Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2004 Solutions
Cayley Contest (Grade 10)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

2004 Cayley Contest Solutions

1. Calculating each term,
$$2^2+1^2+0^2+(-1)^2+(-2)^2=4+1+0+1+4=10$$
Answer: (E)

2. 25% of 2004 is $\frac{1}{4}$ of 2004, or 501.
50% of 4008 is $\frac{1}{2}$ of 4008, or 2004.
50% of 1002 is $\frac{1}{2}$ of 1002, or 501.
100% of 1002 is 1002.
10% of 8016 is $\frac{1}{10}$ of 8016, or 801.6.
20% of 3006 is $\frac{1}{5}$ of 3006, or 601.2.
Answer: (B)

3. To get from A to B, we must go 2 units to the right and 3 units up. Since B is the midpoint of AC, then to get from B to C, we must also go 2 units to the right and 3 units up. Therefore, C has coordinates $(5,7)$.
Answer: (E)

4. Since $x+1-2+3-4=5-6+7-8$, then simplifying both sides we get $x-2=-2$ so $x=0$.
Answer: (C)

5. The first figure has outer perimeter 4. The second figure has outer perimeter 8. The third figure has outer perimeter 12. So from one figure to the next, the outer perimeter increases by 4. So the outer perimeter of the fifth figure is 8 more than that of the third figure, or 20. (We could easily draw out the fifth figure, which would be made up of 9 small squares.)
Answer: (C)

6. Removing a common factor of 7,
$$7x+42y=7(x+6y)=7(17)=119$$
Answer: (E)

7. *Solution 1*
$3^2+3^2+3^2=3\times 3^2=3^1\times 3^2=3^3$. Thus, $a=3$.

Solution 2
$3^2+3^2+3^2=9+9+9=27$, so $3^a=27$ and so $a=3$, since $3^3=27$.
Answer: (B)

8. The circumference of a circle is equal to $2\pi r$, where r is the radius of the circle.
Since the circumference of the outer circle is 24π, then its radius is 12. Thus, $OB=12$.
Since the circumference of the inner circle is 14π, then its radius is 7. Thus, $OA=7$.
Therefore, $AB=OB-OA=5$. Answer: (B)

9. By the Pythagorean Theorem, the length of the rope joining B to C is
$\sqrt{16^2+30^2}=\sqrt{256+900}=\sqrt{1156}=34\,\text{m}$.
We must also determine the length of rope joining A to C. To get from A to C we must go over 16 m and up 12 m (the difference between the heights of the towers), so the rope has length $\sqrt{16^2+12^2}=\sqrt{256+144}=\sqrt{400}=20\,\text{m}$.
Therefore, the total length of rope used is 54 m.

Answer: (A)

10. When the cube is folded, H and I share an edge. At one end of this edge is the face labelled G (so the faces G, H and I meet at a point), and at the other end of this edge is the face labelled J.
Thus, J is opposite G.

Answer: (D)

11. Since each number after the second is the product of the previous two numbers, then the 18 in the fifth position is the product of the 3 and the number in the fourth position. Thus, the number in the fourth position is 6.
So the sequence now reads, x, ____, 3, 6, 18.
Using the rule, 6 equals the product of 3 and the number in the second position. Thus, the sequence reads x, 2, 3, 6, 18.
This tells us that $2x=3$ or $x=\frac{3}{2}$.

Answer: (B)

12. The sum of the numbers in the 1st row is $2x+5$, so the sum of the numbers in any row, column or diagonal is also $2x+5$.
Therefore, the entry in the 2nd row, 1st column must be 5, using the 1st column.
Thus, the entry in the 2nd row, 2nd column is $2x+3$, using the 2nd row.
Looking at the sum of the 2nd column,

$$\begin{aligned}3+(2x+3)+x&=2x+5\\3x+6&=2x+5\\x&=-1\end{aligned}$$

so the sum of the numbers in any row is $2(-1)+5=3$.

Answer: (C)

13. Since the perimeter of the smaller square is 72 cm, its side length is $\frac{1}{4}(72)=18$ cm.
Therefore, the area of the smaller square is $(18\text{ cm})^2=324\text{ cm}^2$.
Since the area inside the larger square is the combined area of the smaller square and the shaded region, then the area of the larger square is $160\text{ cm}^2+324\text{ cm}^2=484\text{ cm}^2$.
Thus, the side length of the larger square is $\sqrt{484}=22\,\text{cm}$, and so the perimeter of the larger square is $4(22\text{ cm})=88\text{ cm}$.

Answer: (B)

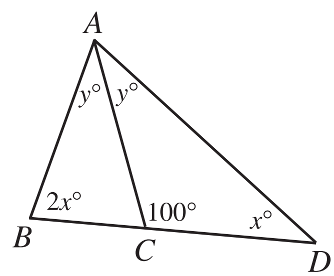
14. The average of 4, 20 and x is equal to $\dfrac{4+20+x}{3}$. The average of y and 16 is equal to $\dfrac{16+y}{2}$.

Since these averages are equal,

$$\begin{aligned}\frac{4+20+x}{3} &= \frac{16+y}{2}\\ 2(4+20+x) &= 3(16+y)\\ 48+2x &= 48+3y\\ 2x &= 3y\\ \frac{x}{y} &= \frac{3}{2}\end{aligned}$$

Answer: (A)

15. In triangle ACD, $x^\circ + y^\circ + 100^\circ = 180^\circ$, so $x + y = 80$ (*).
Since $\angle ACB$ and $\angle ACD$ are supplementary, then $\angle ACB = 180^\circ - \angle ACD = 80^\circ$.
Thus, in triangle ACB, $2x^\circ + y^\circ + 80^\circ = 180^\circ$, so $2x + y = 100$ (**).
Subtracting (*) from (**), we obtain $x = 20$.



Answer: (E)

16. When a player rolls two dice, there are 6 possibilities for the outcome on each die, so there are 36 possibilities for the outcomes when two dice are rolled.
Which possibilities give a score of 3 or less? These are: 1 and 1, 1 and 2, 1 and 3, 2 and 1, 2 and 2, 2 and 3, 3 and 1, 3 and 2, 3 and 3. So 9 of the 36 possibilities give a score of 3 or less.
Thus, the probability is $\frac{9}{36} = \frac{1}{4}$.

Answer: (A)

17. Putting each of the two fractions over a common denominator of mn, we get

$$\begin{aligned}\frac{1}{m}+\frac{1}{n} &= \frac{5}{24}\\ \frac{n}{mn}+\frac{m}{mn} &= \frac{5}{24}\\ \frac{m+n}{mn} &= \frac{5}{24}\\ \frac{20}{mn} &= \frac{5}{24}\\ mn &= \frac{24\times 20}{5}\\ mn &= 96\end{aligned}$$

Answer: (D)

18. This problem requires a fair amount of fiddling around. After some work, we can see that the ant can walk along 9 edges without walking along any edge for a second time. (For example, it could walk A to B to F to E to A to D to C to G to H to D. It is then stuck.)
In fact, 9 edges, or 108 cm, is the maximum.

Justifying this fact takes us into a really neat area of mathematics called "graph theory". (Graph theory was first developed by Euler when he was working on the Königsberg bridge problem.)
What if the ant could actually walk along 10 edges?
If the ant had done this, it would have walked in and out of a vertex 20 times in total (once at each end of each of the 10 edges). The cube has 8 vertices and each vertex has 3 edges meeting at it, so since the ant has not walked along the same edge twice, then it can only have been in and out of any given vertex at most 3 times.
But for 20 ins and outs with 8 vertices in total, there must be 4 vertices that have been used 3 times.
This is impossible, though, because other than the starting vertex and the ending vertex in the path, the ant must go both in and out of a vertex, so each vertex other than the starting and ending vertices must be used an even number of times. So we cannot have more than 2 vertices being used an odd number of times.
Therefore, the ant cannot walk along 10 edges, so 108 cm must indeed be the maximum.

Answer: (D)

19. Here we use the rule for manipulating exponents $\dfrac{2^a}{2^b} = 2^{a-b}$.

Therefore, each of the 2003 fractions after the first fraction is equal to $2^{-1} = \dfrac{1}{2}$.

This gives us 2004 copies of $\dfrac{1}{2}$ being added up, for a total of 1002.

Answer: (A)

20. Let a be the value of an arrow shot into ring A, let b be the value of an arrow shot into ring B, and let c be the value of an arrow shot into ring C.
From the given information about the three archers, we know that

$$c + a = 15$$
$$c + b = 18$$
$$b + a = 13$$

We are interested in calculating $2b$.
Adding up the second and third equations, we get $a + 2b + c = 31$ and so substituting the information from the first equation, we get $2b + 15 = 31$ or $2b = 16$.
(Notice that we could have found the values of a, b and c, but we did not need to do this.)

Answer: (C)

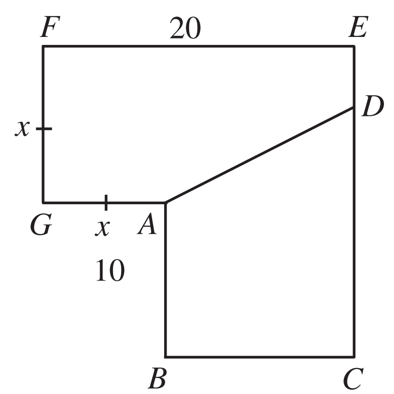
21. Suppose that Laura uses the last blue sheet on day number d.
Then the total number of blue sheets with which she started was d.
Since she uses 3 red sheets per day and has 15 red sheets left over, then she started with $3d+15$ red sheets.
Since the blue and red sheets were initially in the ratio 2 : 7, then

$$\frac{d}{3d+15}=\frac{2}{7}$$
$$7d=6d+30$$
$$d=30$$

Thus, she started with 30 blue sheets and 105 red sheets, or 135 sheets of construction paper in total.

Answer: (C)

22. First, we determine the lengths of the sides of the rooms.

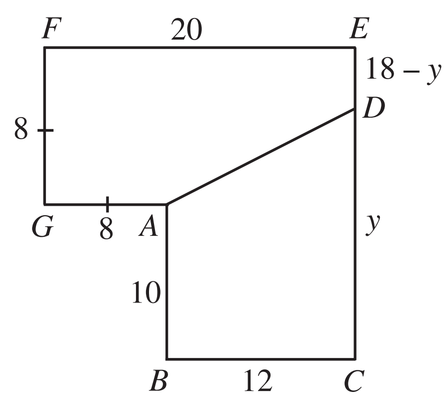


Suppose that $AG=x$. Then $FG=x$.
So the room can be thought of as a rectangular room of width $FE=20$ and length $AB+FG=10+x$, with a rectangular corner of dimensions $AG=x$ by $AB=10$ removed.
Equating the area of the entire room with this way of visualizing it,

$$20(10+x)-10x=280$$
$$10x+200=280$$
$$x=8$$

Therefore, the lengths of the sides of the room are (starting from B and proceeding clockwise) 10, 8, 8, 20, 18 and 12.

Let y now be the distance from C to D, the point where the new wall touches CE.
Now $CBAD$ can be viewed as a trapezoid with base CD and parallel side AB (since the room has square corners).
Also, the height of the trapezoid is $BC=12$. The area of this trapezoid is supposed to be 140, or half of the total area of the large room.
Therefore, since the area of the trapezoid is $\frac{1}{2}(BC)(AB+CD)$, we have



$$\tfrac{1}{2}(12)(10+y)=140$$
$$6(10+y)=140$$
$$10+y=\tfrac{70}{3}$$
$$y=\tfrac{40}{3}$$

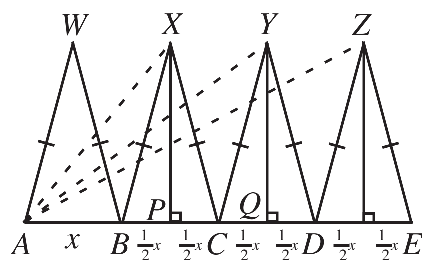
Also, the height of the trapezoid is $BC = 12$. The area of this trapezoid is supposed to be 140, or half of the total area of the large room.
Therefore, since the area of the trapezoid is $\frac{1}{2}(BC)(AB + CD)$, we have
Thus, the distance from C to D is $\frac{40}{3}$.

Answer: (E)

23. The ball is rolling towards Marcos at 4 m/s and he is running towards it at 8 m/s, so he gains 12 metres per second on the ball. Since he starts 30 m from the ball, it will take him $\frac{30}{12} = 2.5$ s to reach the ball.
The ball is rolling away from Michael at 4 m/s and he is running at 9 m/s, so he is gaining 5 m/s on the ball. Since he starts 15 m behind the ball, he would catch up to the ball in 3 s if it continued to roll.
Thus, Marcos gets to the ball first. After 2.5 s, Michael has gained $5(2.5) = 12.5$ m on the ball, so is 2.5 m from the ball when Marcos touches it first.

Answer: (C)

24. First, we determine the lengths of the sides of the new triangle, in terms of x.
We drop perpendiculars from X, Y and Z to points P, Q and R, respectively, on the line AE. Since each of the four triangles is isosceles, then
$BP = PC = CQ = QD = DR = RE = \frac{1}{2}x$.



Consider triangle ARZ, which is right-angled at R. Since $AZ = AE = 4x$, then by the Pythagorean Theorem,

$$AR^2 + RZ^2 = AZ^2$$
$$RZ^2 = (4x)^2 - \left(\tfrac{7}{2}x\right)^2$$
$$RZ^2 = \tfrac{15}{4}x^2$$

so the square of the height of each of the four isosceles triangles is $\frac{15}{4}x^2$.

Thus, $AY^2 = AQ^2 + QY^2 = \left(\frac{5}{2}x\right)^2 + \frac{15}{4}x^2 = 10x^2$, so $AY = \sqrt{10}x$, and
$AX^2 = AP^2 + PX^2 = \left(\frac{3}{2}x\right)^2 + \frac{15}{4}x^2 = 6x^2$, so $AX = \sqrt{6}x$.

Thus, the new triangle has side lengths $\sqrt{6}x$, $\sqrt{10}x$ and $4x$. Since $\left(\sqrt{6}x\right)^2 + \left(\sqrt{10}x\right)^2 = (4x)^2$, then this new triangle is right-angled, with hypotenuse $4x$, and so has area $\frac{1}{2}\left(\sqrt{6}x\right)\left(\sqrt{10}x\right) = \frac{1}{2}\sqrt{60}x^2 = \sqrt{15}x^2$.

We would like the area to be less than 2004, so $\sqrt{15}x^2 < 2004$ or $x < \sqrt{\dfrac{2004}{\sqrt{15}}} \approx 22.747$.

Therefore, the largest integer value of x that works is 22.

Answer: (E)

25. We start by rewriting each expression so that each has the same numerator:

$$\frac{7x+1}{2}=\frac{2+(7x-1)}{2}=1+\frac{7x-1}{2}$$

$$\frac{7x+2}{3}=\frac{3+(7x-1)}{3}=1+\frac{7x-1}{3}$$

$$\vdots$$

$$\frac{7x+300}{301}=\frac{301+(7x-1)}{301}=1+\frac{7x-1}{301}$$

For each of these expressions, the original fraction will be in lowest terms only when the fraction in the new expression is in lowest terms, ie. $\frac{7x+1}{2}$ is in lowest terms only when $\frac{7x-1}{2}$ is in lowest terms.

So the original problem is equivalent to determining the number of positive integers x with $x \leq 60$ such that each of

$$\frac{7x-1}{2},\ \frac{7x-1}{3},\ \ldots,\ \frac{7x-1}{301}$$

is in lowest terms.

This is equivalent to determine the number of positive integers x with $x \leq 60$ for which $7x-1$ has no common factor with any of the integers from 2 to 301, inclusive.

For x from 1 to 43, $7x-1$ will be actually equal to one of the integers from 2 to 301, so there will be a common factor.

So we must examine the integers from 44 to 60.

If x is odd, then $7x-1$ is even, and so has a common factor of 2, for example.

So we must examine the integers 44, 46, 48, 50, 52, 54, 56, 58, and 60.

The values of $7x-1$ for these integers are 307, 321, 335, 349, 363, 377, 391, 405, and 419, respectively. We would like to determine how many of these have no common factors with any of the integers from 2 to 301.

The integers 321, 363 and 405 are divisible by 3, so they can be removed. The integer 335 is divisible by 5 and so can be removed. This leaves us with 307, 349, 377, 391 and 419.

307 is a prime number.

349 is a prime number.

377 is divisible by 13, so can be removed.

391 is divisible by 17, so can be removed.

419 is a prime number.

Each of these prime numbers is divisible only 1 and itself, so has no common factor with any of the integers from 2 to 301.

Therefore, there are 3 integers x with $x \leq 60$ for which the fractions are all in lowest terms.

Answer: (C)