Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2002 Solutions
Fermat Contest (Grade 11)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Awards

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1. If $x = 3$, $5 - 2x^2 = 5 - 2(3)^2 = 5 - 18 = -13$.

ANSWER: (C)

2. Evaluating,

$$\frac{3^3 + 3}{2^2 + 2} = \frac{27 + 3}{4 + 2} = \frac{30}{6} = 5.$$

ANSWER: (E)

3. Since there are 24 hours in a day, then 56 hours is two full days (48 hours) plus an additional 8 hours. So to find the correct time, we add 8 hours to 9:04 a.m., and obtain 5:04 p.m. ANSWER: (B)

4. We look at each of the five statements.
25 is a perfect square, since $25 = 5^2$.
31 is a prime number, since it has no positive factors other than 1 and 31.
3 is not the smallest prime number, since 2 is a prime number.
8 is a perfect cube, since $8 = 2^3$.
15 is the product of two prime numbers, since $15 = 3 \times 5$. ANSWER: (C)

5. The area of the entire poster is $(50 \text{ cm})(100 \text{ cm}) = 5000 \text{ cm}^2$.
The area of the picture of Pierre de Fermat is $(20 \text{ cm})(40 \text{ cm}) = 800 \text{ cm}^2$.
Thus, the percentage of the poster covered by the picture is

$$\frac{800 \text{ cm}^2}{5000 \text{ cm}^2} \times 100\% = \frac{8}{50} \times 100\% = 16\%$$

ANSWER: (B)

6. Let the heights of Gisa, Henry, Ivan, Justina and Katie be G, H, I, J, and K, respectively. From the first sentence, $H < G < J$. From the second sentence, $K < I < G$. So J is bigger than all of G, H, I, and K, so Justina is the tallest. ANSWER: (D)

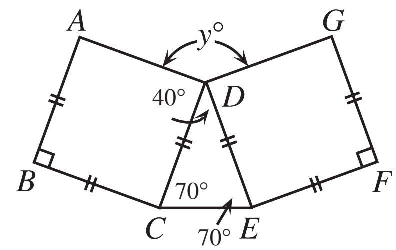
7. We can determine the area of the shaded region by guessing the side lengths of the various rectangles. Let us suppose that the top left rectangle has a width of 2 and a height of 3. Then the top right rectangle has a width of 5, since its height is also 3. Thus, we can conclude that the height of the bottom right rectangle is 5. This tells us that the shaded rectangle is 2 by 5, or has an area of 10. This problem can also be solved with a more algebraic approach, but this is the most straightforward way. ANSWER: (E)

8. Since squares $ABCD$ and $DEFG$ have equal side lengths, then $DC = DE$, ie. ΔCDE is isosceles. Therefore, $\angle DEC = \angle DCE = 70^\circ$ and so

$$\angle CDE = 180^\circ - 70^\circ - 70^\circ = 40^\circ$$

and

$$y^\circ = 360^\circ - \angle ADC - \angle CDE - \angle GDE$$
$$y^\circ = 360^\circ - 90^\circ - 40^\circ - 90^\circ$$
$$y^\circ = 140^\circ$$
$$y = 140$$



ANSWER: (E)

9. There are 20 possible golf balls that can be drawn, and 6 of them have multiples of 3 written on them (ie. 3, 6, 9, 12, 15, 18). Therefore, the probability that the number on the golf ball drawn is a multiple of 3 is $\frac{6}{20}$. ANSWER: (B)

10. Since $ABCD$ is a square, then $AB = BC$, or

$$x + 16 = 3x$$
$$16 = 2x$$
$$x = 8$$

Thus the side length of the square is $x + 16 = 3x = 24$, and the perimeter is $4(24) = 96$.

ANSWER: (C)

11. The slope of the line is

$$\frac{-2-0}{0-1} = 2$$

and its y-intercept is –2 (from the first point given), so it has equation $y = 2x - 2$. Substituting in the point $(7, b)$, we obtain $b = 2(7) - 2 = 12$. ANSWER: (A)

12. We determine the smallest and largest three-digit perfect squares first.
Clearly, the smallest three-digit perfect square is $100 = 10^2$.
Now $\sqrt{1000} \approx 31.6$, so the largest perfect square less than 1000 is $31^2 = 961$ (since $31^2 < 1000$ and $32^2 > 1000$).
Therefore, only the perfect squares between $10^2 = 100$ and $31^2 = 961$ have three digits, which means that exactly 22 three-digit positive integers are perfect squares.

ANSWER: (B)

13. A "double-single" number is of the form *aab*, where *a* and *b* are different digits. There are 9 possibilities for *a* (since *a* cannot be 0). For each of these possibilities, there are 9

possibilities for b (since b can be any digit from 0 to 9, provided that it doesn't equal a). So there are $9 \times 9 = 81$ possibilities in total. ANSWER: (A)

14. If we divide 2002 by 7, we see that $2002 = 7(286)$. Since there are 7 natural numbers in each row, and the last entry in each row is the multiple of 7 corresponding to the row number, then 2002 must lie in the 7th column of the 286th row. So $m = 7$, $n = 286$, and $m + n = 293$. ANSWER: (D)

15. Using the rules for forming the sequence, the third term is $a + 2$ and the fourth term is $a + 2 + (a + 2) = 2(a + 2)$. Similarly, the fifth term is $4(a + 2)$ and the sixth term is $8(a + 2)$. But the sixth term is 56, so $8(a + 2) = 56$ or $a + 2 = 7$ or $a = 5$.

ANSWER: (E)

16. *Solution 1*

We factor out a common factor of a from the first two terms and a common factor of b from the last two terms, and so

$$\begin{aligned} ac + ad + bc + bd &= 68 \\ a(c + d) + b(c + d) &= 68 \\ a(4) + b(4) &= 68 \\ 4(a + b) &= 68 \\ a + b &= 17 \end{aligned}$$

using the fact that $c + d = 4$.
Then $a + b + c + d = (a + b) + (c + d) = 17 + 4 = 21$.

Solution 2

We let $c = d = 2$, since $c + d = 4$, and then substitute these values to find that

$$\begin{aligned} ac + ad + bc + bd &= 68 \\ 2a + 2a + 2b + 2b &= 68 \\ a + b &= 17 \end{aligned}$$

Thus $(a + b) + (c + d) = 17 + 4 = 21$.

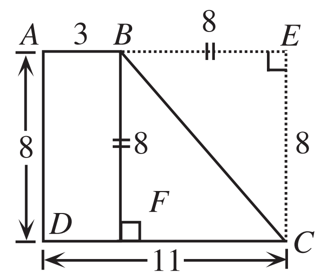
ANSWER: (D)

17. Let the number of females in the group be F.
Then since the average age of the females is 28, then the sum of the ages of the females is $28F$.
There are also $140 - F$ males in the group, and the sum of there ages is $21(140 - F)$.
So since the average of all of the ages is 24, then

$$\frac{\text{Sum of all ages}}{140} = 24$$
$$\frac{28F + 21(140 - F)}{140} = 24$$
$$28F + 21(140) - 21F = 24(140)$$
$$7F = 3(140)$$
$$F = 60$$

Therefore, there are 60 females in the group. ANSWER: (D)

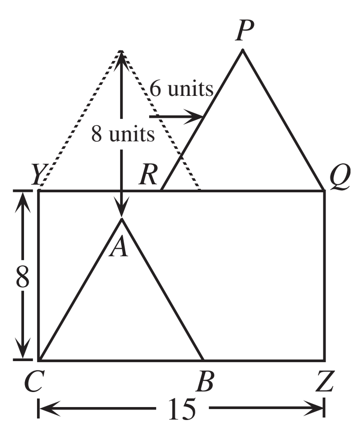
18. Since E is folded over onto F, then BE is equal to BF and $\angle BFC = \angle BEC = 90^\circ$.
So $BECF$ is a rectangle, and since $EC = 8$ cm, then $BF = 8$ cm and so $BE = FC = 8$ cm. Thus, $AB = AE - BE = 11 - 8 = 3$ cm and $BC = \sqrt{BF^2 + FC^2} = \sqrt{64 + 64} = \sqrt{128} \approx 11.31$ cm
Therefore, the perimeter of the trapezoid is approximately $3 + 11.31 + 11 + 8 = 33.31$ cm, which is closest to 33.3 cm.



ANSWER: (A)

19. We rewrite the right side using exponent laws:
$$\begin{aligned} 2^a 3^b &= 8\left(6^{10}\right) \\ &= 2^3\left([(2)(3)]^{10}\right) \\ &= 2^3\left(2^{10}3^{10}\right) \\ &= 2^{13}3^{10} \end{aligned}$$
and so $a = 13$ and $b = 10$, which tells us that $b - a = -3$. ANSWER: (E)

20. Since ΔABC and ΔPQR are both equilateral triangles with side length 9, they are congruent triangles.
So we can think of translating ΔABC until it lies on top of ΔPQR by shifting it up 8 units, until CB lies on YQ (C will coincide with Y) and then shifting it to the right $15 - 9 = 6$ units until B coincides with Q. (This last shift is 6 units to the right since $YQ = 15$ and $CB = 9$, so B will be a distance of 6 from Q before this second shift.)
So we move from A to P by moving 8 units up and 6 to the right, or a distance of $\sqrt{6^2 + 8^2} = 10$ units.



ANSWER: (A)

21. Since

$$\sqrt{\frac{3}{1}\cdot\frac{5}{3}\cdot\frac{7}{5}\cdot\ \cdots\ \cdot\frac{2n+1}{2n-1}}=9$$

then

$$\frac{3}{1}\cdot\frac{5}{3}\cdot\frac{7}{5}\cdot\ \cdots\ \cdot\frac{2n+1}{2n-1}=81$$

In the product on the left-hand side, the numerator in each fraction but the last appears as the denominator in the next fraction and so can be divided out. (This is called a "telescoping product".) After doing this cancellation, all that is left is the denominator of 1 from the first fraction and the numerator of $2n+1$ from the last fraction. Therefore, $\frac{2n+1}{1}=2n+1=81$ or $n=40$.

ANSWER: (C)

22. Using the equation and information given,

$$f(2)=f(1+1)=f(1)+f(1)+2(1)(1)=4+4+2=10.$$

(Notice that there is nothing else we could have actually calculated at this stage.)

Continuing,

$$f(4)=f(2+2)=f(2)+f(2)+2(2)(2)=10+10+8=28$$

$$f(8)=f(4+4)=f(4)+f(4)+2(4)(4)=28+28+32=88$$

Alternatively, we could have used the functional equation to calculate the values of $f(2)$, $f(3)=f(2+1)$, $f(4)=f(3+1)$, etc., all the way up to $f(8)$.

ANSWER: (C)

23. Once the m eights and k nines are added to the list, we have $9+m+k$ numbers in total and the sum of these numbers is

$$1+2+3+4+5+6+7+8+9+8m+9k=45+8m+9k.$$

So we calculate the average and equate with the given value

$$\begin{aligned}\frac{\text{Sum of numbers in list}}{\text{Number of numbers in list}}&=7.3\\ \frac{45+8m+9k}{9+m+k}&=\frac{73}{10}\\ 450+80m+90k&=657+73m+73k\\ 7m+17k&=207\end{aligned}$$

Next we note that m and k are both positive integers, and that $k<13$, since $13(17)=221>207$. So we try the possibilities for k from 1 to 12, and see that for m to be a positive integer, the only possibility is $k=6$, which gives $m=15$.

Thus, $k+m=21$.

ANSWER: (B)

24. First, we calculate the volumes of the two cylindrical containers:

$$V_{\text{large}} = \pi(6)^2(20) = 720\pi \text{ cm}^3$$

$$V_{\text{small}} = \pi(5)^2(18) = 450\pi \text{ cm}^3$$

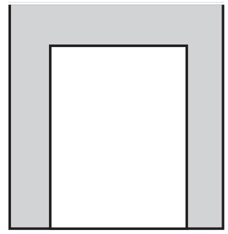


Figure 3

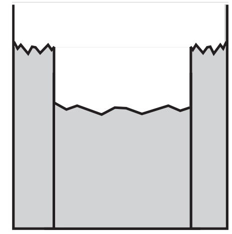


Figure 4

The volume of water initially contained in the large cylinder is

$$V_{\text{water, initial}} = \pi(6)^2(17) = 612\pi \text{ cm}^3$$

The easiest way to determine the final depth of water in the small cylinder is as follows. Imagine putting a lid on the smaller container and lowering it all the way to the bottom of the larger container, as shown in Figure 3. So there will be water beside and above the smaller container. Note that the larger container will be filled to the brim (since the combined volume of the small container and the initial water is greater than the volume of the large container) and some water will have spilled out of the larger container.

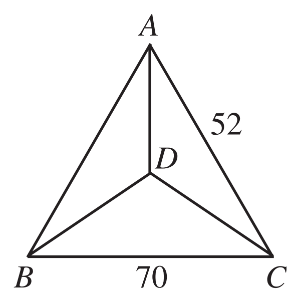
Now if the lid on the small container is removed, all of the water in the large container above the level of the brim of the small container will spill into the small container, as shown in Figure 4. This water occupies a cylindrical region of radius 6 cm and height 2 cm, and so has a volume of $\pi(6)^2(2) = 72\pi \text{ cm}^3$. This is the volume of water that is finally in the small container. Since the radius of the small container is 5 cm, then the depth of water is

$$\text{Depth} = \frac{72\pi \text{ cm}^3}{\pi(5 \text{ cm})^2} = \frac{72}{25} \text{ cm} = 2.88 \text{ cm}$$

ANSWER: (D)

25. The important fact to remember to solve this problem is that if a, b, c are the side lengths of a triangle, then $a+b>c$, $a+c>b$, and $b+c>a$. This is called the "Triangle Inequality". We can simplify this slightly by saying that if a, b, c are the side lengths of a triangle with $a \le b \le c$, then we need $a+b>c$ to be true. (The other two inequalities are automatically satisfied since c is the longest side.)

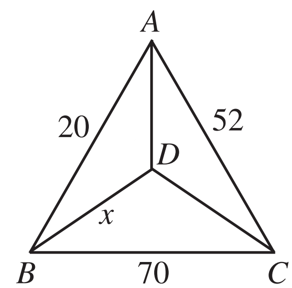
Let x represent the length of the unknown edge of the tetrahedron.
We represent the tetrahedron schematically as in the diagram.
We can think here of triangle ABC as being the "base" of the tetrahedron, and vertex D as being the "top" vertex of the tetrahedron. This diagram also emphasizes that the tetrahedron is formed by 4 triangles ABC, ABD, ADC, and BDC.



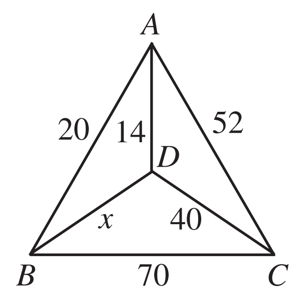
Let $BC = 70$. Now BC is an edge in two triangles (ABC and BDC) and so one of these triangles must be formed entirely from known edge lengths. Of the known edge lengths, only 40 and 52, or 20 and 52 can form a triangle with a third side of 70, by the Triangle Inequality. So let $AC = 52$ (the common edge length of these two triangles).

If $AB = 40$, then the three remaining edge lengths are 14, 20 and x. 14 and 20 cannot together make a triangle with any of 40, 52, or 70. So there are no possibilities for x in this case.
If $AB = 20$, the three remaining edge lengths are 14, 40 and x. 14 and 40 cannot together make a triangle with 70, so either $BD = x$ or $DC = x$.

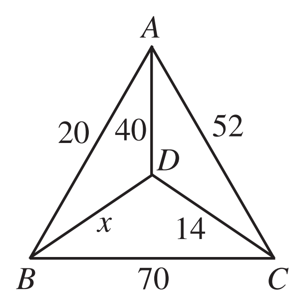
If $DC = x$, then triangle ABD must have side lengths 20, 14 and 40, which is impossible, so $BD = x$.



<u>Case 1</u>: $AD = 14$ and $DC = 40$
From ΔABD, $20 + 14 > x$ and $14 + x > 20$, so since x is an integer, $7 \le x \le 33$.
From ΔBDC, $40 + 70 > x$ and $x + 40 > 70$, so since x is an integer, $31 \le x \le 109$.
Combining these conditions, $31 \le x \le 33$.



<u>Case 1</u>: $DC = 14$ and $AD = 40$
From ΔABD, $20 + 40 > x$ and $20 + x > 40$, so since x is an integer, $21 \le x \le 59$.
From ΔBDC, $14 + 70 > x$ and $x + 14 > 70$, so since x is an integer, $57 \le x \le 83$.
Combining these conditions, $57 \le x \le 59$.



So there are 6 possibilities for x.
[Note: In fact, all six of these possibilities do in fact form tetrahedra with the other 5 edge lengths. As we discovered as we were creating this question, it is possible to come up with side lengths that do form 4 triangles, but which cannot be put together to form a tetrahedron. This is a very interesting problem to work on.] ANSWER: (E)