Concours
canadien de
mathématiques

Une activité du Centre d'éducation
en mathématiques et en informatique,
Université de Waterloo, Waterloo, Ontario

2000 Solutions
Concours Cayley (10^e - Sec. IV)

pour les prix

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Part A:

1. The value of $2(5-2)-5^2$ is

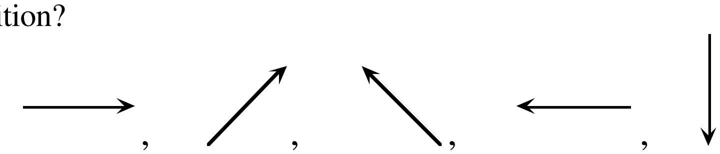
(**A**) -19 (**B**) -4 (**C**) 1 (**D**) -11 (**E**) -17

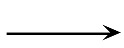
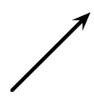
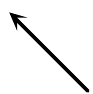
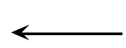
Solution
When evaluating this expression, we use 'order of operations' in the standard way. Doing so, we find,

$$
\begin{aligned}
&2(5-2)-5^2\\
&=2(3)-25\\
&=6-25\\
&=-19.
\end{aligned}
$$

ANSWER: (A)

2. If the following sequence of five arrows repeats itself continuously, what arrow would be in the 48th position?



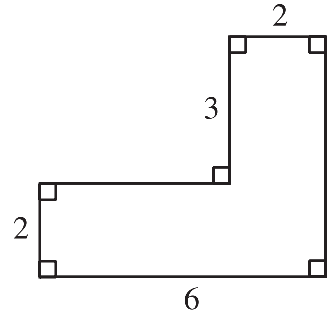
(**A**)  (**B**)  (**C**)  (**D**)  (**E**) 

Solution
Since this sequence repeats itself, once it has completed nine cycles it will be the same as starting at the beginning. Thus the 48th arrow will be the same as the third one.

ANSWER: (C)

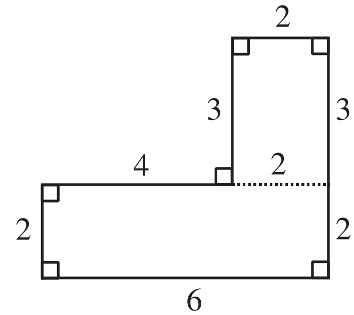
3. In the given diagram, the numbers shown are the lengths of the sides. What is the perimeter of the figure?

(**A**) 13 (**B**) 18 (**C**) 22
(**D**) 21 (**E**) 19



Solution

The easiest way to determine the length of the missing side is by drawing in a line to form two rectangles. Since opposite sides in a rectangle are equal we can label the sides as shown. The required perimeter is thus $2+3+2+6+2+4+3=22$.



ANSWER: (C)

4. A farmer has 7 cows, 8 sheep and 6 goats. How many more goats should be bought so that half of her animals will be goats?

(**A**) 18 (**B**) 15 (**C**) 21 (**D**) 9 (**E**) 6

Solution 1

If the cows and sheep were themselves goats we would have 15 goats. This means that she would need nine extra goats.

Solution 2

Let the number of goats added be x.

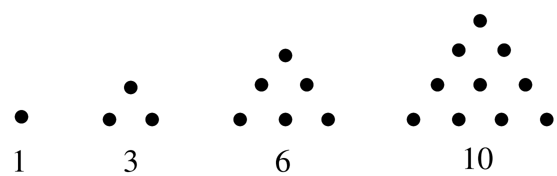
Therefore, $\dfrac{6+x}{21+x}=\dfrac{1}{2}$.

Cross multiplying gives,
$$\begin{aligned} 2(6+x) &= 21+x \\ 12+2x &= 21+x \\ x &= 9. \end{aligned}$$

As in solution 1, she would add 9 goats.

ANSWER: (D)

5. The first four triangular numbers 1, 3, 6, and 10 are illustrated in the diagram. What is the tenth triangular number?



(**A**) 55 (**B**) 45 (**C**) 66
(**D**) 78 (**E**) 50

Solution

From our diagram, it can be seen that the fifth triangular number is found by adding a row with five dots. This number is thus, $1+2+3+4+5=\dfrac{5\times 6}{2}=15$. The sixth triangular number is $1+2+3+4+5+6=\dfrac{6\times 7}{2}=21$. If we follow this to its conclusion, the nth triangular number is

$1+2+3+\ldots+n=\frac{(n)(n+1)}{2}$. The tenth triangular number will be $\frac{(10)(11)}{2}=55$.

ANSWER: (A)

6. The sum of the digits of an even ten digit integer is 89. The last digit is

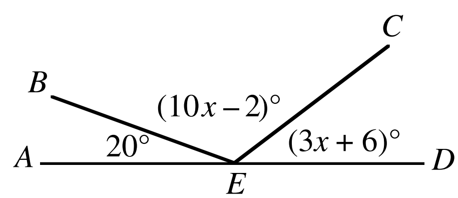
(**A**) 0 (**B**) 2 (**C**) 4 (**D**) 6 (**E**) 8

Solution

89 is a large number to be the sum of the digits of a ten digit number. In fact, the largest possible digital sum is 10×9 or 90. Since 89 is only 1 less than 90, the number in question must be composed of nine 9's and one 8. In order that the number be divisible by 2, the last digit must be 8.

ANSWER: (E)

7. If AD is a straight line segment and E is a point on AD, determine the measure of $\angle CED$.



(**A**) 20° (**B**) 12° (**C**) 42°
(**D**) 30° (**E**) 45°

Solution

Since there are 180° in a straight line, we can form the equation,

$$20°+(10x-2)°+(3x+6)°=180°$$
$$20+10x-2+3x+6=180 \text{ (in degrees)}$$
$$13x+24=180$$
$$13x=156$$
$$x=12.$$

Therefore $\angle CED=\left(3(12)+6\right)°=42°$.

ANSWER: (C)

8. On a 240 kilometre trip, Corey's father drove $\frac{1}{2}$ of the distance. His mother drove $\frac{3}{8}$ of the total distance and Corey drove the remaining distance. How many kilometres did Corey drive?

(**A**) 80 (**B**) 40 (**C**) 210 (**D**) 30 (**E**) 55

Solution

If Corey's father and mother drove $\frac{1}{2}$ and $\frac{3}{8}$ the total distance, respectively, altogether they drove $\frac{1}{2}+\frac{3}{8}$ or $\frac{7}{8}$th the total distance. Thus Corey must have driven $\frac{1}{8}\times 240$ or 30 kilometres.

ANSWER: (D)

9. Evaluate $(-50)+(-48)+(-46)+\ldots+54+56$.

(**A**) 156 (**B**) 10 (**C**) 56 (**D**) 110 (**E**) 162

Solution

If we add some terms to this series, we would have the following:

$$(-50)+(-48)+(-46)+\ldots+48+50+52+54+56.$$

Each of the negative integers has its opposite included in the sum and each pair of these sums is 0. This implies that, $(-50)+(-48)+(-46)+\ldots+46+48+50$ is 0. The overall sum is now just $52+54+56$ or 162. ANSWER: (E)

10. The ages of three contestants in the Cayley Contest are 15 years, 9 months; 16 years, 1 month; and 15 years, 8 months. Their average (mean) age is

(**A**) 15 years, 8 months (**B**) 15 years, 9 months (**C**) 15 years, 10 months
(**D**) 15 years, 11 months (**E**) 16 years

Solution 1

Consider one of the ages, say the youngest, as a base age. The other two contestants are one month and five months older respectively. Since $\frac{0+1+5}{3}=2$, this implies that the average age is two months greater than the youngest. This gives an average age of 15 years, 10 months.

Solution 2

This second solution involves a little more calculation but still gives the same correct answer. Since there are twelve months in a year, the age of the first contestant, in months, is $15\times 12+9$ or 189 months. Similarly, the ages of the other two students would be 193 and 188 months. The average age would thus be $\frac{189+193+188}{3}$ or 190 months. The average age is then 15 years, 10 months because $190=12\times 15+10$. ANSWER: (C)

Part B: Each correct answer is worth 6.

11. A store had a sale on T-shirts. For every two T-shirts purchased at the regular price, a third T-shirt was bought for \$1.00. Twelve T-shirts were bought for \$120.00. What was the regular price for one T-shirt?

(**A**) \$10.00 (**B**) \$13.50 (**C**) \$14.00 (**D**) \$14.50 (**E**) \$15.00

Solution

We will start this question by representing the regular price of one T-shirt as x dollars. If a person bought a ‘lot’ of three T-shirts, they would thus pay $(2x+1)$ dollars. Since the cost of twelve T-shirts is \$120.00, this implies that a single ‘lot’ would cost \$30. This allows us to write the equation, $2x+1=30$ or $x=14.50$. The regular price of a T-shirt is \$14.50.

ANSWER: (D)

12. Natural numbers are equally spaced around a circle in order from 1 to n. If the number 5 is directly opposite the number 14, then n is

(**A**) 14 (**B**) 15 (**C**) 16 (**D**) 18 (**E**) 20

Solution

If 5 is opposite 14 then each of the eight numbers between and including 6 and 13 are each opposite a natural number. These eight numbers would be matched giving a total of 2×8 or 16 numbers. If we add 5 and 14 the total is 18.

ANSWER: (D)

13. The average of 19 consecutive integers is 99. The largest of these integers is

(**A**) 118 (**B**) 108 (**C**) 109 (**D**) 117 (**E**) 107

Solution

If the average of the 19 consecutive numbers is 99 the middle number is 99 which is the tenth number. If the tenth number is 99, the nineteenth number will be 108.

ANSWER: (B)

14. A positive integer is to be placed in each box. The product of any four adjacent integers is always 120. What is the value of x?

		2			4		x				3		

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution

Since the product of any four integers is 120, $a_1a_2a_3a_4 = a_2a_3a_4a_5 = 120$ where a_n represents the number in the nth box. Therefore, $a_1 = a_5$ and similarly $a_2 = a_6$, $a_3 = a_7$, $a_4 = a_8$ or more generally, $a_n = a_{n+4}$. Thus the boxes can be filled as follows:

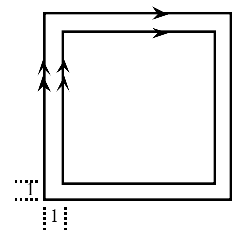
x	4	2	3	x	4	2	3	x	4	2	3	x	4

Therefore, $(4)(2)(3)(x) = 120$

$$x = \frac{120}{24} = 5.$$

ANSWER: (E)

15. Eight squares with the same centre have parallel sides and are one unit apart. The two largest squares are shown. If the largest square has a perimeter of 96, what is the perimeter of the smallest square?

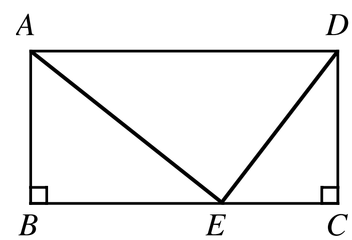


(**A**) 40 (**B**) 68 (**C**) 32
(**D**) 64 (**E**) 89

Solution
Since the largest square has perimeter 96, it has a side length of $\frac{96}{4}$ or 24. From the diagram, the side length of the next square is $24 - 2$ or 22. Continuing thus, the side lengths of the eight squares form the sequence: 24, 22, 20, 18, 16, 14, 12, 10. The side length of the eighth square will be 10 giving a perimeter of $4 \times 10 = 40$.

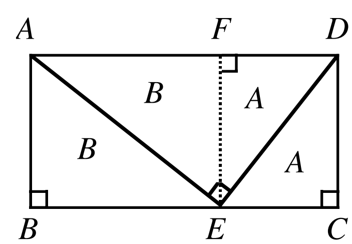
ANSWER: (A)

16. In the diagram, $ABCD$ is a rectangle with $AD = 13$, $DE = 5$ and $EA = 12$. The area of $ABCD$ is



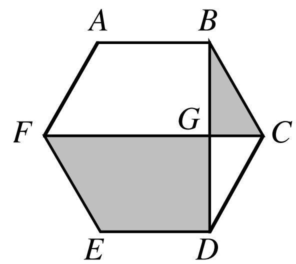
(**A**) 39 (**B**) 60 (**C**) 52
(**D**) 30 (**E**) 25

Solution
Since $13^2 = 12^2 + 5^2$ we use the converse of Pythagorus' Theorem to conclude that $\angle AED = 90°$.
The area of ΔAED is then $\frac{1}{2}(5)(12) = 30$. Through *E*, we draw a line parallel to CD and BA. Since the area of ΔFDE equals the area of ΔCDE we label each of these areas *A*. Similarly, the area of ΔAFE equals the area of ΔBAE and so each of these areas can be labelled *B*. Since $A + B = 30$, the area of the rectangle is $2(A + B)$ or $2(30) = 60$.



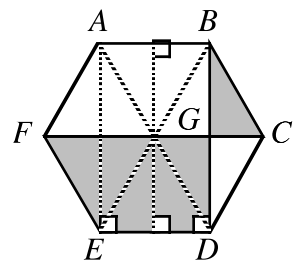
ANSWER: (B)

17. In the regular hexagon $ABCDEF$, two of the diagonals, FC and BD, intersect at G. The ratio of the area of quadrilateral $FEDG$ to ΔBCG is

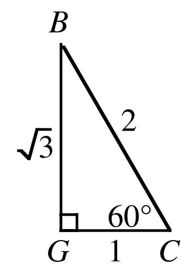


(**A**) $3\sqrt{3}:1$ (**B**) 4:1 (**C**) 6:1
(**D**) $2\sqrt{3}:1$ (**E**) 5:1

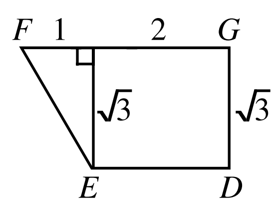
Solution 1
Join E to B and D to A as shown. Also join E to A and draw a line parallel to AE through the point of intersection of BE and AD. Quadrilateral $FEDG$ is now made up of five triangles each of which has the same area as ΔBCG. The required ratio is 5:1.



Solution 2
For convenience, assume that each side of the hexagon has a length of 2 units. Each angle in the hexagon equals 120° so $\angle BCG = \frac{1}{2}(120^\circ) = 60^\circ$. Now label ΔBCG as shown. Using the standard ratios for a $30^\circ - 60^\circ - 90^\circ$ triangle we have $BG = \sqrt{3}$ and $GC = 1$.



The area of $\Delta BCG = \frac{1}{2}(1)\sqrt{3} = \frac{\sqrt{3}}{2}$. Dividing the quadrilateral $FGDE$ as illustrated, it will have an area of
$2\left(\sqrt{3}\right) + \frac{1}{2}(1)\left(\sqrt{3}\right) = \frac{5\sqrt{3}}{2}$.
The required ratio is $\frac{5\sqrt{3}}{2} : \frac{\sqrt{3}}{2}$ or 5:1, as in solution 1.



ANSWER: (E)

18. If a, b and c are distinct positive integers such that $abc = 16$, then the largest possible value of $a^b - b^c + c^a$ is

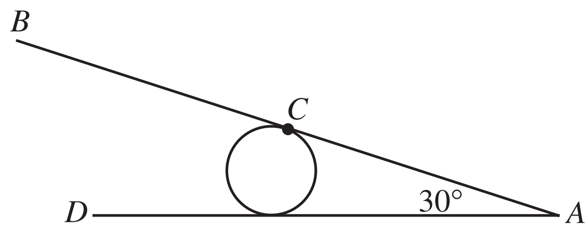
(**A**) 253 (**B**) 63 (**C**) 249 (**D**) 263 (**E**) 259

Solution

If a, b and c are distinct then the correct factorization is $16 = 1 \times 2 \times 8$. Since a, b and c must be some permutation of 1, 2 and 8 there are exactly six possibilities which give the values –247, -61, 65, 249, 263, and 63. Of these, $8^1 - 1^2 + 2^8$ or 263 is the largest.

ANSWER: (D)

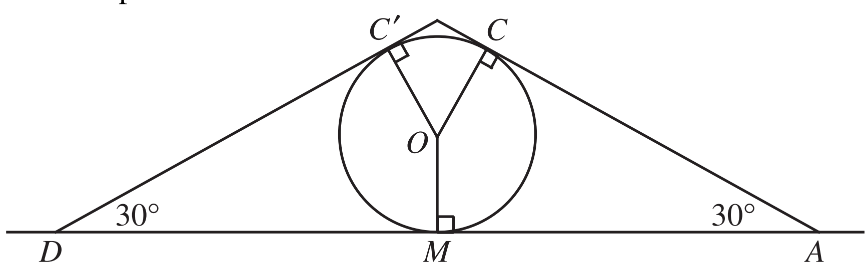
19. A metal rod with ends A and B is welded at its middle, C, to a cylindrical drum of diameter 12. The rod touches the ground at A making a $30°$ angle. The drum starts to roll along AD in the direction of D. How far along AD must the drum roll for B to touch the ground?



(**A**) π (**B**) 2π (**C**) 3π (**D**) 4π (**E**) 5π

Solution

The drum rolls so that point C moves to C'.



Since a line drawn from the centre of the circle makes angles of $90°$ with tangents drawn to a circle, $\angle COM = 360° - 90° - 90° - 30° = 150°$. By symmetry, $\angle C'OM = 150°$ and thus $\angle C'OC = 360° - 150° - 150° = 60°$. Since $\angle C'OC = 60°$, this implies that point C will have to travel $\frac{1}{6}$th the circumference of the circle or $\frac{1}{\cancel{6}_1}(\cancel{12}^{2}\pi) = 2\pi$.

ANSWER: (B)

20. Twenty pairs of integers are formed using each of the integers 1, 2, 3, ..., 40 once. The positive difference between the integers in each pair is 1 or 3. (For example, 5 can be paired with 2, 4, 6 or 8.) If the resulting differences are added together, the greatest possible sum is

(**A**) 50 (**B**) 54 (**C**) 56 (**D**) 58 (**E**) 60

Solution

Since we have twenty pairings, it is possible to have nineteen differences of 3 and one difference of 1. The maximum sum of these differences is thus, $3(19)+1=58$. The pairings can be achieved in the following way: $\{(1,4), (2,5), (3,6), (7,10), (8,11), (9,12), (13,16), (14,17), (15,18), (19,22), (20,23), (21,24), (25,28), (26,29), (27,30), (31,34), (32,35), (33,36), (37,40), (38,39)\}$. Note that there is just one pair, $(38,39)$, that differs by one.

ANSWER: (D)

Part C: Each correct answer is worth 8.

21. A wooden rectangular prism has dimensions 4 by 5 by 6. This solid is painted green and then cut into 1 by 1 by 1 cubes. The ratio of the number of cubes with exactly two green faces to the number of cubes with three green faces is

(**A**) 9:2 (**B**) 9:4 (**C**) 6:1 (**D**) 3:1 (**E**) 5:2

Solution

The cubes with two green faces are the cubes along the edges, not counting the corner cubes. In each dimension, we lost two cubes to the corners so we then have four edges with 4 cubes, four with 3 cubes and four with 2 cubes. The total number of cubes with paint on two edges is then $4(4)+4(3)+4(2)=36$. The number of cubes that have paint on three sides are the corner cubes of which there are eight. The required ratio is then 36:8 or 9:2.

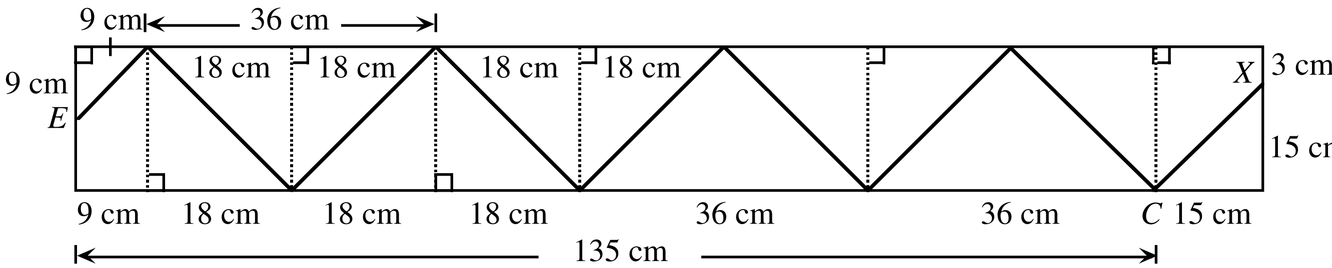
ANSWER: (A)

22. An ant walks inside a 18 cm by 150 cm rectangle. The ant's path follows straight lines which always make angles of $45°$ to the sides of the rectangle. The ant starts from a point X on one of the shorter sides. The first time the ant reaches the opposite side, it arrives at the midpoint. What is the distance, in centimetres, from X to the nearest corner of the rectangle?

(**A**) 3 (**B**) 4 (**C**) 6 (**D**) 8 (**E**) 9

Solution

If we took a movie of the ant's path and then played it backwards, the ant would now start at the point E and would then end up at point X. Since the ant now 'starts' at a point nine cm from the corner, the 'first' part of his journey is from E to B. This amounts to nine cm along the length of the rectangle since ΔBAE is an isosceles right-angled triangle. This process continues as illustrated, until the ant reaches point C. By the time the ant has reached C, it has travelled $9+18+3\times 36$ or 135 cm along the length of the rectangle. To travel from C to X, the ant must travel 15 cm along the length of the rectangle which puts the ant 3 cm from the closest vertex.



ANSWER: (A)

23. The left most digit of an integer of length 2000 digits is 3. In this integer, any two consecutive digits must be divisible by 17 or 23. The 2000th digit may be either 'a' or 'b'. What is the value of $a+b$?

(**A**) 3 (**B**) 7 (**C**) 4 (**D**) 10 (**E**) 17

Solution
We start by noting that the two-digit multiples of 17 are 17, 34, 51, 68, and 85. Similarly we note that the two-digit multiples of 23 are 23, 46, 69, and 92. The first digit is 3 and since the only two-digit number in the two lists starting with 3 is 34, the second digit is 4. Similarly the third digit must be 6. The fourth digit, however, can be either 8 or 9. Let's consider this in two cases.

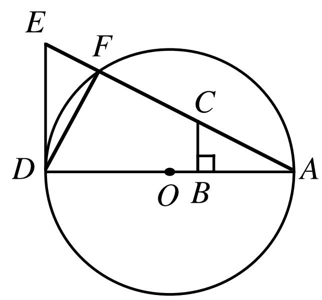
Case 1
If the fourth digit is 8, the number would be 3468517 and would stop here since there isn't a number in the two lists starting with 7.

Case 2
If the fourth digit is 9, the number would be 34692 34692 34 ... and the five digits '34692' would continue repeating indefinitely as long as we choose 9 to follow 6.

If we consider a 2000 digit number, its first 1995 digits must contain 399 groups of '34692'. The last groups of five digits could be either 34692 or 34685 which means that the 2000th digit may be either 2 or 5 so that $a+b=2+5=7$.

ANSWER: (B)

24. In the diagram shown, $\angle ABC = 90°$, $CB \| ED$, $AB = DF$, $AD = 24$, $AE = 25$ and O is the centre of the circle. Determine the perimeter of $CBDF$.



(**A**) 39 (**B**) 40 (**C**) 42
(**D**) 43 (**E**) 44

Solution
We start by showing that $\Delta ABC \cong \Delta DFE$.
Since $ED \| CB$ this implies that $\angle DEF = \angle BCA$ because of corresponding angles. Also, $\angle DFA = 90°$ because it is an angle in a semicircle which also means that $\angle DFE$ is $90°$. Thus the two triangles are equiangular. Since $AB = DF$, $\Delta ABC \cong \Delta DFE$ (ASA). Therefore, $EF = CB$ and $DF = BA$. Using Pythagorus in ΔADE, $DE^2 = 25^2 - 24^2 \Rightarrow DE = 7 = CA$.
Thus $CE = 25 - 7 = 18$.
The required perimeter is,

$$\begin{aligned} CB + BD + DF + FC = EF + (BD + DF) + FC &= (EF + FC) + (BD + DF) \\ &= (EF + FC) + (BD + BA), \text{ since } DF = BA \\ &= CE + AD \\ &= 18 + 24 = 42. \end{aligned}$$

ANSWER: (C)

25. For the system of equations $x^2 + x^2y^2 + x^2y^4 = 525$ and $x + xy + xy^2 = 35$, the sum of the real y values that satisfy the equations is

(**A**) 20 (**B**) 2 (**C**) $\frac{3}{2}$ (**D**) $\frac{55}{2}$ (**E**) $\frac{5}{2}$

Consider the system of equations
$$x^2 + x^2y^2 + x^2y^4 = 525 \qquad (1)$$
and
$$x + xy + xy^2 = 35 \qquad (2)$$

The expression on the left side of equation (1) can be rewritten as,

$$\begin{aligned} x^2 + x^2y^2 + x^2y^4 &= \left(x + xy^2\right)^2 - x^2y^2 \\ &= \left(x + xy^2 - xy\right)\left(x + xy^2 + xy\right) \end{aligned}$$

Thus, $\left(x + xy^2 - xy\right)\left(x + xy^2 + xy\right) = 525$

Substituting from (2) gives, $\left(x + xy^2 - xy\right)(35) = 525$

or, $x + xy^2 - xy = 15 \qquad (3)$

Now subtracting (3) from (2), $2xy = 20$, $x = \frac{10}{y}$

Substituting for x in (3) gives,

$$\begin{aligned}\frac{10}{y}+10y-10&=15\\10y^2-25y+10&=0\\2y^2-5y+2&=0\\(2y-1)(y-2)&=0\\y=\frac{1}{2}\quad\text{or}\quad y&=2\end{aligned}$$

The sum of the real y values satisfying the system is $\frac{5}{2}$.

ANSWER: (E)