2006 Hypatia Contest (Grade 11)

Thursday, April 20, 2006

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1. The odd positive integers are arranged in rows in the triangular pattern, as shown.

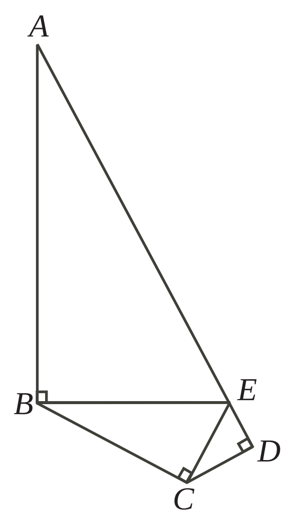
$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 3 & & 5 & & & \\
 & & 7 & & 9 & & 11 & & \\
 & 13 & & 15 & & 17 & & 19 & \\
21 & & 23 & & \cdots & & & &
\end{array}
$$

   (a) What is the 25th odd positive integer? In which row of the pattern will this integer appear?

   (b) What is the 19th integer that appears in the 21st row? Explain how you got your answer.

   (c) Determine the row and the position in that row where the number 1001 occurs. Explain how you got your answer.

2. In the diagram, $\triangle ABE$, $\triangle BCE$ and $\triangle CDE$ are right-angled, with $\angle AEB = \angle BEC = \angle CED = 60°$, and $AE = 24$.



   (a) Determine the length of CE.

   (b) Determine the perimeter of quadrilateral $ABCD$.

   (c) Determine the area of quadrilateral $ABCD$.

3. A line ℓ passes through the points $B(7, -1)$ and $C(-1, 7)$.

   (a) Determine the equation of this line.

   (b) Determine the coordinates of the point P on the line ℓ so that P is equidistant from the points $A(10, -10)$ and $O(0, 0)$ (that is, so that $PA = PO$).

   (c) Determine the coordinates of all points Q on the line ℓ so that $\angle OQA = 90°$.

4. The abundancy index $I(n)$ of a positive integer n is $I(n) = \dfrac{\sigma(n)}{n}$, where $\sigma(n)$ is the sum of all of the positive divisors of n, including 1 and n itself.

   For example, $I(12) = \dfrac{1+2+3+4+6+12}{12} = \dfrac{7}{3}$.

   (a) Prove that $I(p) \leq \frac{3}{2}$ for every prime number p.

   (b) For every odd prime number p and for all positive integers k, prove that $I(p^k) < 2$.

   (c) If p and q are different prime numbers, determine $I(p^2)$, $I(q)$ and $I(p^2q)$, and prove that $I(p^2)I(q) = I(p^2q)$.

   (d) Determine, with justification, the smallest odd positive integer n such that $I(n) > 2$.