Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2007 Cayley Contest

(Grade 10)

Tuesday, February 20, 2007

Solutions

1. Calculating, $8 + 2(3^2) = 8 + 2(9) = 8 + 18 = 26$.

   Answer: (A)

2. Calculating, $\dfrac{7+21}{14+42} = \dfrac{28}{56} = \dfrac{1}{2}$.

   Answer: (C)

3. If $3x - 2x + x = 3 - 2 + 1$, then $2x = 2$ or $x = 1$.

   Answer: (B)

4. In 3 hours, Leona earns \$24.75, so she makes $\$24.75 \div 3 = \$8.25$ per hour.
   Therefore, in a 5 hour shift, Leona earns $5 \times \$8.25 = \$41.25$.

   Answer: (E)

5. The value of $\frac{1}{4}$ of 100 is $\frac{1}{4} \times 100 = 25$.
   We evaluate each of the five given possibilities:

   (A): 20% of 200 is $0.2(200) = 40$
   (B): 10% of 250 is $0.1(250) = 25$
   (C): 15% of 100 is 15
   (D): 25% of 50 is $0.25(50) = 12.5$
   (E): 5% of 300 is $0.05(300) = 15$

   Therefore, the correct answer is (B).
   (Note that we did not have to evaluate (C), (D) and (E) after seeing that (B) was correct.)

   Answer: (B)

6. Evaluating each of the five given possibilities with $a = 2$ and $b = 5$,
   (A) $\dfrac{a}{b} = \dfrac{2}{5}$ (B) $\dfrac{b}{a} = \dfrac{5}{2}$ (C) $a - b = 2 - 5 = -3$ (D) $b - a = 5 - 2 = 3$ (E) $\frac{1}{2}a = \frac{1}{2}(2) = 1$
   The largest of these five is 3, or (D).

   Answer: (D)

7. The mean of 6, 9 and 18 is $\dfrac{6+9+18}{3} = \dfrac{33}{3} = 11$.
   Since the mean of 12 and y is thus 11, then the sum of 12 and y is $2(11) = 22$, so $y = 10$.

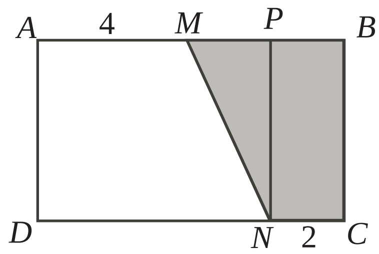
   Answer: (C)

8. In $\triangle ABC$, $\angle ABC = \angle BAC$, so $AC = BC$.
   In $\triangle BCD$, $\angle CBD = \angle CDB$, so $CD = BC$.
   Since the perimeter of $\triangle CBD$ is 19 and $BD = 7$, then $7 + BC + CD = 19$ or $2(BC) = 12$ or $BC = 6$.
   Since the perimeter of $\triangle ABC$ is 20, $BC = 6$, and $AC = BC$, then $AB + 6 + 6 = 20$ or $AB = 8$.

   Answer: (D)

9. *Solution 1*
   Since the area of rectangle $ABCD$ is 40 and $AB = 8$, then $BC = 5$.
   Therefore, $MBCN$ is a trapezoid with height 5 and parallel bases of lengths 4 and 2, so has area $\frac{1}{2}(5)(4 + 2) = 15$.

   Solution 2
   Since the area of rectangle $ABCD$ is 40 and $AB = 8$, then $BC = 5$.
   Draw a line from N to AB parallel to BC (and so perpendicular to AB) meeting AB at P.



This line divides $MBCN$ into a rectangle $PBCN$ of width 2 and height 5, and a triangle MPN with base MP of length 2 and height PN of height 5.
The area of $MBCN$ is sum of the areas of these two parts, or $2(5) + \frac{1}{2}(2)(5) = 10 + 5 = 15$.

Answer: (A)

10. *Solution 1*
To get from one term to the next, we double and add 4.
Since the first term is x, the second term is $2x + 4$.
Since the second term is $2x + 4$, the third term is $2(2x + 4) + 4 = 4x + 12$.
Since the third term equals 52, then $4x + 12 = 52$ so $4x = 40$ or $x = 10$.

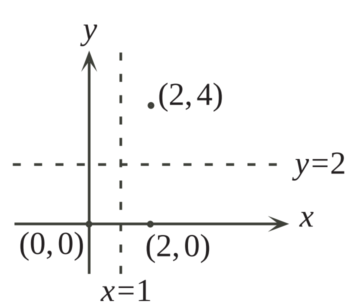
Solution 2
To get from one term to the next, we double and add 4.
Therefore, to get from one term to the previous term, we subtract 4 and divide by 2.
Since the third term is 52, then subtracting 4 gives 48 and dividing by 2 gives 24, which is the second term.
Since the second term is 24, then subtracting 4 gives 20 and dividing by 2 gives 10, which is the first term.
Thus, $x = 10$.

Answer: (D)

11. Suppose that Ivan ran a distance of x km on Monday.
Then on Tuesday, he ran $2x$ km, on Wednesday, he ran x km, on Thursday, he ran $\frac{1}{2}x$ km, and on Friday he ran x km.
The shortest of any of his runs was on Thursday, so $\frac{1}{2}x = 5$ or $x = 10$.
Therefore, his runs were 10 km, 20 km, 10 km, 5 km, and 10 km, for a total of 55 km.

Answer: (A)

12. When $(0, 0)$ is reflected in the line $x = 1$, the image is $(2, 0)$.



When $(2, 0)$ is reflected in the line $y = 2$, the image is $(2, 4)$.

Answer: (E)

13. *Solution 1*
Since the ratio $AN : AC$ equals the ratio $AP : PB$ (each is $1 : 2$) and $\angle A$ is common in $\triangle APN$ and $\triangle ABC$, then $\triangle APN$ is similar to $\triangle ABC$.
Since the ratio of side lengths between these two triangles is $1 : 2$, then the ratio of areas is $1 : 2^2 = 1 : 4$.
Thus, the area of $\triangle ABC$ is $4 \times 2 = 8$ cm^2.

Solution 2
Since the ratio $AN : AC$ equals the ratio $AP : PB$ (each is $1 : 2$) and $\angle A$ is common in $\triangle APN$ and $\triangle ABC$, then $\triangle APN$ is similar to $\triangle ABC$.
Therefore, $\angle ANP = \angle ACB = 90^\circ$.
Similarly, $\triangle PMB$ is similar to $\triangle ACB$, and $\angle PMB = \angle ACB = 90^\circ$.
Therefore, $AN = NC = PM = \frac{1}{2}AC$ and $NP = CM = MB = \frac{1}{2}CB$.
Thus, $\triangle PMB$ is congruent to $\triangle ANP$, so has area 2 cm^2.
Rectangle $NPMC$ has the same width and height as $\triangle ANP$, so has double the area of $\triangle APN$ or 4 cm^2.
Thus, the area of $\triangle ACB$ is $2 + 4 + 2 = 8$ cm^2.

Solution 3
Join C to P.
Since CP is the diagonal of the rectangle, then $\triangle CPN$ and $\triangle PCM$ are congruent, so have equal areas.
Since $AN = NC$ and $\triangle PNA$ and $\triangle PNC$ have equal height (PN), then the areas of $\triangle PNA$ and $\triangle PNC$ are equal.
Similarly, the areas of $\triangle PMC$ and $\triangle PMB$ are equal.
In summary, the areas of the four small triangles are equal.
Since the area of $\triangle APN$ is 2 cm^2, then the total area of $\triangle ABC$ is 8 cm^2.

Answer: (A)

14. Using a common denominator, $\dfrac{3}{x-3} + \dfrac{5}{2x-6} = \dfrac{6}{2x-6} + \dfrac{5}{2x-6} = \dfrac{11}{2x-6}$.
Therefore, $\dfrac{11}{2x-6} = \dfrac{11}{2}$, so $2x - 6 = 2$.

Answer: (A)

15. Since $\triangle ABC$ and $\triangle PQR$ are equilateral, then $\angle ABC = \angle ACB = \angle RPQ = 60^\circ$.
Therefore, $\angle YBP = 180^\circ - 65^\circ - 60^\circ = 55^\circ$ and $\angle YPB = 180^\circ - 75^\circ - 60^\circ = 45^\circ$.
In $\triangle BYP$, we have $\angle BYP = 180^\circ - \angle YBP - \angle YPB = 180^\circ - 55^\circ - 45^\circ = 80^\circ$.
Since $\angle XYC = \angle YBP$, then $\angle XYC = 80^\circ$.
In $\triangle CXY$, we have $\angle CXY = 180^\circ - 60^\circ - 80^\circ = 40^\circ$.

Answer: (C)

16. Since 60% of the 10 000 students are in Arts, 6000 students are in Arts, and so 4000 students are in Science.
Since 40% of the Science students are male, then $0.4(4000) = 1600$ of the Science students are male.
Since half of the total of 10 000 students are male, then $5000 - 1600 = 3400$ of the male students are in Arts.
Since there are 6000 students in Arts, then $6000 - 3400 = 2600$ of the Arts students are female, or $\dfrac{2600}{6000} \times 100\% \approx 43.33\%$ of the Arts students are female.

Answer: (E)

17. First, we note that no matter how many Heroes are present, all four would always reply "Hero" when asked "Are you a Hero or a Villain?". (This is because Heroes will tell the truth and answer "Hero" and Villains will lie and answer "Hero".)
When each is asked "Is the person on your right a Hero or a Villain?", all four reply "Villain", so any Hero that is at the table must have a Villain on his right (or he would have answered "Hero") and any Villain at the table must have a Hero on his right (or he would have had a Villain on his right and answered "Hero").
In other words, Heroes and Villains must alternate around the table, so there are 2 Heroes and 2 Villains.
(It is worth checking that when 2 Heroes and 2 Villains sit in alternate seats, the answers are indeed as claimed.)

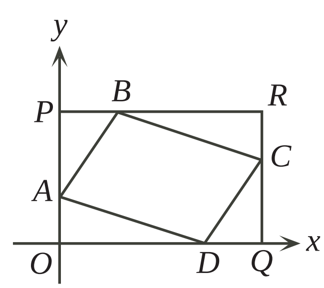
Answer: (C)

18. *Solution 1*
Suppose there are x balls in total in the bag.
Then there are $\frac{1}{3}x$ red balls and $\frac{2}{7}x$ blue balls.
This tells us that the number of green balls is $x - \frac{1}{3}x - \frac{2}{7}x = \frac{21}{21}x - \frac{7}{21}x - \frac{6}{21}x = \frac{8}{21}x$.
But we know that the number of green balls is $2 \times \frac{2}{7}x - 8$.
Thus, $\frac{8}{21}x = 2 \times \left(\frac{2}{7}x\right) - 8$ or $\frac{8}{21}x = \frac{12}{21}x - 8$ or $\frac{4}{21}x = 8$ or $x = 42$.
Since $x = 42$, the number of green balls is $\frac{8}{21}x = \frac{8}{21}(42) = 16$.

Solution 2
Suppose that there were 21 balls in the bag. (We choose 21 since there are fractions with denominator 3 and fractions with denominator 7 in the problem.)
Since $\frac{1}{3}$ of the balls are red, then 7 balls are red.
Since $\frac{2}{7}$ of the balls are blue, then 6 balls are red.
Thus, there are $21 - 7 - 6 = 8$ green balls in the bag. However, this is only 4 less than twice the number of blue balls, so there cannot be 21 balls in the bag.
To get from "4 less" to "8 less", we try doubling the number of balls in the bag.
If there are 42 balls in the bag, then 14 are red and 12 are blue, so 16 are green, which is 8 less than twice the number of blue balls.
Therefore, the number of green balls is 16.

Answer: (B)

19. We draw a horizontal line through B (meeting the y-axis at P) and a vertical line through C (meeting the x-axis at Q). Suppose the point of intersection of these two lines is R.



We know that P has coordinates $(0, 3)$ (since B has y-coordinate 3) and Q has coordinates $(5, 0)$ (since C has x-coordinate 5), so R has coordinates $(5, 3)$.
Using the given coordinates, $OA = 1$, $AP = 2$, $PB = 1$, $BR = 4$, $RC = 1$, $CQ = 2$, $QD = 1$,

and $DO = 4$.
The area of $ABCD$ equals the area of $PRQO$ minus the areas of triangles APB, BRC, CQD, and DOA.
$PRQO$ is a rectangle, so has area $3 \times 5 = 15$.
Triangles APB and CQD have bases PB and QD of length 1 and heights AP and CQ of length 2, so each has area $\frac{1}{2}(1)(2) = 1$.
Triangles BRC and DOA have bases BR and DO of length 4 and heights CR and AO of length 1, so each has area $\frac{1}{2}(4)(1) = 2$.
Thus, the area of $ABCD$ is $15 - 1 - 1 - 2 - 2 = 9$.
(Alternatively, we could notice that $ABCD$ is a parallelogram. Therefore, if we draw the diagonal AC, the area is split into two equal pieces. Dropping a perpendicular from C to Q on the x-axis produces a trapezoid $ACQO$ from which only two triangles need to be removed to determine half of the area of $ABCD$.)

Answer: (A)

20. Since $3(n^{2007}) < 3^{4015}$, then $n^{2007} < \dfrac{3^{4015}}{3} = 3^{4014}$.
But $3^{4014} = (3^2)^{2007} = 9^{2007}$ so we have $n^{2007} < 9^{2007}$.
Therefore, $n < 9$ and so the largest integer n that works is $n = 8$.

Answer: (D)

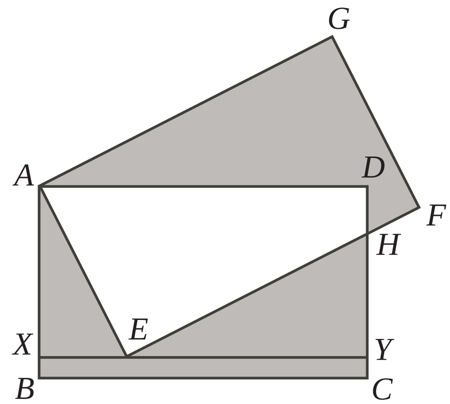
21. Since T has played 5 matches so far, then T has played P, Q, R, S, and W (ie. each of the other teams).
Since P has played only 1 match and has played T, then P has played no more matches.
Since S has played 4 matches and has not played P, then S must have played each of the remaining 4 teams (namely, Q, R, T, and W).
Since Q has played only 2 matches and has played T and S, then Q has played no more matches.
Since R has played 3 matches, has played T and S but has not played P or Q, then R must have played W as well.
Therefore, W has played T, S and R, or 3 matches in total. (Since we have considered all possible opponents for W, then W has played no more matches.)

Answer: (C)

22. Suppose the first integer in the list is n.
Then the remaining four integers are $n + 3$, $n + 6$, $n + 9$, and $n + 12$.
Since the fifth number is a multiple of the first, then $\dfrac{n+12}{n} = \dfrac{n}{n} + \dfrac{12}{n} = 1 + \dfrac{12}{n}$ is an integer.
Since $1 + \dfrac{12}{n}$ is an integer, then $\dfrac{12}{n}$ is an integer, or n is a positive divisor of 12.
The positive divisors of 12 are 1, 2, 3, 4, 6, and 12, so there are 6 possible values of n and so 6 different lists.
(We can check that each of 6 values of n produces a different list, each of which has the required property.)

Answer: (D)

23. Since the same region ($AEHD$) is unshaded inside each rectangle, then the two shaded regions have equal area, since the rectangles have equal area.
Thus, the total shaded area is twice the area of $AEHCB$.
Draw a horizontal line through E, meeting AB at X and HC at Y.



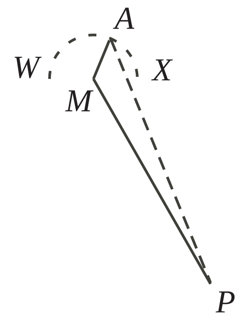
We know $\angle BAE = 30^\circ$ and $AE = 12$, so $EX = \frac{1}{2}AE = 6$ and $AX = \sqrt{3}EX = 6\sqrt{3}$, by the ratios in a 30°-60°-90° triangle.
Thus, the area of $\triangle EXA$ is $\frac{1}{2}(6)(6\sqrt{3}) = 18\sqrt{3}$.
Since $AB = 12$ and $AX = 6\sqrt{3}$, then $XB = 12 - 6\sqrt{3}$.
Therefore, rectangle $BXYC$ has height $12 - 6\sqrt{3}$ and width 18, so has area $18(12 - 6\sqrt{3})$ or $216 - 108\sqrt{3}$.
Since $XY = BC = 18$ and $EX = 6$, then $EY = 12$.
Since $\angle AEB = 60^\circ$ and $\angle AEH = 90^\circ$, then $\angle HEY = 30^\circ$.
Since $EY = 12$ and $\angle HEY = 30^\circ$, then $HY = \dfrac{12}{\sqrt{3}} = 4\sqrt{3}$.
Thus, $\triangle EYH$ has area $\frac{1}{2}(12)(4\sqrt{3}) = 24\sqrt{3}$.
So the total area of $AEHCB$ is $18\sqrt{3} + (216 - 108\sqrt{3}) + 24\sqrt{3} = 216 - 66\sqrt{3}$.
Therefore, the total shaded area is $2(216 - 66\sqrt{3}) = 432 - 132\sqrt{3} \approx 203.369$.

Answer: (C)

24. To form such a collection of integers, our strategy is to include some integers larger than 1 whose product is 2007 and then add enough 1s to the collection to make the sum 2007.
In order to make n as small as possible, we would like to include as few 1s as possible, and so make the sum of the initial integers (whose product is 2007) as large as possible.
Since we would like to consider integers whose product is 2007, we should find the divisors of 2007.
We see that $2007 = 3 \times 669 = 3 \times 3 \times 223$, and 223 is a prime number.
So the collections of positive integers larger than 1 whose product is 2007 are $\{3, 669\}$, $\{3, 3, 223\}$, and $\{9, 223\}$.
The collection with the largest sum is $\{3, 669\}$ whose sum is 672. To get a sum of 2007, we must add $2007 - 672 = 1335$ copies of 1, which means that we are using $1335 + 2 = 1337$ integers.
Therefore, the smallest value of n is 1337.

Answer: (B)

25. Since $\angle WAX = 90^\circ$ regardless of the position of square $ABCD$, then A always lies on the semi-circle with diameter WX.
The centre of this semi-circle is the midpoint, M, of WX.
To get from P to M, we must go up 4 units and to the left 3 units (since $WX = 2$), so $PM^2 = 3^2 + 4^2 = 25$ or $PM = 5$.
Since the semi-circle with diameter WX has diameter 2, it has radius 1, so $AM = 1$.
So we have $AM = 1$ and $MP = 5$.



Therefore, the maximum possible length of AP is $5 + 1 = 6$, when A, M, P lie on a straight line.

ANSWER: (E)