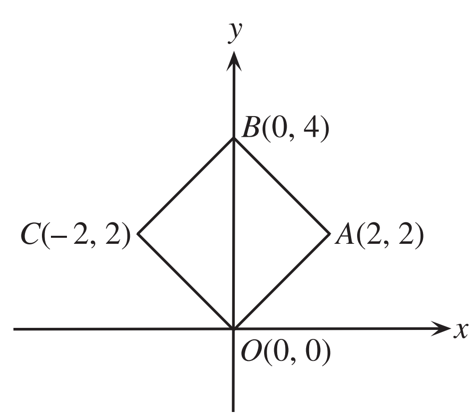
If we pair the series from front to back the sum of each pair is -3.

Including all the pairs $548 - 551$, $541 - 544$ and so on there would be 79 pairs which give a sum of -237.

If the last term, -551, were omitted we would have a positive sum.

Therefore we need all 79 pairs or 158 terms.

5. (a) A square $OABC$ is drawn with vertices as shown. Find the equation of the circle with largest area that can be drawn inside the square.



Solution

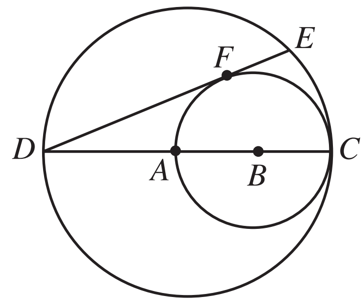
The square has a side length of $2\sqrt{2}$.

The diameter of the inscribed circle is $2\sqrt{2}$, so its radius is $\sqrt{2}$.

The centre of the circle is $(0, 2)$.

The required equation is $x^2 + (y-2)^2 = 2$ or $x^2 + y^2 - 4y + 2 = 0$.

(b) In the diagram, DC is a diameter of the larger circle centred at A, and AC is a diameter of the smaller circle centred at B. If DE is tangent to the smaller circle at F, and $DC = 12$, determine the length of DE.



Solution

Join B to F and C to E.

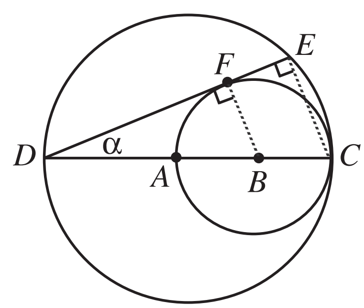
$FB \perp DE$ and DFE is a tangent.

Since DC is a diameter, $\angle DEC = 90°$.

Thus $FB \parallel EC$.

By Pythagoras, $DF = \sqrt{9^2 - 3^2} = \sqrt{72}$.

Using similar triangles (or the side splitting theorem) we have,



$$\frac{DE}{DF} = \frac{DC}{DB}$$

$$\frac{DE}{6\sqrt{2}} = \frac{4}{3}$$

$$DE = 8\sqrt{2} \text{ or } \sqrt{128}$$

OR

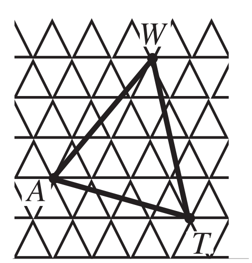
$$\frac{EC}{FB} = \frac{12}{9}$$

$$EC = \tfrac{4}{3}FB$$

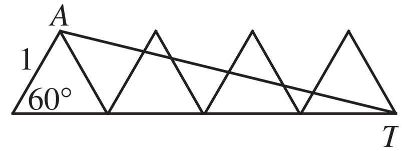
$$EC = 4$$

By Pythagoras, $DE = 8\sqrt{2}$ or $\sqrt{128}$.

6. (a) In the grid, each small equilateral triangle has side length 1. If the vertices of ΔWAT are themselves vertices of small equilateral triangles, what is the area of ΔWAT?



Solution 1

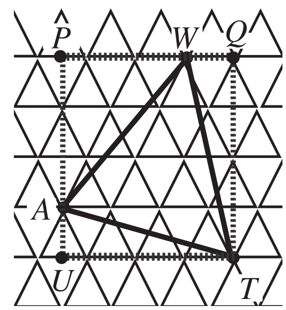


$AT^2 = 1^2 + 4^2 - 2(1)(4)\cos 60° = 13$

Since ΔWAT is an equilateral triangle with a side of length $\sqrt{13}$, its height will be $\frac{\sqrt{3}}{2}\left(\sqrt{13}\right)$. The area of ΔWAT is thus, $\frac{1}{2}\left[\left(\frac{\sqrt{3}}{2}\right)\left(\sqrt{13}\right)\right]\sqrt{13} = \frac{13}{4}\sqrt{3}$. It is also possible to use the formula for the area of a triangle,

Area $= \frac{1}{2}ab\sin c$. Since the triangle is equilateral, area of $\Delta WAT = \frac{\sqrt{3}AT^2}{4} = \frac{13\sqrt{3}}{4}$.

Solution 2



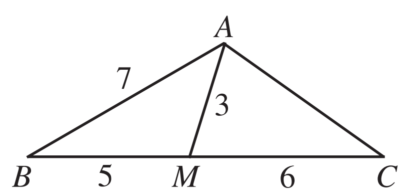
Since the small triangles have sides 1, they have a height of $\frac{\sqrt{3}}{2}$.

Consider rectangle $PQTU$.

Then

$$\begin{aligned}|\Delta WAT| &= |PQTU| - |\Delta APW| - |\Delta WQT| - |\Delta TUA| \\ &= (PQ)(QT) - \tfrac{1}{2}(AP)(PW) - \tfrac{1}{2}(WQ)(QT) - \tfrac{1}{2}(TU)(UA) \\ &= (3.5)\left(2\sqrt{3}\right) - \tfrac{1}{2}\left(\tfrac{3\sqrt{3}}{2}\right)(2.5) - \tfrac{1}{2}(1)\left(2\sqrt{3}\right) - \tfrac{1}{2}(3.5)\left(\tfrac{\sqrt{3}}{2}\right) \\ &= 7\sqrt{3} - \frac{15\sqrt{3}}{4} \\ &= \frac{13\sqrt{3}}{4}\end{aligned}$$

(b) In ΔABC, M is a point on BC such that $BM = 5$ and $MC = 6$. If $AM = 3$ and $AB = 7$, determine the exact value of AC.



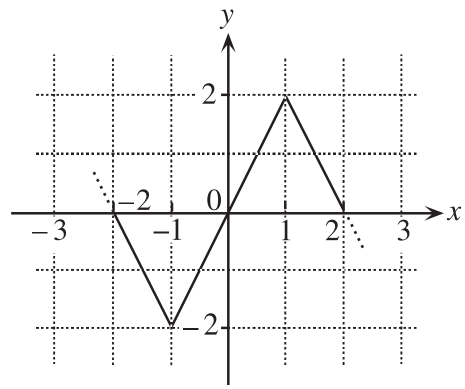
Solution

From ΔABM, $\cos < B = \frac{3^2 - 7^2 - 5^2}{-2(7)(5)} = \frac{13}{14}$.

From ΔABC, $AC^2 = 7^2 + 11^2 - 2(7)(11)\left(\frac{13}{14}\right) = 27$.

Therefore, $AC = \sqrt{27}$.

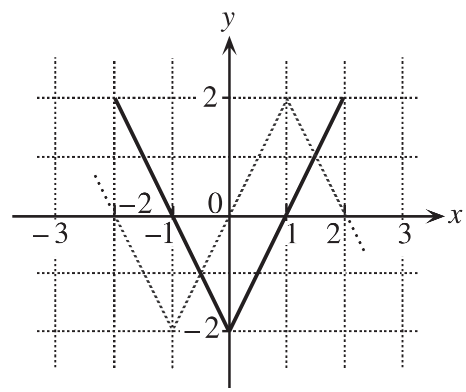
7. (a) The function $f(x)$ has period 4. The graph of one period of $y = f(x)$ is shown in the diagram. Sketch the graph of $y = \frac{1}{2}[f(x-1) + f(x+3)]$, for $-2 \le x \le 2$.



Solution 1

(a)

x	$f(x)$	$f(x-1)$	$f(x+3)$	$\frac{1}{2}[f(x-1)+f(x+3)]$
-2	0	2	2	2
-1	-2	0	0	0
0	0	-2	-2	-2
1	2	0	0	0
2	0	2	2	2



Now plot the points and join them with straight line segments.

Solution 2

Since $f(x)$ has period 4, $f(x+3) = f(x-1)$.

Therefore, $y = \frac{1}{2}[f(x-1) + f(x+3)] = \frac{1}{2}[f(x-1) + f(x-1)] = f(x-1)$.

The required graph is that of $y = f(x-1)$ which is formed by shifting the given graph ***1 unit*** to the right.

(b) If *x* and *y* are real numbers, determine all solutions (x, y) of the system of equations

$$x^2 - xy + 8 = 0$$
$$x^2 - 8x + y = 0.$$

Solution 1

Subtracting,

$$\begin{aligned} x^2 - xy + 8 &= 0 \\ x^2 - 8x + y &= 0 \\ \hline -xy + 8x + 8 - y &= 0 \\ 8(1+x) - y(1+x) &= 0 \\ (8-y)(1+x) &= 0 \end{aligned}$$

$y = 8 \quad \text{or} \quad x = -1$

If $y=8$, both equations become $x^2-8x+8=0$, $x=4\pm 2\sqrt{2}$.
If $x=-1$ both equations become $y+9=0$, $y=-9$.
The solutions are $(-1,-9),\left(4+2\sqrt{2},8\right)$ and $\left(4-2\sqrt{2},8\right)$.

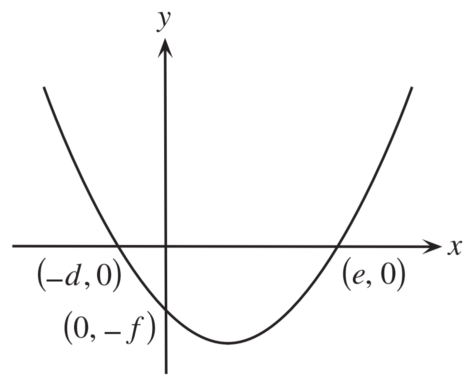
Solution 2
If $x^2-xy+8=0$, $y=\frac{x^2+8}{x}$.

And $x^2-8x+y=0$ implies $y=8x-x^2$.

Equating, $\frac{x^2+8}{x}=8x-x^2$

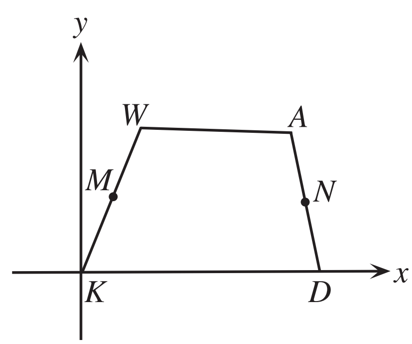
or, $x^3-7x^2+8=0$.
By inspection, $x=-1$ is a root.
By division, $x^3-7x^2+8=(x+1)\left(x^2-8x+8\right)$.
As before, the solutions are $(-1,-9)$, $\left(4\pm 2\sqrt{2},8\right)$.

8. (a) In the graph, the parabola $y=x^2$ has been translated to the position shown. Prove that $de=f$.



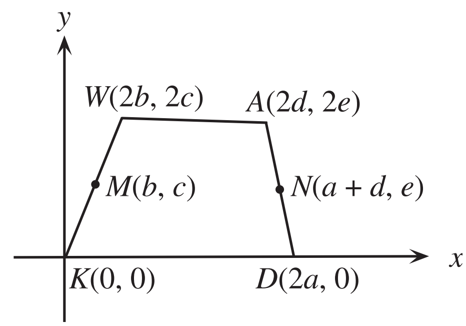
Solution
Since the given graph is congruent to $y=x^2$ and has x-intercepts $-d$ and e, its general form is $y=(x+d)(x-e)$.
To find the y-intercept, let $x=0$. Therefore y-intercept = $-de$.
We are given that the y-intercept is $-f$.
Therefore $-f=-de$ or $f=de$.

(b) In quadrilateral $KWAD$, the midpoints of KW and AD are M and N respectively. If $MN=\frac{1}{2}(AW+DK)$, prove that WA is parallel to KD.



Solution 1

Establish a coordinate system with $K(0,0)$, $D(2a,0)$ on the *x*-axes. Let *W* be $(2b,2c)$ and *A* be $(2d,2e)$.
Thus *M* is (b,c) and *N* is $(a+d,e)$.



KD has slope 0 and slope $WA = \dfrac{e-c}{d-b}$.

Since $MN = \frac{1}{2}(AW + DK)$

$$\sqrt{(a+d-b)^2+(e-c)^2}$$
$$= \tfrac{1}{2}\left(2a+\sqrt{(2d-2b)^2+(2e-2c)^2}\right)$$
$$= \tfrac{1}{2}\left(2a+2\sqrt{(d-b)^2+(e-c)^2}\right)$$

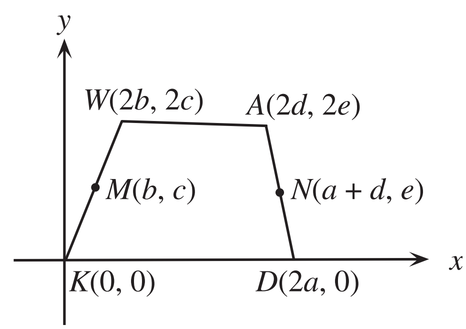
Squaring both sides gives,

$$(a+d-b)^2+(e-c)^2 = a^2+2a\sqrt{(d-b)^2+(e-c)^2}+(d-b)^2+(e-c)^2$$
$$a^2+2a(d-b)+(d-b)^2 = a^2+2a\sqrt{(d-b)^2+(e-c)^2}+(d-b)^2$$

Simplifying and dividing by $2a$ we have, $d-b=\sqrt{(d-b)^2+(e-c)^2}$.
Squaring, $(d-b)^2=(d-b)^2+(e-c)^2$.
Therefore $(e-c)^2=0$ or $e=c$.
Since $e=c$ then slope of WA is 0 and $KD \parallel AW$.

Solution 2

Join *A* to *K* and call *P* the mid-point of AK.
Join *M* to *P*, *N* to *P* and *M* to *N*.
In ΔKAW, *P* and *M* are the mid-points of KA and KW.
Therefore, $MP = \frac{1}{2}WA$.
Similarly in ΔKAD, $PN = \frac{1}{2}KD$.
Therefore $MP + PN = MN$.



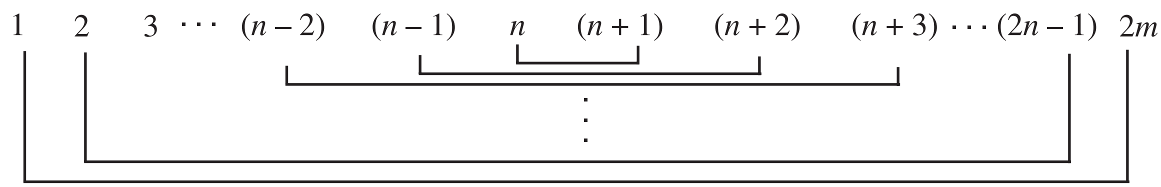
As a result *M*, *P* and *N* cannot form the vertices of a triangle but must form a straight line.
So if MPN is a straight line with $MP \parallel WA$ and $PN \parallel KD$ then $WA \parallel KD$ as required.

Solution 3

We are given that $\overrightarrow{AN} = \overrightarrow{ND}$ and $\overrightarrow{WM} = \overrightarrow{MK}$.
Using vectors,
(1) $\overrightarrow{MN} = \overrightarrow{MW} + \overrightarrow{WA} + \overrightarrow{AN}$ (from quad. $MWAN$)
(2) $\overrightarrow{MN} = \overrightarrow{MK} + \overrightarrow{KD} + \overrightarrow{DN}$ (from quad. $KMND$)
It is also possible to write, $\overrightarrow{MN} = -\overrightarrow{MW} + \overrightarrow{KD} - \overrightarrow{AN}$,
(3) (This comes from taking statement (2) and making appropriate substitutions.)

If we add (1) and (3) we find, $2\overrightarrow{MN} = \overrightarrow{WA} + \overrightarrow{KD}$.
But it is given that $2\left|\overrightarrow{MN}\right| = \left|\overrightarrow{AW}\right| + \left|\overrightarrow{DK}\right|$.
From these two previous statements, $\overrightarrow{MN}$ must be parallel to $\overrightarrow{WA}$ and $\overrightarrow{KD}$ otherwise $2\left|\overrightarrow{MN}\right| < \left|\overrightarrow{AW}\right| + \left|\overrightarrow{DK}\right|$.
Therefore, $WA \parallel KD$.

9. Consider the first $2n$ natural numbers. Pair off the numbers, as shown, and multiply the two members of each pair. Prove that there is no value of n for which two of the n products are equal.



Solution 1
The sequence is $1(2n), 2(2n-1), 3(2n-2), ..., k(2n-k+1), ..., p(2n-p+1), ..., n(n+1)$.
In essence we are asking the question, 'is it possible that $k(2n-k+1) = p(2n-p+1)$ where p and k are both less than or equal to n?'

$$k(2n-k+1) = p(2n-p+1) \qquad \text{(supposing them to be equal)}$$
$$2nk - k^2 + k = 2np - p^2 + p$$
$$p^2 - k^2 + 2nk - 2np + k - p = 0$$
$$(p-k)(p+k) + 2n(k-p) + (k-p) = 0$$
$$(p-k)[(p+k) - 2n - 1] = 0$$
$$(p-k)(p+k-2n-1) = 0$$

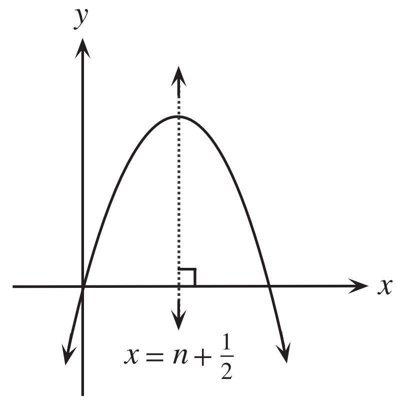
Since p and k are both less than or equal to n, it follows $p+k-2n-1 \neq 0$. Therefore $p = k$ and they represent the same pair. Thus the required is proven.

Solution 2
The products are $1(2n+1-1)$, $2(2n+1-2)$, $3(2n+1-3), ..., n(2n+1-n)$.
Consider the function, $y = x(2n+1-x) = -x^2 + (2n+1)x = f(x)$.

The graph of this function is a parabola, opening down, with its vertex at $x = n + \frac{1}{2}$.

The products are the y-coordinates of the points on the parabola corresponding to $x = 1, 2, 3, ..., n$. Since all the points are to the left of the vertex, no two have the same y-coordinate.

Thus the products are distinct.



Solution 3

The sum of these numbers is $\frac{2n(2n+1)}{2}$ or $n(2n+1)$.

Their average is $\frac{n(2n+1)}{2n} = n + \frac{1}{2}$.

The $2n$ numbers can be rewritten as,

$$n + \frac{1}{2} - \left(\frac{2n-1}{2}\right), \cdots, n + \frac{1}{2} - \frac{3}{2}, n + \frac{1}{2} - \frac{1}{2}, n + \frac{1}{2} + \frac{1}{2}, n + \frac{1}{2} + \frac{3}{2}, \cdots, n + \frac{1}{2} + \left(\frac{2n-1}{2}\right).$$

The product pairs, starting from the middle and working outward are

$$P_1 = \left(n + \tfrac{1}{2}\right)^2 - \tfrac{1}{4}$$

$$P_2 = \left(n + \tfrac{1}{2}\right)^2 - \tfrac{9}{4}$$

$$\vdots$$

$$P_n = \left(n + \frac{1}{2}\right)^2 - \left(\frac{2n-1}{2}\right)^2$$

Each of the numbers $\left(\frac{2k-1}{2}\right)^2$ is distinct for $k = 1, 2, 3, ..., n$ and hence no terms of P_k are equal.

Solution 4

The sequence is $1(2n), 2(2n-1), 3(2n-2), ..., n[2n-(n-1)]$.

This sequence has exactly n terms.

When the kth term is subtracted from the $(k+1)$th term the difference is $(k+1)[2n-k] - k[2n-(k-1)] = 2(n-k)$. Since $n > k$, this is a positive difference.

Therefore each term is greater than the term before, so no two terms are equal.

10. The equations $x^2 + 5x + 6 = 0$ and $x^2 + 5x - 6 = 0$ ***each*** have integer solutions whereas only one of the equations in the pair $x^2 + 4x + 5 = 0$ and $x^2 + 4x - 5 = 0$ has integer solutions.

    (a) Show that if $x^2 + px + q = 0$ and $x^2 + px - q = 0$ ***both*** have integer solutions, then it is possible to find integers a and b such that $p^2 = a^2 + b^2$. (i.e. (a, b, p) is a Pythagorean triple).

(b) Determine q in terms of a and b.

Solution

(a) We have that $x^2 + px + q = 0$ and $x^2 + px - q = 0$ both have integer solutions.

For $x^2 + px + q = 0$, its roots are $\dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$.

In order that these roots be integers, $p^2 - 4q$ must be a perfect square.

Therefore, $p^2 - 4q = m^2$ for some positive integer *m*.

Similarly for $x^2 + px - q = 0$, it has roots $\dfrac{-p \pm \sqrt{p^2 + 4q}}{2}$ and in order that these roots be integers $p^2 + 4q$ must be a perfect square.

Thus $p^2 + 4q = n^2$ for some positive integer *n*.

Adding gives $2p^2 = m^2 + n^2$ (with $n \geq m$ since $n^2 = p^2 + 4q$
$\geq p^2 - 4q = m^2$)

And so $p^2 = \frac{1}{2}m^2 + \frac{1}{2}n^2 = \left(\frac{n+m}{2}\right)^2 + \left(\frac{n-m}{2}\right)^2$.

We note that *m* and *n* have the same parity since $m^2 = p^2 - 4q \equiv p^2 (\text{mod } 2)$ and $n^2 \equiv p^2 + 4q \equiv p^2 (\text{mod } 2)$.

Since $\frac{n+m}{2}$ and $\frac{n-m}{2}$ are positive integers then $p^2 = a^2 + b^2$ where $a = \frac{n+m}{2}$ and $b = \frac{n-m}{2}$.

(b) From (a), $a = \frac{n+m}{2}$ and $b = \frac{n-m}{2}$ or $n = a + b$ and $m = a - b$.

From before,

$$
\begin{aligned}
p^2 + 4q &= n^2 \\
4q^2 &= n^2 - p^2 \\
&= (a+b)^2 - \left(a^2 + b^2\right) \\
4q &= 2ab.
\end{aligned}
$$

Therefore, $q = \dfrac{ab}{2}$.