Canadian Mathematics Competition
An activity of the Centre for Education in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2006 Euclid Contest

Wednesday, April 19, 2006

Solutions

1. (a) Answer: 0

Solution 1

Since $3x - 3y = 24$, then $x - y = 8$.
To determine the x-intercept we set $y = 0$ and obtain $x = 8$.
To determine the y-intercept we set $x = 0$ and obtain $y = -8$.
Thus, the sum of the intercepts is $8 + (-8) = 0$.

Solution 2

To determine the x-intercept we set $y = 0$ and obtain $3x = 24$ or $x = 8$.
To determine the y-intercept we set $x = 0$ and obtain $-3y = 24$ or $y = -8$.
Thus, the sum of the intercepts is $8 + (-8) = 0$.

Solution 3

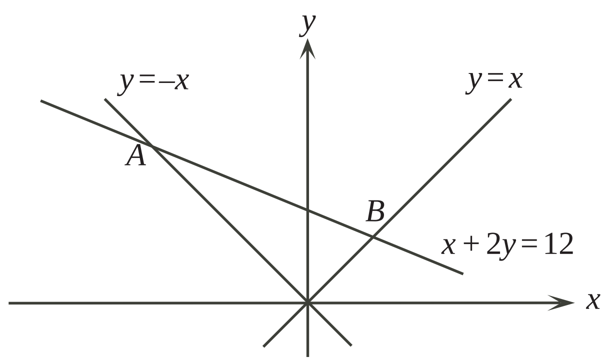
Since $3x - 3y = 24$, then $x - y = 8$ or $y = x - 8$.
This tells us immediately that the y-intercept of the line is $y = -8$ and that the x-intercept (obtained by setting $y = 0$) is $x = 8$.
Thus, the sum of the intercepts is $8 + (-8) = 0$.

(b) Answer: 20

Since $(1, 1)$ is the point of intersection of the two lines, then it must satisfy the equation of each line.
From the first line, $p(1) = 12$ or $p = 12$.
From the second line, $2(1) + q(1) = 10$ or $q = 8$.
Therefore, $p + q = 20$.

(c) *Solution 1*

To determine B, the point of intersection of the lines $y = x$ and $x + 2y = 12$, we set $y = x$ in the second equation to obtain $x + 2x = 12$ or $3x = 12$ or $x = 4$.



Since $y = x$, B has coordinates $(4, 4)$.
To determine A, the point of intersection of the lines $y = -x$ and $x + 2y = 12$, we set $y = -x$ in the second equation to obtain $x - 2x = 12$ or $-x = 12$ or $x = -12$.
Since $y = -x$, A has coordinates $(-12, 12)$.
The length of AB equals the distance between A and B, or

$$\sqrt{(4 - (-12))^2 + (4 - 12)^2} = \sqrt{16^2 + (-8)^2} = \sqrt{320} = 8\sqrt{5}$$

Solution 2

We determine the coordinates of A and B as in Solution 1.

Since the slopes of $y = x$ (slope of 1) and $y = -x$ (slope of -1) are negative reciprocals, then these lines are perpendicular, so $\angle AOB = 90^\circ$.
Since B has coordinates $(4, 4)$, then OB has length $\sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$.
Since A has coordinates $(-12, 12)$, then OA has length $\sqrt{(-12)^2 + 12^2} = \sqrt{288} = 12\sqrt{2}$.
Using the Pythagorean Theorem on triangle AOB,

$$AB = \sqrt{OB^2 + OA^2} = \sqrt{32 + 288} = \sqrt{320} = 8\sqrt{5}$$

2. (a) Answer: 9
For the average of two digits to be 5, their sum must be 10.
The two-digit positive integers whose digits sum to 10 are 19, 28, 37, 46, 55, 64, 73, 82, 91, of which there are 9.

(b) Answer: $n = 45$
Solution 1
Suppose that n has digits AB. Then $n = 10A + B$.
The average of the digits of n is $\frac{A + B}{2}$.
Putting a decimal point between the digits of n is equivalent to dividing n by 10, so the resulting number is $\frac{10A + B}{10}$.
So we want to determine A and B so that

$$\begin{aligned}\frac{10A + B}{10} &= \frac{A + B}{2} \\ 10A + B &= 5(A + B) \\ 5A &= 4B\end{aligned}$$

Since A and B are digits such that $5A = 4B$, then $A = 4$ and $B = 5$ is the only possibility.
Therefore, $n = 45$.
(We can quickly check that the average of the digits of n is 4.5, the number obtained by putting a decimal point between the digits of n.)

Solution 2
When we compute the average of two digits, the result is either an integer or a half-integer (ie. a decimal number of the form $a.5$).
Therefore, the possible averages are 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 7.5, 8.0, 8.5, 9.0. (0.0 is not possible as an two-digit integer cannot start with 0.)
From this list, the only one equal to the average of the two digits forming it is 4.5.
Therefore, $n = 45$ (formed by removing the decimal point from 4.5).

(c) *Solution 1*
When the average of three integers is 28, their sum is $3(28) = 84$.
When the average of five integers is 34, their sum is $5(34) = 170$.
In this case, the difference between the sum of the five integers and the sum of the three integers is $s + t$ which must equal $170 - 84 = 86$.
Therefore, $s + t = 86$ and so the average of s and t is $\frac{s + t}{2} = 43$.

Solution 2
Suppose the first three integers are a, b and c.

Then $\frac{a+b+c}{3} = 28$ or $a+b+c = 84$.
Also, $\frac{a+b+c+s+t}{5} = 34$ or $a+b+c+s+t = 170$.
Thus, $s+t = (a+b+c+s+t) - (a+b+c) = 170 - 84 = 86$ and so the average of s and t is $\frac{s+t}{2} = 43$.

Solution 3
Suppose that the average of s and t is A.
Since the average of the initial three numbers is 28, the average of s and t is A, and the average of all five numbers is 34, then 34 must be $\frac{2}{5}$ of the way from 28 to A. The difference between 34 and 28 is 6, so the total difference between A and 28 must be $\frac{5}{2}(6) = 15$.
Thus, A, the average of s and t, is $28 + 15 = 43$.

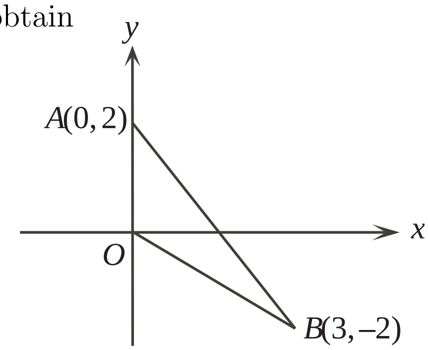
3. (a) Answer: $(21, -1)$
Solution 1
The x-intercepts of the given parabola are $x = 20$ and $x = 22$.
The x-coordinate of the vertex of the parabola is the average of the x-intercepts, or $\frac{1}{2}(20 + 22) = 21$.
When $x = 21$, $y = (21 - 20)(21 - 22) = -1$.
Thus, the coordinates of the vertex are $(21, -1)$.

Solution 2
We expand the right side of the equation of the parabola to obtain $y = x^2 - 42x + 440$.
Next we complete the square to obtain

$$y = x^2 - 2(21)x + 21^2 - 21^2 + 440 = (x-21)^2 - 21^2 + 440 = (x-21)^2 - 441 + 440 = (x-21)^2 - 1$$

From this form, we immediately see that the coordinates of the vertex are $(21, -1)$.

(b) Consider the parabola $y = x^2 + 2 = (x - 0)^2 + 2$.
The coordinates of its vertex are $A(0, 2)$.
Consider the parabola $y = x^2 - 6x + 7$.
Completing the square, we obtain $y = (x - 3)^2 - 9 + 7 = (x - 3)^2 - 2$.
The coordinates of its vertex are $B(3, -2)$.
Therefore, the vertices of $\triangle OAB$ are $O(0, 0)$, $A(0, 2)$, $B(3, -2)$.
Sketching this triangle, we obtain

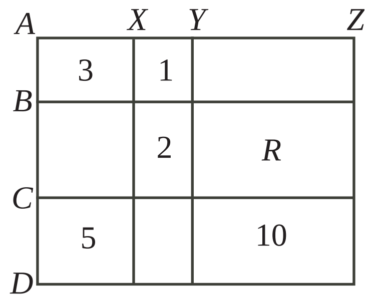


We can consider this triangle as having base OA (of length 2) and height, equal to the distance from B to the y-axis, of 3.
Thus, $\triangle OAB$ has area $\frac{1}{2}(2)(3) = 3$.

4. (a) ANSWER: $R = 12$

Solution 1

We label some of the points in the diagram.



Looking at the middle column of rectangles, each has the same width, so the ratio of their areas equals the ratio of their heights. Thus, $AB : BC = 1 : 2$.

Looking at the rectangles in the first column, the area of the middle rectangle must be twice the area of the top rectangle, or 6.

Thus, $BC : CD = 6 : 5$ by the reasoning above.

So, looking at the third column, $R : 10 = 6 : 5$ or $R = 12$.

Solution 2

Let the width of the first column be x.

Since the area of the top left rectangle is 3, the height of the first row is $\frac{3}{x}$.

Since the area of the bottom left rectangle is 5, the height of the third row is $\frac{5}{x}$.

Since the height of the first row is $\frac{3}{x}$ and the area of the top middle rectangle is 1, the width of the middle column is $\frac{x}{3}$.

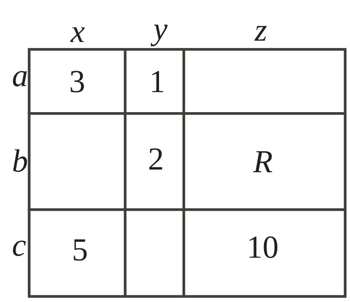
Thus, the height of the middle row is $\frac{6}{x}$, since the area of the middle rectangle is 2.

Since the height of the third row is $\frac{5}{x}$ and the area of the bottom right rectangle is 10, then the width of the third column is $2x$.

Since the rectangle labelled R has height $\frac{6}{x}$ and width $2x$, then it has area 12.

Solution 3

We label some of the lengths in the diagram.

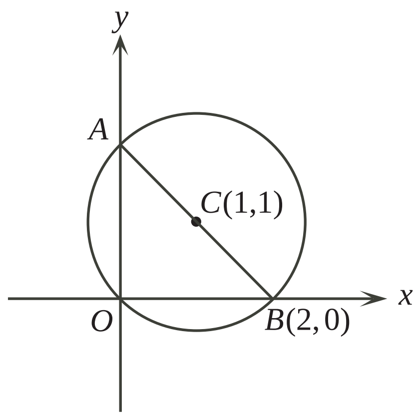


From the given information, $ax = 3$, $ay = 1$, $by = 2$, $bz = R$, $cx = 5$ and $cz = 10$.

We want to determine bz.

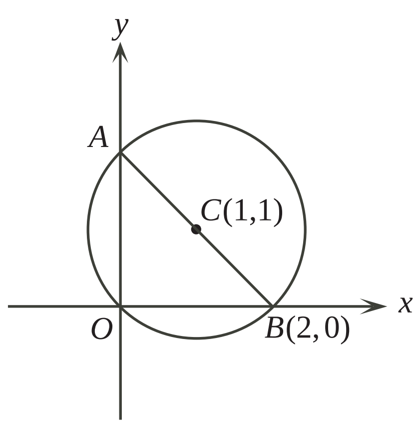
But $bz = \frac{(ax)(by)(cz)}{(ay)(cx)} = \frac{(3)(2)(10)}{(1)(5)} = 12$, so $R = 12$.

(b) *Solution 1*
Since $\angle AOB = 90^\circ$, AB is a diameter of the circle.
Join AB.

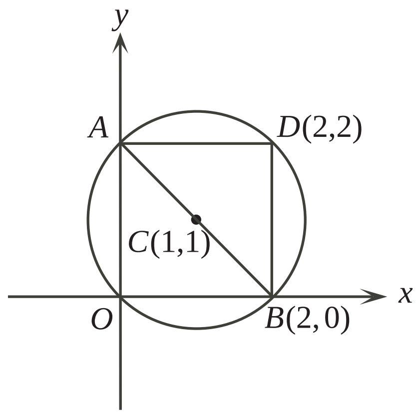


Since C is the centre of the circle and AB is a diameter, then C is the midpoint of AB, so A has coordinates $(0,2)$.
Therefore, the area of the part of the circle inside the first quadrant is equal to the area of $\triangle AOB$ plus the area of the semi-circle above AB.
The radius of the circle is equal to the distance from C to B, or $\sqrt{(1-2)^2+(1-0)^2}=\sqrt{2}$, so the area of the semi-circle is $\frac{1}{2}\pi(\sqrt{2})^2=\pi$.
The area of $\triangle AOB$ is $\frac{1}{2}(OB)(AO)=\frac{1}{2}(2)(2)=2$.
Thus, the area of the part of the circle inside the first quadrant is $\pi+2$.

Solution 2
Since $\angle AOB = 90^\circ$, AB is a diameter of the circle.
Join AB.



Since C is the centre of the circle and AB is a diameter, then C is the midpoint of AB, so A has coordinates $(0,2)$.
Thus, $AO = BO$.
We "complete the square" by adding point $D(2,2)$, which is on the circle, by symmetry.



The area of the square is 4.
The radius of the circle is equal to the distance from C to B, or $\sqrt{(1-2)^2+(1-0)^2}=\sqrt{2}$, so the area of the circle is $\pi(\sqrt{2})^2=2\pi$.
The area of the portion of the circle outside the square is thus $2\pi-4$. This area is divided into four equal sections (each of area $\frac{1}{4}(2\pi-4)=\frac{1}{2}\pi-1$), two of which are the only portions of the circle outside the first quadrant.
Therefore, the area of the part of the circle inside the the first quadrant is $2\pi-2(\frac{1}{2}\pi-1)=\pi+2$.

Two additional ways to find the coordinates of A:

- The length of OC is $\sqrt{1^2+1^2}=\sqrt{2}$.
  Since C is the centre of the circle and O lies on the circle, then the circle has radius $\sqrt{2}$.
  Since the circle has centre $(1,1)$ and radius $\sqrt{2}$, its equation is $(x-1)^2+(y-1)^2=2$.
  To find the coordinates of A, we substitute $x=0$ to obtain $(0-1)^2+(y-1)^2=2$ or $(y-1)^2=1$, and so $y=0$ or $y=2$.
  Since $y=0$ gives us the point O, then $y=2$ gives us A, ie. A has coordinates $(0,2)$.
- Since O and A are both on the circle and each has a horizontal distance of 1 from C, then their vertical distances from C must be same, ie. must each be 1.
  Thus, A has coordinates $(0,2)$.

5. (a) Answer: $\frac{2}{5}$

Since there are 5 choices for a and 3 choices for b, there are fifteen possible ways of choosing a and b.
If a is even, a^b is even; if a is odd, a^b is odd.
So the choices of a and b which give an even value for a^b are those where a is even, or 6 of the choices (since there are two even choices for a and three ways of choosing b for each of these). (Notice that in fact the value of b does not affect whether a^b is even or odd, so the probability depends only on the choice of a.)
Thus, the probability is $\frac{6}{15}=\frac{2}{5}$.

(b) Starting with 4 blue hats and 2 green hats, the probability that Julia removes a blue hat is $\frac{4}{6}=\frac{2}{3}$. The result would be 3 blue hats and 3 green hats, since a blue hat is replaced with a green hat.
In order to return to 4 blue hats and 2 green hats from 3 blue and 3 green, Julia would need remove a green hat (which would be replaced by a blue hat). The probability of her

removing a green hat from 3 blue and 3 green is $\frac{3}{6} = \frac{1}{2}$.
Summarizing, the probability of choosing a blue hat and then a green hat is $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$.

Starting with 4 blue hats and 2 green hats, the probability that Julia removes a green hat is $\frac{2}{6} = \frac{1}{3}$. The result would be 5 blue hats and 1 green hat, since a green hat is replaced with a blue hat.
In order to return to 4 blue hats and 2 green hats from 5 blue and 1 green, Julia would need remove a blue hat (which would be replaced by a green hat). The probability of her removing a green hat from 5 blue and 1 green is $\frac{5}{6}$.
Summarizing, the probability of choosing a green hat and then a blue hat is $\frac{1}{3} \times \frac{5}{6} = \frac{5}{18}$.

These are the only two ways to return to 4 blue hats and 2 green hats after two turns – removing a blue hat then a green, or removing a green then a blue.
Therefore, the total probability of returning to 4 blue hats and 2 green hats after two turns is $\frac{1}{3} + \frac{5}{18} = \frac{11}{18}$.

6. (a) Answer: $a = 1$
Adding the two equations, we obtain

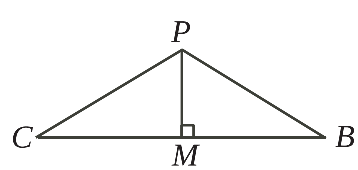
$$\begin{aligned} \sin^2 x + \cos^2 x + \sin^2 y + \cos^2 y &= \frac{3}{2}a + \frac{1}{2}a^2 \\ 2 &= \frac{3}{2}a + \frac{1}{2}a^2 \\ 4 &= 3a + a^2 \\ 0 &= a^2 + 3a - 4 \\ 0 &= (a+4)(a-1) \end{aligned}$$

and so $a = -4$ or $a = 1$.
However, $a = -4$ is impossible, since this would give $\sin^2 x + \cos^2 y = -6$, whose left side is non-negative and whose right side is negative.
Therefore, the only possible value for a is $a = 1$.

(We can check that angles $x = 90^\circ$ and $y = 45^\circ$ give $\sin^2 x + \cos^2 y = \frac{3}{2}$ and $\cos^2 x + \sin^2 y = \frac{1}{2}$, so $a = 1$ is indeed possible.)

(b) From the given information, $PC = PB$.
If we can calculate the length of PC, we can calculate the value of h, since we already know the length of AC.

Now $\triangle CPB$ is isosceles with $PC = PB$, $BC = 2$ and $\angle BPC = 120^\circ$.
Since $\triangle CPB$ is isosceles, $\angle PCB = \angle PBC = 30^\circ$.



Join P to the midpoint, M, of BC.
Then PM is perpendicular to BC, since $\triangle PCB$ is isosceles.

Therefore, $\triangle PMC$ is right-angled, has $\angle PCM = 30^\circ$ and has $CM = 1$.
Thus, $PC = \frac{2}{\sqrt{3}}$.
(There are many other techniques that we can use to calculate the length of PC.)

Returning to $\triangle APC$, we see $AP^2 = AC^2 - PC^2$ or $h^2 = 2^2 - \left(\frac{2}{\sqrt{3}}\right)^2 = 4 - \frac{4}{3} = \frac{8}{3}$, and so $h = \sqrt{\frac{8}{3}} = 2\sqrt{\frac{2}{3}} = \frac{2\sqrt{6}}{3} \approx 1.630$.
Therefore, the height is approximately 1.63 m or 163 cm.

7. (a) ANSWER: $k = 233$

Solution 1

We calculate the first 15 terms, writing each as an integer times a power of 10:

$$2, 5, 10, 5 \times 10, 5 \times 10^2, 5^2 \times 10^3, 5^3 \times 10^5, 5^5 \times 10^8, 5^8 \times 10^{13}, 5^{13} \times 10^{21}, 5^{21} \times 10^{34},$$

$$5^{34} \times 10^{55}, 5^{55} \times 10^{89}, 5^{89} \times 10^{144}, 5^{144} \times 10^{233}$$

Since the 15th term equals an odd integer times 10^{233}, then the 15th term ends with 233 zeroes.

Solution 2

To obtain the 6th term, we calculate $50 \times 500 = 25 \times 1000$.
Each of the 4th and 5th terms equals an odd integer followed by a number of zeroes, so the 6th term also equals an odd integer followed by a number of zeroes, where the number of zeroes is the sum of the numbers of zeroes at the ends of the 4th and 5th terms.
This pattern will continue. Thus, starting with the 6th term, the number of zeroes at the end of the term will be the sum of the number of zeroes at the ends of the two previous terms.
This tells us that, starting with the 4th term, the number of zeroes at the ends of the terms is

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233$$

Therefore, the 15th term ends with 233 zeroes.

(b) *Solution 1*

Since a, b and c are consecutive terms in an arithmetic sequence, then $b = a + d$ and $c = a + 2d$ for some number d.
Therefore,

$$\begin{aligned} a^2 - bc &= a^2 - (a+d)(a+2d) = a^2 - a^2 - 3ad - 2d^2 = -3ad - 2d^2 \\ b^2 - ac &= (a+d)^2 - a(a+2d) = a^2 + 2ad + d^2 - a^2 - 2ad = d^2 \\ c^2 - ab &= (a+2d)^2 - a(a+d) = a^2 + 4ad + 4d^2 - a^2 - ad = 3ad + 4d^2 \end{aligned}$$

Thus,

$$(b^2 - ac) - (a^2 - bc) = d^2 - (-3ad - 2d^2) = 3d^2 + 3ad$$

and

$$(c^2 - ab) - (b^2 - ac) = (3ad + 4d^2) - d^2 = 3d^2 + 3ad$$

Therefore, $(b^2 - ac) - (a^2 - bc) = (c^2 - ab) - (b^2 - ac)$, so the sequence $a^2 - bc$, $b^2 - ac$ and $c^2 - ab$ is arithmetic.

Solution 2
Since a, b and c are consecutive terms in an arithmetic sequence, then $a = b - d$ and $c = b + d$ for some number d.
Therefore,

$$\begin{aligned} a^2 - bc &= (b-d)^2 - b(b+d) = b^2 - 2bd + d^2 - b^2 - bd = -3bd + d^2 \\ b^2 - ac &= b^2 - (b-d)(b+d) = b^2 - b^2 + d^2 = d^2 \\ c^2 - ab &= (b+d)^2 - (b-d)b = b^2 + 2bd + d^2 - b^2 + bd = 3bd + d^2 \end{aligned}$$

Thus,

$$(b^2 - ac) - (a^2 - bc) = d^2 - (-3bd + d^2) = 3bd$$

and

$$(c^2 - ab) - (b^2 - ac) = (3bd + d^2) - d^2 = 3bd$$

Therefore, $(b^2 - ac) - (a^2 - bc) = (c^2 - ab) - (b^2 - ac)$, so the sequence $a^2 - bc$, $b^2 - ac$ and $c^2 - ab$ is arithmetic.

Solution 3
To show that $a^2 - bc$, $b^2 - ac$ and $c^2 - ab$ form an arithmetic sequence, we can show that $(c^2 - ab) + (a^2 - bc) = 2(b^2 - ac)$.
Since a, b and c form an arithmetic sequence, then $a + c = 2b$.
Now

$$\begin{aligned} (c^2 - ab) + (a^2 - bc) &= c^2 + a^2 - b(a+c) \\ &= c^2 + a^2 + 2ac - b(a+c) - 2ac \\ &= (c+a)^2 - b(a+c) - 2ac \\ &= (c+a)(a+c-b) - 2ac \\ &= 2b(2b-b) - 2ac \\ &= 2b^2 - 2ac \\ &= 2(b^2 - ac) \end{aligned}$$

as required.

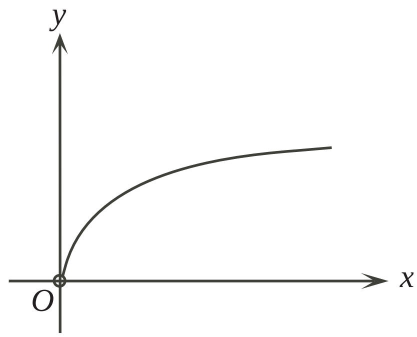
8. (a) We use logarithm rules to rearrange the equation to solve for y:

$$\begin{aligned} \log_2 x - 2\log_2 y &= 2 \\ \log_2 x - \log_2(y^2) &= 2 \\ \log_2\left(\frac{x}{y^2}\right) &= 2 \\ \frac{x}{y^2} &= 2^2 \\ \tfrac{1}{4}x &= y^2 \\ y &= \pm\tfrac{1}{2}\sqrt{x} \end{aligned}$$

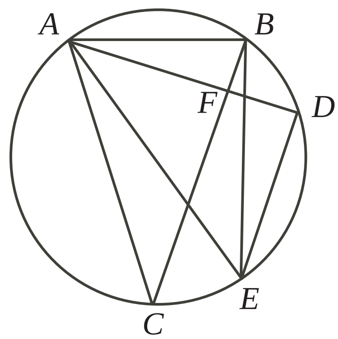
But since the domain of the $\log_2$ function is all positive real numbers, we must have $x > 0$ and $y > 0$, so we can reject the negative square root to obtain

$$y = \tfrac{1}{2}\sqrt{x}, \quad x > 0$$

The graph of this function is:

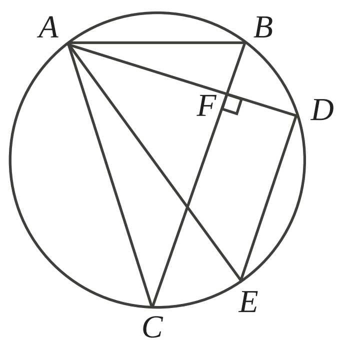


(b) *Solution 1*
Join A to E and C, and B to E.



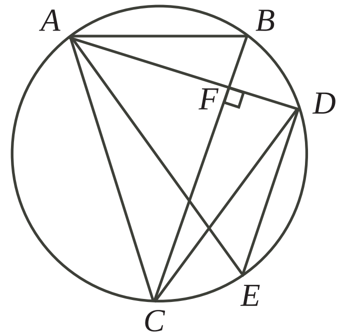
Since DE is parallel to BC and AD is perpendicular to BC, then AD is perpendicular to DE, ie. $\angle ADE = 90^\circ$.
Therefore, AE is a diameter.
Now $\angle EAC = \angle EBC$ since both are subtended by EC.
Therefore, $\angle EAC + \angle ABC = \angle EBC + \angle ABC = \angle EBA$ which is indeed equal to 90° as required, since AE is a diameter.

Solution 2
Join A to E and C.



Since DE is parallel to BC and AD is perpendicular to BC, then AD is perpendicular to DE, ie. $\angle ADE = 90^\circ$.
Therefore, AE is a diameter.
Thus, $\angle ECA = 90^\circ$.
Now $\angle ABC = \angle AEC$ since both are subtended by AC.
Now $\angle EAC + \angle ABC = \angle EAC + \angle AEC = 180^\circ - \angle ECA$ using the sum of the angles in $\triangle AEC$.
But $\angle ECA = 90^\circ$, so $\angle EAC + \angle AEC = 90^\circ$.

Solution 3
Join A to E and C, and C to D.



Since DE is parallel to BC and AD is perpendicular to BC, then AD is perpendicular to DE, ie. $\angle ADE = 90^\circ$.
Therefore, AE is a diameter.
Now $\angle ABC = \angle ADC$ since both are subtended by AC.
Also $\angle EAC = \angle EDC$ since both are subtended by EC.
So $\angle EAC + \angle ABC = \angle EDC + \angle ADC = \angle ADE = 90^\circ$.

9. (a) *Solution 1*
Since $\sin^2 x + \cos^2 x = 1$, then $\cos^2 x = 1 - \sin^2 x$, so

$$\begin{aligned} f(x) &= \sin^6 x + (1-\sin^2 x)^3 + k(\sin^4 x + (1-\sin^2 x)^2) \\ &= \sin^6 x + 1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x + k(\sin^4 x + 1 - 2\sin^2 x + \sin^4 x) \\ &= (1+k) - (3+2k)\sin^2 x + (3+2k)\sin^4 x \end{aligned}$$

Therefore, if $3+2k=0$ or $k=-\frac{3}{2}$, then $f(x) = 1+k = -\frac{1}{2}$ for all x and so is constant.
(If $k \neq -\frac{3}{2}$, then we get

$$\begin{aligned} f(0) &= 1+k \\ f\left(\tfrac{1}{4}\pi\right) &= (1+k) - (3+2k)\left(\tfrac{1}{2}\right) + (3+2k)\left(\tfrac{1}{4}\right) = \tfrac{1}{4} + \tfrac{1}{2}k \\ f\left(\tfrac{1}{6}\pi\right) &= (1+k) - (3+2k)\left(\tfrac{1}{4}\right) + (3+2k)\left(\tfrac{1}{16}\right) = \tfrac{7}{16} + \tfrac{5}{8}k \end{aligned}$$

which cannot be all equal for any single value of k, so $f(x)$ is not constant if $k \neq -\frac{3}{2}$.)

Solution 2
Since $\sin^2 x + \cos^2 x = 1$, then

$$\begin{aligned} f(x) &= (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x) + k(\sin^4 x + \cos^4 x) \\ &= (\sin^4 + 2\sin^2 x\cos^2 x + \cos^4 x - 3\sin^2 x\cos^2 x) \\ &\qquad k(\sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x - 2\sin^2 x\cos^2 x) \\ &= ((\sin^2 x + \cos^2 x)^2 - 3\sin^2 x\cos^2 x) + k((\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x) \\ &= 1 - 3\sin^2 x\cos^2 x + k(1 - 2\sin^2 x\cos^2 x) \\ &= (1+k) - (3+2k)\sin^2 x\cos^2 x \end{aligned}$$

Therefore, if $3+2k=0$ or $k=-\frac{3}{2}$, then $f(x) = 1+k = -\frac{1}{2}$ for all x and so is constant.
(We can check as in Solution 1 that if $k \neq -\frac{3}{2}$, then $f(x)$ is not constant.)

Solution 3
For $f(x)$ to be constant, we need $f'(x) = 0$ for all values of x.
Calculating using the Chain Rule,

$$\begin{aligned} f'(x) &= 6\sin^5 x\cos x - 6\cos^5 x\sin x + k(4\sin^3 x\cos x - 4\cos^3 x\sin x) \\ &= 2\sin x\cos x(3(\sin^4 x - \cos^4 x) + 2k(\sin^2 x - \cos^2 x)) \\ &= 2\sin x\cos x(\sin^2 x - \cos^2 x)(3(\sin^2 x + \cos^2 x) + 2k) \\ &= 2\sin x\cos x(\sin^2 x - \cos^2 x)(3 + 2k) \end{aligned}$$

If $3 + 2k = 0$ or $k = -\frac{3}{2}$, then $f'(x) = 0$ for all x, so $f(x)$ is constant.
(If $3 + 2k \neq 0$, then choosing $x = \frac{1}{6}\pi$ for example gives $f'(x) \neq 0$ so $f(x)$ is not constant.)

(b) *Solution 1*
Using the simplified version of $f(x)$ from Solution 1 of (a), we have

$$f(x) = (1 + k) - (3 + 2k)\sin^2 x + (3 + 2k)\sin^4 x$$

and so we want to solve

$$\begin{aligned} 0.3 - (1.6)\sin^2 x + (1.6)\sin^4 x &= 0 \\ 16\sin^4 x - 16\sin^2 x + 3 &= 0 \\ (4\sin^2 x - 3)(4\sin^2 x - 1) &= 0 \end{aligned}$$

Therefore, $\sin^2 x = \frac{1}{4}, \frac{3}{4}$, and so $\sin x = \pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}$.
Therefore,

$$x = \tfrac{1}{6}\pi + 2\pi k, \tfrac{5}{6}\pi + 2\pi k, \tfrac{7}{6}\pi + 2\pi k, \tfrac{11}{6}\pi + 2\pi k, \tfrac{1}{3}\pi + 2\pi k, \tfrac{2}{3}\pi + 2\pi k, \tfrac{4}{3}\pi + 2\pi k, \tfrac{5}{3}\pi + 2\pi k$$

for $k \in \mathbb{Z}$.

Solution 2
Using the simplified version of $f(x)$ from Solution 2 of (a), we have

$$f(x) = (1 + k) - (3 + 2k)\sin^2 x\cos^2 x$$

and so we want to solve

$$\begin{aligned} 0.3 - (1.6)\sin^2 x\cos^2 x &= 0 \\ 0.3 - (1.6)\sin^2 x(1 - \sin^2 x) &= 0 \\ 1.6\sin^4 x - 1.6\sin^2 x + 0.3 &= 0 \end{aligned}$$

and the solution concludes as in Solution 1.

Solution 3
Using the simplified version of $f(x)$ from Solution 2 of (a), we have

$$f(x) = (1 + k) - (3 + 2k)\sin^2 x\cos^2 x$$

Using the fact that $\sin 2x = 2\sin x\cos x$, we can further simplify $f(x)$ to

$$f(x) = (1 + k) - \tfrac{1}{4}(3 + 2k)\sin^2 2x$$

and so we want to solve

$$\begin{aligned} 0.3 - \tfrac{1}{4}(1.6)\sin^2 2x &= 0 \\ 4\sin^2 2x &= 3 \\ \sin^2 2x &= \tfrac{3}{4} \end{aligned}$$

and so $\sin 2x = \pm\frac{\sqrt{3}}{2}$.
Therefore,

$$2x = \tfrac{1}{3}\pi + 2\pi k, \tfrac{2}{3}\pi + 2\pi k, \tfrac{4}{3}\pi + 2\pi k, \tfrac{5}{3}\pi + 2\pi k$$

for $k \in \mathbb{Z}$, and so

$$x = \tfrac{1}{6}\pi + \pi k, \tfrac{1}{3}\pi + \pi k, \tfrac{2}{3}\pi + \pi k, \tfrac{5}{6}\pi + \pi k$$

for $k \in \mathbb{Z}$.
(Note that this solution, while appearing different, does agree with that from Solution 1, since here each of the four families of solutions has "$+\pi k$" and in Solution 1 each of the eight families has "$+2\pi k$".)

(c) *Solution 1*
Using the simplified version of $f(x)$ from Solution 1 of (a), we have

$$f(x) = (1+k) - (3+2k)\sin^2 x + (3+2k)\sin^4 x$$

We want to determine the values of k for which there is an a such that $f(a) = 0$.
From (a), if $k = -\frac{3}{2}$, $f(x)$ is constant and equal to $-\frac{1}{2}$, so has no roots.
Let $u = \sin^2 x$.
Then u takes all values between 0 and 1 as $\sin x$ takes all values between -1 and 1.
Then we want to determine for which k the equation

$$(3+2k)u^2 - (3+2k)u + (1+k) = 0 \qquad (*)$$

has a solution for u with $0 \leq u \leq 1$.
First, we must ensure that the equation (*) has real solutions, ie.

$$\begin{aligned} (3+2k)^2 - 4(3+2k)(1+k) &\geq 0 \\ (3+2k)(3+2k-4(1+k)) &\geq 0 \\ (3+2k)(-1-2k) &\geq 0 \\ (3+2k)(1+2k) &\leq 0 \end{aligned}$$

This is true if and only if $-\frac{3}{2} < k \leq -\frac{1}{2}$. (We omit $k = -\frac{3}{2}$ because of the earlier comment.)
Next, we have to check for which values of k the equation (*) has a solution u with $0 \leq u \leq 1$. We may assume that $-\frac{3}{2} < k \leq -\frac{1}{2}$.
To do this, we solve the equation (*) using the quadratic formula to obtain

$$u = \frac{(3+2k) \pm \sqrt{(3+2k)^2 - 4(3+2k)(1+k)}}{2(3+2k)}$$

or

$$u = \frac{(3+2k) \pm \sqrt{-(3+2k)(1+2k)}}{2(3+2k)} = \frac{1}{2} \pm \frac{1}{2}\sqrt{-\frac{1+2k}{3+2k}}$$

Since $k > -\frac{3}{2}$ then $3 + 2k > 0$.
For u to be between 0 and 1, we need to have

$$0 \leq \sqrt{-\frac{1+2k}{3+2k}} \leq 1$$

Thus

$$0 \leq -\frac{1+2k}{3+2k} \leq 1$$

Since $-\frac{3}{2} < k \leq -\frac{1}{2}$ then $3 + 2k > 0$ and $1 + 2k \leq 0$, so the left inequality is true.
Therefore, we need $-\dfrac{1+2k}{3+2k} \leq 1$ or $-(1 + 2k) \leq (3 + 2k)$ (we can multiply by $(3 + 2k)$ since it is positive), and so $-4 \leq 4k$ or $k \geq -1$.
Combining with $-\frac{3}{2} < k \leq -\frac{1}{2}$ gives $-1 \leq k \leq -\frac{1}{2}$.

Solution 2
Using the simplified version of $f(x)$ from Solution 3 of (b), we have

$$f(x) = (1 + k) - \tfrac{1}{4}(3 + 2k)\sin^2 2x$$

If we tried to solve $f(x) = 0$, we would obtain

$$(1 + k) - \tfrac{1}{4}(3 + 2k)\sin^2 2x = 0$$

or

$$\sin^2 2x = \frac{4(1+k)}{3+2k}$$

(From (a), if $k = -\frac{3}{2}$, $f(x)$ is constant and equal to $-\frac{1}{2}$, so has no roots.)
In order to be able to solve this (first for $\sin 2x$, then for $2x$ then for x), we therefore need

$$0 \leq \frac{4(1+k)}{3+2k} \leq 1$$

If $3 + 2k > 0$, we can multiply the inequality by $3 + 2k$ to obtain

$$0 \leq 4(1 + k) \leq 3 + 2k$$

and so we get $k \geq -1$ from the left inequality and $k \leq -\frac{1}{2}$ from the right inequality.
Combining these with $-\frac{3}{2} < k$, we obtain $-1 \leq k \leq -\frac{1}{2}$.

If $3 + 2k < 0$, we would obtain $0 \geq 4(1 + k) \geq 3 + 2k$ which would give $k \leq -1$ and $k \geq -\frac{1}{2}$, which are inconsistent.

Therefore, $-1 \leq k \leq -\frac{1}{2}$.

There were many other clever approaches to be taken in this problem:

- Deriving either of the expressions

$$f(x) = \left(k + \frac{3}{2}\right)(\sin^4 x + \cos^4 x) - \frac{1}{2}$$

or

$$f(x) = (1 + k) - \frac{1}{4}(3 + 2k)\sin^2 2x$$

led to some simpler algebra.

– In (c), deriving the equation

$$(1-\sin^2 x)(\sin^2 x) = \frac{k+1}{2k+3}$$

and rewriting the left side as $-(\sin^2 x - \frac{1}{2})^2 + \frac{1}{4}$ allowed one student to conclude that the right side lies between 0 and $\frac{1}{4}$, thus quickly obtaining the range of values for k.

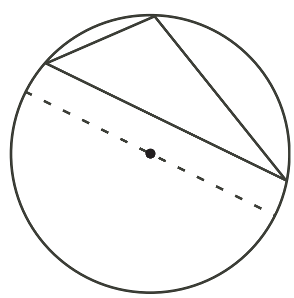
10. Through this solution, we will use the following facts:

When an acute triangle is inscribed in a circle:

- each of the three angles of the triangle is the angle inscribed in the major arc defined by the side of the triangle by which it is subtended,
- each of the three arcs into which the circle is divided by the vertices of the triangles is less than half of the circumference of the circle, and
- it contains the centre of the circle.

Why are these facts true?

- Consider a chord of a circle which is not a diameter.
  Then the angle subtended in the major arc of this circle is an acute angle and the angle subtended in the minor arc is an obtuse angle.
  Now consider an acute triangle inscribed in a circle.
  Since each angle of the triangle is acute, then each of the three angles is inscribed in the major arc defined by the side of the triangle by which it is subtended.
- It follows that each arc of the circle that is outside the triangle must be a minor arc, thus less than the circumference of the circle.
- Lastly, if the centre was outside the triangle, then we would be able to draw a diameter of the circle with the triangle entirely on one side of the diameter.

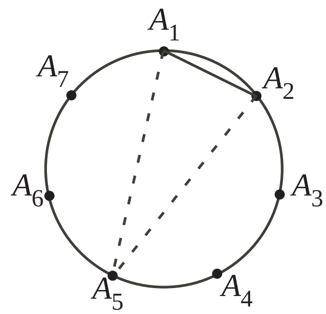


In this case, one of the arcs of the circle cut off by one of the sides of the triangle would have to be a major arc, which cannot happen, because of the above.
Therefore, the centre is contained inside the triangle.

(a) Since there are $N = 7$ points from which the triangle's vertices can be chosen, there are $\binom{7}{3} = 35$ triangles in total.
We compute the number of acute triangles.

Fix one of the vertices of such a triangle at A_1.
We construct the triangle by choosing the other two vertices in ascending subscript order.
We choose the vertices by considering the arc length from the previous vertex – each of

these arc lengths must be smaller than half the total circumference of the circle.
Since there are 7 equally spaced points on the circle, we assume the circumference is 7, so the arc length formed by each side must be at most 3.
Since the first arc length is at most 3, the second point can be only A_2, A_3 or A_4.
If the second point is A_2, then since the second and third arc lengths are each at most 3, then the third point must be A_5. (Since the second arc length is at most 3, then the third point cannot be any further along than A_5. However, the arc length from A_5 around to A_1 is 3, so it cannot be any closer than A_5.)



If the second point is A_3, the third point must be A_5 or A_6.
If the second point is A_4, the third point must be A_5 or A_6 or A_7.
Therefore, there are 6 acute triangles which include A_1 as one of its vertices.

How many acute triangles are there in total?
We can repeat the above process for each of the 6 other points, giving $7 \times 6 = 42$ acute triangles.
But each triangle is counted three times here, as it has been counted once for each of its vertices.
Thus, there are $\frac{7 \times 6}{3} = 14$ acute triangles.
Therefore, the probability that a randomly chosen triangle is acute if $\frac{14}{35} = \frac{2}{5}$.

(b) *Solution 1*
Since there are $N = 2k$ points from which the triangles vertices can be chosen, there are $\binom{2k}{3} = \frac{2k(2k-1)(2k-2)}{6}$ triangles in total.
We compute the number of acute triangles as in (a) by counting the number of acute triangles with one vertex fixed at A_1 and then multiply by $2k$ and divide by 3 as in (a).

Fix one vertex at A_1 and suppose that the circumference of the circle is $2k$.
The diametrically opposite point from A_1 is A_{k+1}.
Since the triangle cannot be entirely on one side of a diameter, then the second vertex must be between A_2 and A_k inclusive and the other between A_{k+2} and A_{2k} inclusive. This will ensure that the first and third arcs are shorter than half of the circumference of the circle.
If the second vertex is at A_2, then the third vertex must be no further than A_{k+1} for the second arc length to be shorter than k. Thus, there are no possibilities.
If the second vertex is at A_3, then the third vertex must be no further than A_{k+2}. (In this case, the arc between the second and third vertices has length $k-1$, as does the arc between the third and first vertices.) Thus, there is one possibility.
If the second vertex is at A_4, then the third vertex must be no further than A_{k+3}. In this

case, if the third vertex is A_{k+3}, the arc between the second and third vertices has length $k-1$ and the arc between the third and first vertices has length $k-2$. If the third vertex is A_{k+2}, these lengths are reversed. If the third vertex is neither of these points, then one of the two arcs will have length greater than $k-1$. Thus, there are two possibilities (A_{k+2} and A_{k+3}).

In general, if the second vertex is at A_j (with $2 \leq j \leq k$), then the third vertex must be no further than A_{j+k-1}, so there are $j+k-1-(k+1) = j-2$ possibilities. (This is because if the third vertex is A_i, then $i-j<k$ and $(2k+1)-i<k$ for the two arc lengths to be less than k. Therefore, $i<j+k$ and $i>k+1$ so i runs from $k+2$ to $j+k-1$, which is $j-2$ possibilities in total.)

As j runs from 2 to k, then $j-2$ runs from 0 to $k-2$, for a total of $0+1+2+\cdots+(k-2) = \frac{1}{2}(k-2)(k-1)$ acute triangles with one vertex fixed at A_1.

So in total there are $\frac{1}{3}(2k) \times \frac{1}{2}(k-2)(k-1) = \frac{1}{6}(2k)(k-2)(k-1)$ acute triangles.

Therefore, the probability is $\dfrac{\frac{1}{6}(2k)(k-2)(k-1)}{\frac{1}{6}(2k)(2k-1)(2k-2)} = \dfrac{k-2}{4k-2}$.

Solution 2

Again, we note that the total number of triangles is $\dbinom{2k}{3}$ if $N=2k$.

We calculate the number of acute triangles with one vertex fixed at A_1.

Let the circumference of the circle be $2k$.

The vertices of any triangle partition this circumference into lengths a, b and c, reading clockwise from A_1.

Our strategy is to count the number of acute triangles in this configuration, and then multiply by $\dfrac{2k}{3}$ as in Solution 1 to obtain the total number of acute triangles.

We look for solutions to the equation $a+b+c=2k$ with $a,b,c \geq 1$. (These are the partitions of the circumference of the circle.) Since we are looking for acute triangles, we must also have $a,b,c<k$ as in the preamble. Each different solution to the equation subject to these restrictions gives us different triangle and vice-versa.

So we count the number of integer solutions to $a+b+c=2k$ with $1 \leq a,b,c<k$.

Consider the transformation $a'=k-a$, $b'=k-b$ and $c'=k-c$.

Since $a,b,c<k$, then $a',b',c'>0$.

Also, $a+b+c=2k$ if and only if $3k-(a+b+c)=k$ if and only if $(k-a)+(k-b)+(k-c)=k$ if and only if $a'+b'+c'=k$.

So this transformation gives us a one-to-one correspondence between the acute triangles between the acute triangles on $2k$ vertices with one vertex fixed at A_1, and *all* triangles on k vertices with one vertex fixed at A_1. (Since we can "undo" this tranformation (ie. find its inverse), then it is a one-to-one correspondence (ie. a bijection).)

The total number of triangles on k points with one vertex fixed is $\dbinom{k-1}{2}$ since there are 2 vertices to choose from the remaining $k-1$ points.

Therefore, the total number of acute triangles with one vertex fixed at A_1 is also $\dbinom{k-1}{2}$, so the total number of acute triangles is

$$\frac{2k}{3}\binom{k-1}{2} = \frac{2k}{3}\frac{(k-1)(k-2)}{2} = \frac{k(k-1)(k-2)}{3}$$

To obtain the probability, we divide by $\binom{2k}{3}$ to obtain

$$\frac{6}{(2k)(2k-1)(2k-2)}\frac{k(k-1)(k-2)}{3} = \frac{k-2}{4k-2}$$

(c) From (b), the probability is $\frac{k-2}{4k-2}$.

We want to determine the values of k for which $\frac{k-2}{4k-2} = \frac{a}{2007}$ for some positive integer a.

Cross-multiplying, $2007(k-2) = a(4k-2)$.

Since the right is even, the left side must be even, so $k-2$ is even, so k is even, say $k = 2m$ for some positive integer $m \geq 1$.

Then $2007(2m-2) = a(8m-2)$ or $2007(m-1) = a(4m-1)$.

Since

$$(4m-1) - 4(m-1) = 3 \qquad (*)$$

then the possible positive common divisors of $4m-1$ and $m-1$ are 1 and 3 (since any common divisor of $4m-1$ and $m-1$ must also divide into 3 by (*)).

In other words, $\gcd(4m-1, m-1) = 1$ or $\gcd(4m-1, m-1) = 3$.

If $\gcd(4m-1, m-1) = 1$ then $4m-1$ and $m-1$ have no common factors larger than 1.

Since $2007(m-1) = a(4m-1)$, then $4m-1$ divides into $2007(m-1)$ and so $4m-1$ divides into 2007, since $4m-1$ and $m-1$ have no common factors.

Now $2007 = 9 \times 223 = 3^2 \times 223$, so the positive divisors of 2007 are 1, 3, 9, 223, 669, 2007.

The divisors having the form $4m-1$ for some positive integer m are 3, 233 and 2007, giving:

$4m-1$	3	233	2007
m	1	56	502
a	0	495	2004
k		112	1004

($m = 1$ gives $a = 0$ which is inadmissible, so there is no value for k.)

If $\gcd(4m-1, m-1) = 3$, then $m-1$ is divisible by 3, so we write $m-1 = 3p$ for some non-negative integer p.

Thus, $4m-1 = 12p+3$ and so $2007(m-1) = a(4m-1)$ becomes $2007(3p) = a(12p+3)$ or $2007p = a(4p+1)$.

Note that $\gcd(4p+1, p) = 1$, since $\gcd(12p+3, 3p) = 3$.

Since $4p+1$ divides into $2007p$ and has no common factors with p, then $4p+1$ divides into 2007.

The divisors of 2007 having the form $4p+1$ are 1, 9 and 669, giving:

$4p+1$	1	9	669
p	0	2	167
m	1	7	502
a	0	446	501
k		14	1004

($m = 1$ gives $a = 0$ which is inadmissible, so there is no value for k.)

Thus, the possible values of k are 14, 112 and 1004.