Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2007 Galois Contest

Wednesday, April 18, 2007

Solutions

1. (a) Let A be the price of an Apple in cents, B the price of a Banana in cents, and C the price of a Cherry in cents.
From the given information, $A + C = 62$ and $B + C = 66$.
Since each combination includes a Cherry and one other piece of fruit, then the price of a Banana must be $66 - 62 = 4$ cents higher than that of an Apple.
(We could have also subtracted these equations to obtain $B - A = 4$.)
Therefore, the difference in prices of an Apple and a Banana is 4 cents, with the Banana having a higher price.

(b) *Solution 1*
Let M be the price of a Mango in cents, N the price of a Nectarine in cents, and P the price of a Pear in cents.
From the given information, $M + N = 60$, $P + N = 60$ and $M + P = 68$.
Looking at the first two equations, M and P must be equal (since $M = 60 - N$ and $P = 60 - N$).
Looking then at the third equation, using $M = P$, we get $2P = 68$, so $P = 34$, or the price of a Pear is 34 cents.

Solution 2
Let M be the price of a Mango in cents, N the price of a Nectarine in cents, and P the price of a Pear in cents.
From the given information, $M + N = 60$, $P + N = 60$ and $M + P = 68$.
Adding the second and third equations, we obtain $2P + M + N = 60 + 68 = 128$.
Since $M + N = 60$ (from the first equation), then $2P + 60 = 128$ or $2P = 68$ or $P = 34$.
Therefore, the price of a Pear is 34 cents.

Solution 3
Let M be the price of a Mango in cents, N the price of a Nectarine in cents, and P the price of a Pear in cents.
From the given information, $M + N = 60$, $P + N = 60$ and $M + P = 68$.
Adding the three equations together, we obtain $2M + 2N + 2P = 188$.
Dividing by 2, we obtain $M + N + P = 94$.
Since $M + N = 60$ and $M + N + P = 94$, then $P = 94 - 60 = 34$.
Therefore, the price of a Pear is 34 cents.

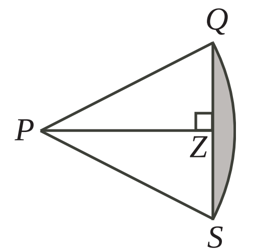
(c) Let T be the price of a Tangerine in cents, L the price of a Lemon in cents, and G the price of a Grapefruit in cents.
From the given information, $T + L = 60$, $T - G = 6$ and $G + T + L = 94$.
Using the first equation to substitute into the third equation, $G + 60 = 94$, so $G = 34$.
Since $G = 34$ and $T - G = 6$, then $T = 34 + 6 = 40$.
Since $T = 40$ and $T + L = 60$, then $L = 20$, so the price of a Lemon is 20 cents.
(There are lots of other ways to combine these equations to get $L = 20$.)

2. (a) In the diagram, the radius of the sector is 12 so $OA = OB = 12$.
Since the angle of the sector is 60°, then the sector is $\dfrac{60^\circ}{360^\circ} = \dfrac{1}{6}$ of the total circle.
Therefore, arc AB is $\frac{1}{6}$ of the total circumference of a circle of radius 12, so has length $\frac{1}{6}(2\pi(12)) = 4\pi$.
Therefore, the perimeter of the sector is $12 + 12 + 4\pi = 24 + 4\pi$.

(b) Each of sector ABD and BDC is one-sixth of a full circle of radius 12, so has area one-sixth of the area of a circle of radius 12.
Therefore, each sector has area $\frac{1}{6}(\pi(12^2)) = \frac{1}{6}(144\pi) = 24\pi$.
Thus, the area of figure $ABCD$ is $2(24\pi) = 48\pi$.

(c) Since OY is a radius of the circle with centre O, then $OY = 12$.
To find the length of XY, we must find the length of OX.
Since $OA = OB$, then $\triangle OAB$ is isosceles.
Since $\angle AOB = 60^\circ$, then $\angle OAB = \frac{1}{2}(180^\circ - 60^\circ) = 60^\circ$.
Therefore, $\angle AOX = 180^\circ - 60^\circ - 90^\circ = 30^\circ$, so $\triangle OAX$ is a 30°-60°-90° triangle.
Since $OA = 12$, then $AX = \frac{1}{2}OA = 6$ and $OX = \sqrt{3}AX = 6\sqrt{3}$.
Thus, $XY = OY - OX = 12 - 6\sqrt{3} \approx 1.61$.

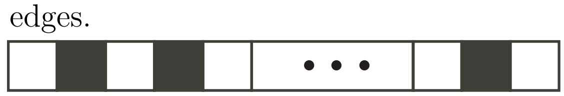
(d) By symmetry, the areas of the two parts of the shaded region are equal.
Consider the right part of the shaded region and the left triangle.



The shaded area is equal to the area of sector PQS minus the area of triangle PQS.
The area of sector PQS is 24π as in part (b).
Using our work from part (c), $\triangle PQS$ is equilateral, so $QS = 12$.
Draw a perpendicular from P to Z on QS.
From (c) again, $PZ = 6\sqrt{3}$.
Therefore, we can consider $\triangle PQS$ as having a base QS of length 12 and height PZ of length $6\sqrt{3}$.
Therefore, the area of $\triangle PQS$ is $\frac{1}{2}(12)(6\sqrt{3}) = 36\sqrt{3}$.
Therefore, the area of the right part of the shaded region is $24\pi - 36\sqrt{3}$, so the area of the entire shaded region is $2(24\pi - 36\sqrt{3}) = 48\pi - 72\sqrt{3} \approx 26.1$.

3. (a) The 1 by 1 by 1 cubes that have at least two painted faces are those along the edges of the 5 by 5 by 5 including those at the corner.
The large cube has 12 edges and has 3 of the small cubes along each edge which are not at the corners.
The large cube has 8 corners.
Therefore, there are $3 \times 12 + 8 = 44$ small cubes that have at least two painted faces.

(b) i. The small cubes that have exactly two white faces are those along the edges of the larger cubes that are not at the corners.
Each of the 12 edges of the large cube has $2k + 1$ smaller cubes along it.
The "white cubes" (that is, white on two faces) that we want are those that are not at the end of one of the edges.
Consider one of these edges.



If we remove the white cube at one end of the edge of $2k + 1$ cubes, half of the remaining cubes (that is, k of the remaining cubes) are white, and all but one of them

is not at the end of the edge.
Therefore, there are $k-1$ cubes not at the end of this edge, so there are $k-1$ cubes on this edge that are white on exactly two faces.
Since there are 12 edges, there are exactly $12(k-1) = 12k-12$ small cubes in the large cube that have exactly two white faces.

ii. The cubes that have at least two white faces are those from part (i) that have exactly two white faces and those with exactly three white faces.
The cubes with exactly three white faces are the 8 cubes at the corners of the large cube.
Therefore, there are $12k-12+8 = 12k-4$ cubes with at least two white faces.
Could $12k-4$ be equal to 2006?
If so, $12k-4 = 2006$ or $12k = 2010$ or $k = 167.5$, which is not an integer.
Therefore, there is no value of k for which the number of cubes having at least two white faces is 2006.

4. (a) Suppose that we use x yellow and y green rods.
We thus want $5x+3y = 62$ and each of x and y is a non-negative integer or $3y = 62-5x$.
The easiest approach is to try the possible values of x and check if y is an integer. We make a table:

x	$62-5x$	$62-5x$ divisible by 3?	y
0	62	No	
1	57	Yes	19
2	52	No	
3	47	No	
4	42	Yes	14
5	37	No	
6	32	No	
7	27	Yes	9
8	22	No	
9	17	No	
10	12	Yes	4
11	7	No	
12	2	No	

Therefore, there are 4 different sets (1 yellow and 19 green, 4 yellow and 14 green, 7 yellow and 9 green, 10 yellow and 4 green) that can be used.

(b) Green rods are 3 cm in length, yellow rods are 5 cm in length, black rods are 7 cm in length, and red rods are 9 cm in length.
If we use a green rods and b red rods, then the total length of the pole is $3a+9b$ cm, which is always divisible by 3.
Therefore, we cannot make a pole of length 62 cm with green and red rods, since 62 is not divisible by 3.
(We can check that each other pair of colours will allow us to make a pole of length 62 cm: 19 green and 1 yellow, 16 green and 2 black, 11 yellow and 1 black, 7 yellow and 3 red, and 5 black and 3 red rods each make a pole of length 62 cm.)

(c) Since we use at least 81 of each colour of rods, then let us suppose that we use $81+a$ green, $81+b$ pink, $81+c$ violet, and $81+d$ red rods, where each of a, b, c, and d is a non-negative integer.

For the total length to be 2007 cm, we must have

$$\begin{aligned} 3(81+a)+4(81+b)+8(81+c)+9(81+d) &= 2007 \\ 3a+4b+8c+9d+81(3+4+8+9) &= 2007 \\ 3a+4b+8c+9d &= 2007-1944 \\ 3a+4b+8c+9d &= 63 \end{aligned}$$

We want to find the total number of non-negative integer solutions to this equation.
To do this, we group the terms on the left-hand side as $3(a+3d)+4(b+2c)$ and let $x=a+3d$ and $y=b+2c$ and look at the equation $3x+4y=63$, where x and y are non-negative integers.
The possible solutions to this are $(x,y)=(1,15),(5,12),(9,9),(13,6),(17,3),(21,0)$.
If $x=a+3d=1$, then $(a,d)=(1,0)$, so there is 1 combination.
If $x=a+3d=5$, then $(a,d)=(5,0),(2,1)$, so there are 2 combinations.
If $x=a+3d=9$, then $(a,d)=(9,0),(6,1),(3,2),(0,3)$, so there are 4 combinations.
If $x=a+3d=13$, then $(a,d)=(13,0),(10,1),(7,2),(4,3),(1,4)$, so there are 5 combinations.
If $x=a+3d=17$, then $(a,d)=(17,0),(14,1),(11,2),(8,3),(5,4),(2,5)$, so there are 6 combinations.
If $x=a+3d=21$, then $(a,d)=(21,0),(18,1),(15,2),(12,3),(9,4),(6,5),(3,6),(0,7)$, so there are 8 combinations.
Similarly, if $y=b+2c$ runs through the possible values of 15, 12, 9, 6, 3, and 0, there are 8, 7, 5, 4, 2, and 1 combinations, respectively, for b and c.

Now we must combine the combinations.
If $x=1$ and $y=15$, there is 1 combination for a and d, and 8 combinations for b and c, so there are $1\times 8=8$ combinations for a, b, c, and d.
If $x=5$ and $y=12$, there are 2 combinations for a and d, and 7 combinations for b and c, so there are $2\times 7=14$ combinations for a, b, c, and d (since each combination for b and c works with each combination for a and d).
With the remaining pairs for x and y, there are $4\times 5=20$, $5\times 4=20$, $6\times 2=12$, $8\times 1=8$ combinations.
In total, there are thus $8+14+20+20+12+8=82$ combinations for a, b, c, and d, so there are 82 such sets of rods.