Canadian Mathematics Competition

An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

Euclid Contest

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING Awards

Tuesday, April 19, 2005

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Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphics displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 2 marks each (questions 1-2) or 3 marks each (questions 3-7). **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.

Instructions for SHORT ANSWER parts:

1. **SHORT ANSWER** parts are indicated like this: .
2. **Enter the answer in the appropriate box in the answer booklet.**
   For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

Instructions for FULL SOLUTION parts:

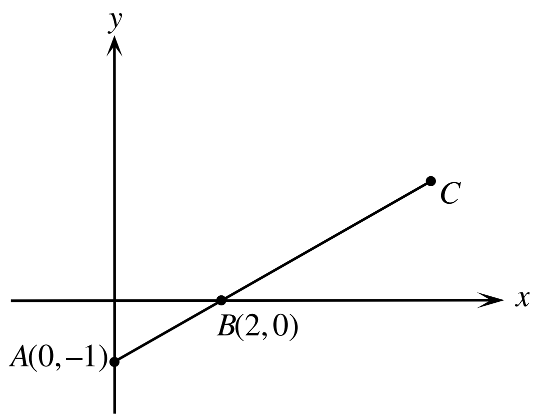
1. **FULL SOLUTION** parts are indicated like this: .
2. **Finished solutions must be written in the appropriate location in the answer booklet.**
   Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

NOTE: At the completion of the Contest, insert the information sheet inside the answer booklet.

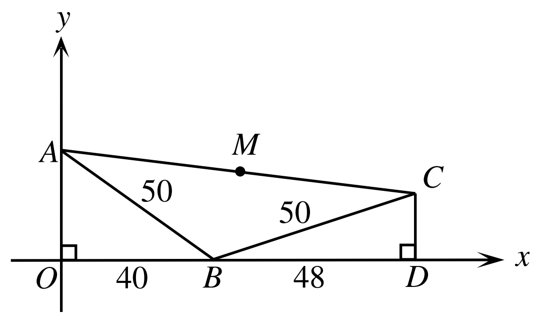
NOTES:	1. Please read the instructions on the front cover of this booklet. 2. Write all answers in the answer booklet provided. 3. For questions marked "💡", full marks will be given for a correct answer placed in the appropriate box in the answer booklet. **If an incorrect answer is given, marks may be given for work shown.** Students are strongly encouraged to show their work. 4. All calculations and answers should be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., except where otherwise indicated.

1. (a) If the point (a,a) lies on the line $3x - y = 10$, what is the value of a?

   (b) In the diagram, points A, B and C lie on a line such that $BC = 2AB$. What are the coordinates of C?



   (c) In the diagram, triangles AOB and CDB are right-angled and M is the midpoint of AC. What are the coordinates of M?



2. (a) If $y = 2x + 3$ and $4y = 5x + 6$, what is the value of x?

   (b) If a, b and c are numbers such that

$$\begin{aligned} -3b + 7c &= -10 \\ b - 2c &= 3 \\ a + 2b - 5c &= 13 \end{aligned}$$

   what is the value of a?

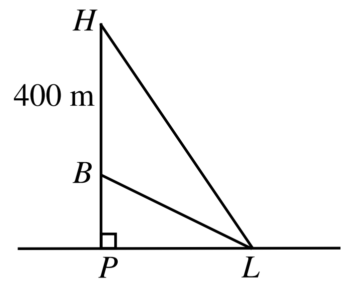
   (c) John and Mary wrote the Euclid Contest. Two times John's score was 60 more than Mary's score. Two times Mary's score was 90 more than John's score. Determine the average of their two scores.

3. (a) If $2^x = 2(16^{12}) + 2(8^{16})$, what is the value of x?

   (b) If $f(x) = 2x - 1$, determine all real values of x such that $\left(f(x)\right)^2 - 3f(x) + 2 = 0$.

4. (a) Six tickets numbered 1 through 6 are placed in a box. Two tickets are randomly selected and removed together. What is the probability that the smaller of the two numbers on the tickets selected is less than or equal to 4?

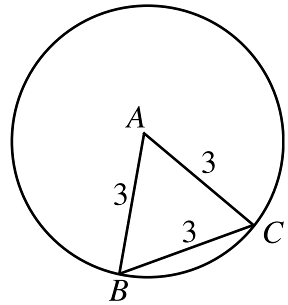
(b) A helicopter hovers at point H, directly above point P on level ground. Lloyd sits on the ground at a point L where $\angle HLP = 60°$. A ball is droppped from the helicopter. When the ball is at point B, 400 m directly below the helicopter, $\angle BLP = 30°$. What is the distance between L and P?



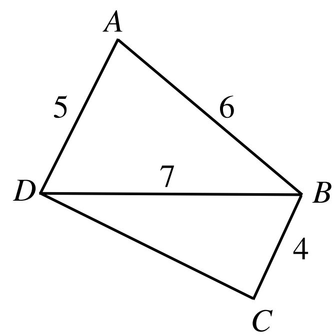
5. (a) A goat starts at the origin $(0,0)$ and then makes several moves. On move 1, it travels 1 unit up to $(0,1)$. On move 2, it travels 2 units right to $(2,1)$. On move 3, it travels 3 units down to $(2,-2)$. On move 4, it travels 4 units to $(-2,-2)$. It continues in this fashion, so that on move n, it turns $90°$ in a clockwise direction from its previous heading and travels n units in this new direction. After n moves, the goat has travelled a total of 55 units. Determine the coordinates of its position at this time.

(b) Determine all possible values of r such that the three term geometric sequence 4, $4r$, $4r^2$ is also an arithmetic sequence.
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9, 11 is an arithmetic sequence.)

6. (a) Equilateral triangle ABC has side length 3, with vertices B and C on a circle of radius 3, as shown. The triangle is then rotated clockwise inside the circle: first about C until A reaches the circle, and then about A until B reaches the circle, and so on. Eventually the triangle returns to its original position and stops. What is the total distance travelled by the point B?



(b) In the diagram, $ABCD$ is a quadrilateral in which $\angle A + \angle C = 180°$. What is the length of CD?

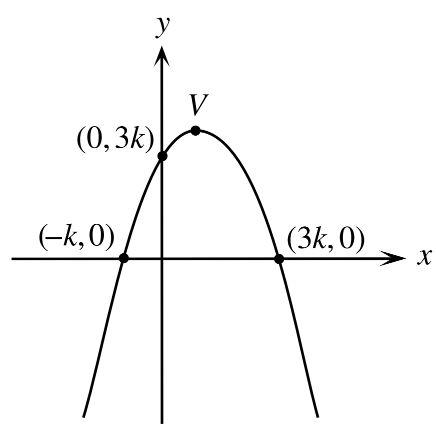


7. (a) If $f(x) = \sin^2 x - 2\sin x + 2$, what are the minimum and maximum values of $f(x)$?

(b) In the diagram, the parabola

$$y = -\frac{1}{4}(x-r)(x-s)$$

intersects the axes at three points. The vertex of this parabola is the point V. Determine the value of k and the coordinates of V.

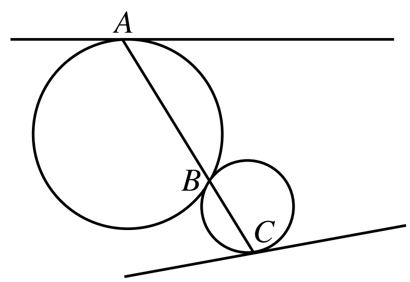


8\. (a) A function is defined by

$$f(x) = \begin{cases} 4 & \text{if } x < -4 \\ -x & \text{if } -4 \leq x \leq 5 \\ -5 & \text{if } x > 5 \end{cases}$$

On the grid in the answer booklet, sketch the graph $g(x) = \sqrt{25 - [f(x)]^2}$. State the shape of each portion of the graph.

(b) In the diagram, two circles are tangent to each other at point B. A straight line is drawn through B cutting the two circles at A and C, as shown. Tangent lines are drawn to the circles at A and C. Prove that these two tangent lines are parallel.



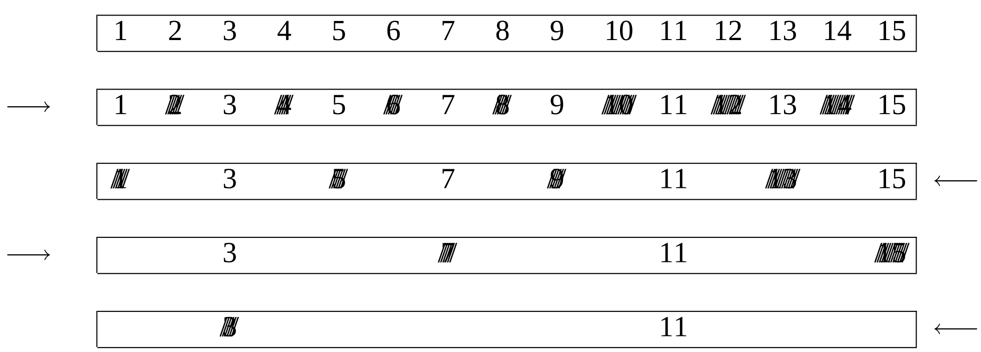
9\. The circle $(x-p)^2 + y^2 = r^2$ has centre C and the circle $x^2 + (y-p)^2 = r^2$ has centre D. The circles intersect at two *distinct* points A and B, with x-coordinates a and b, respectively.

(a) Prove that $a + b = p$ and $a^2 + b^2 = r^2$.

(b) If r is fixed and p is then found to maximize the area of quadrilateral $CADB$, prove that either A or B is the origin.

(c) If p and r are integers, determine the minimum possible distance between A and B. Find positive integers p and r, each larger than 1, that give this distance.

10\. A school has a row of n open lockers, numbered 1 through n. After arriving at school one day, Josephine starts at the beginning of the row and closes every second locker until reaching the end of the row, as shown in the example below. Then on her way back, she closes every second locker that is still open. She continues in this manner along the row, until only one locker remains open. Define $f(n)$ to be the number of the last open locker. For example, if there are 15 lockers, then $f(15) = 11$ as shown below:



(a) Determine $f(50)$.

(b) Prove that there is no positive integer n such that $f(n) = 2005$.

(c) Prove that there are infinitely many positive integers n such that $f(n) = f(2005)$.

Canadian Mathematics Competition

For students...

Thank you for writing the 2005 Euclid Contest!
In 2004, more than 15 000 students around the world registered to write the Euclid Contest.

If you are graduating from secondary school, good luck in your future endeavours!
If you will be returning to secondary school next year, encourage your teacher to register you for the 2005 Canadian Open Mathematics Challenge, which will be written in late November.
Visit our website

www.cemc.uwaterloo.ca

to find

- **More information about the Canadian Open Mathematics Challenge**
- **Free copies of past Contests**
- **Workshops to help you prepare for future Contests**
- **Information about our publications for math enrichment and Contest preparation**
- **Information about careers in math**

For teachers...

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- **Obtain information about our 2005-6 Contests**
- **Learn about workshops and resources we offer for teachers**
- **Find your school results**