Canadian Mathematics Competition
An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2008 Fryer Contest

Wednesday, April 16, 2008

Solutions

1. (a) (i) The sum of the nine integers is

$$11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 = 135$$

(ii) The nine numbers in the magic square add up to 135.
The sum of the numbers in each row is the same, so will be $\frac{1}{3}$ of the total sum, or $\frac{1}{3}(135) = 45$.
Therefore, the magic constant is 45.

(iii) The sum of the numbers in each row, in each column, and on each of the two main diagonals should be 45.
Since we are given two numbers in the first row and the top left to bottom right diagonal, we can complete these to get

18	11	16
	15	
		12

, since $45 - 18 - 11 = 16$ and $45 - 18 - 12 = 15$.

We can now complete the second and third columns to get

18	11	16
	15	17
	19	12

.

We can now finish the magic square to get

18	11	16
13	15	17
14	19	12

.

(b) (i) The sum of the sixteen integers is

$$1 + 2 + 3 + 4 + \cdots + 13 + 14 + 15 + 16 = 136$$

(We can pair up the integers 1 with 16, 2 with 15, and so on, to get 8 pairs, each of which add to 17.)

(ii) The sixteen numbers in the magic square add up to 136.
The sum of the numbers in each row is the same, so will be $\frac{1}{4}$ of the total sum, or $\frac{1}{4}(136) = 34$.
Therefore, the magic constant is 34.

(iii) The sum of the numbers in each row, in each column, and on each of the two main diagonals should be 34.
Since we are given three numbers in the first and fourth rows, first, third and fourth columns, and on both diagonals, we can complete these to get

16	3	2	13
5	10	11	8
9		7	12
4	15	14	1

, since $34 - 16 - 3 - 13 = 2$, and so on.

We can now complete the magic square to get

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

.

2. (a) After playing 5 more games, the Sharks had played $10 + 5 = 15$ games in total and had won $8 + 1 = 9$ of these.
Their final winning percentage was $\dfrac{9}{15} \times 100\% = 60\%$.

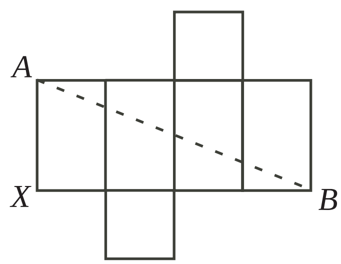
(b) The Emus played $10 + x$ games in total and won $4 + x$ games in total.
Since their final winning percentage was 70%, then $\dfrac{4+x}{10+x} \times 100\% = 70\%$ or $\dfrac{4+x}{10+x} = \dfrac{7}{10}$.

Therefore, $10(4 + x) = 7(10 + x)$, so $40 + 10x = 70 + 7x$ or $3x = 30$ or $x = 10$.
Therefore, the Emus played $10 + x = 10 + 10 = 20$ games in total.

(c) Suppose the Pink Devils played and lost y more games so that they had won exactly $\frac{2}{7}$ of their games at that point.
Then they had played $7 + 3 + y = 10 + y$ games in total and won 7 of them.
Therefore, $\frac{7}{10 + y} = \frac{2}{7}$ or $7(7) = 2(10 + y)$ or $2y + 20 = 49$ or $2y = 29$.
But y is not an integer, so there was no point when the Pink Devils had won exactly $\frac{2}{7}$ of their games.
(We could have also argued starting with the equation $7(7) = 2(10 + y)$ that the left side is even and the right side is odd, so there is no integer y that works.)

3. (a) The box that can be formed by folding the net in Figure 1 has dimensions 4 by 4 by 7, and so has volume $4 \times 4 \times 7 = 112$.
We can use the net more directly to calculate the surface area. The net shows that the box will have two faces that are 4 by 4, and four faces that are 4 by 7.
Therefore, the surface area is $2(4 \times 4) + 4(4 \times 7) = 32 + 112 = 144$.

(b) We label point X on Figure 3.



Now AX is the height of the box, so $AX = 6$.
Also, $\angle AXB = 90^\circ$, since the box is rectangular.
Next, we see that $XB = 2 + 2 + 2 + 2 = 8$, since the four edges of the original box that form XB are those around the bottom face of the original box.
Therefore, by the Pythagorean Theorem,

$$AB^2 = AX^2 + XB^2 = 6^2 + 8^2 = 36 + 64 = 100$$

so $AB = 10$, since $AB > 0$.

(c) We want to find the shortest path from A to G along the surface of the block. First, we make a few observations:

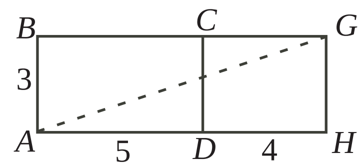
- Any path along the surface from A to G can be traced on an unfolded version of the block (that is, on a net like that in (a) or (b)).
- Given a particular way of unfolding the block, the shortest distance from A to G will be a straight line. The length of this straight line can be found using the Pythagorean Theorem, using the "rise" and the "run" from A to G, as in (b).
- In the given block, there is no single face that uses each of the vertices A and G. Thus, it is impossible for the caterpillar to travel from A to G along one face only, and so the caterpillar uses at least two faces.
- Any straight line path (and thus any path) that uses more than 2 faces will be longer than the possible straight line paths using exactly 2 faces. This is true because the rise and the run will be at least as long as in any of the 2 face paths below.

- Paths that include sections along edges do not need to be considered, as they will not form straight lines on the unfolded box.

So we need to examine all of the possible paths from A to G using exactly two faces. These combinations of faces are

- $ABCD$ then $DCGH$
- $ABCD$ then $BCGF$
- $ADHE$ then $DCGH$
- $ADHE$ then $EHGF$
- $ABFE$ then $BCGF$
- $ABFE$ then $EHGF$

Consider the straight line path across $ABCD$ and $DCGH$.



The path is the hypotenuse of a right-angled triangle with legs of lengths 3 and 9, and so has length $\sqrt{3^2 + 9^2} = \sqrt{90}$.
In a similar way, the lengths of the paths are:

- $ABCD$ then $DCGH$: $\sqrt{90}$
- $ABCD$ then $BCGF$: $\sqrt{74}$
- $ADHE$ then $DCGH$: $\sqrt{80}$
- $ADHE$ then $EHGF$: $\sqrt{74}$
- $ABFE$ then $BCGF$: $\sqrt{80}$
- $ABFE$ then $EHGF$: $\sqrt{90}$

(We note that these lengths occur in three pairs of equal lengths, so we could have calculated three lengths and used symmetry to deal with the other three.)
Since the length of the shortest path is one of these lengths, then the shortest path has length $\sqrt{74}$.

4. (a) First, we examine the digits in a general palindrome.
If a palindrome P has an even number of digits (like 1221), then each digit from 0 to 9 will occur an even number of times in P, as each digit in the first half can be matched with a digit in the second half.
If a palindrome P has an odd number of digits (like 12321), then there will be one digit from 0 to 9 that occurs an odd number of times in P and every other digit will occur an even number of times. This is because P has a "middle" digit. If this digit is removed, the resulting number is a palindrome with an even number of digits and so has an even number of each of 0 to 9. Adding the "middle" digit back in, one of the digits occurs an odd number of times.
Any collection of digits, in which each occurs an even number of times, can be arranged to form a palindrome with an even number of digits by building the palindrome in pairs of digits from the centre out.
Similarly, any collection of digits in which each but one occurs an even number of times can be arranged to form a palindrome, by starting with one occurrence of the digit that

occurs the odd number of times in the middle, and building outwards.
(No other collections of digits can be arranged to form a palindrome.)

Now, we examine x.
The integer x contains digits in the following distribution:

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	3	13	13	4	3	3	3	3	3	3

If we wanted to create a palindrome P with an even number of digits from the digits of x, we would need to reduce the number of occurrences of each of 0 to 9 so that each occurred an even number of times. To do this by removing the minimum number of digits, we can remove one each of 0, 1, 2, 4, 5, 6, 7, 8, 9. (To make an odd number even by subtracting the smallest amount, we subtract 1.) This ensures that each digit occurs an even number of times. Here, we have removed 9 digits in total.
If we wanted to create a palindrome P with an odd number of digits from the digits of x, we would need to reduce the number of occurrences of each of 0 to 9 so that all (but one) of them occurred an even number of times. To do this removing the minimum number of digits, we can remove 1 each of all but one of 0, 1, 2, 4, 5, 6, 7, 8, 9. (For the sake of argument, suppose that we do not remove a 9.) This ensures that each digit but 9 occurs an even number of times and 9 occurs an odd number of times. Here, we have removed 8 digits in total and formed a palindrome.
Therefore, the minimum number of digits that must be removed is 8.

(b) From the chart above, the digits of x add to

$$3(0) + 13(1) + 13(2) + 4(3) + 3(5) + 3(6) + 3(7) + 3(8) + 3(9) = 168$$

To obtain a sum of 130, we must remove digits that have a sum of $168 - 130 = 38$.
We want to remove the minimum number of digits that have a sum of 38.
We try to do this by removing the largest digits first.
Removing three 9's, we have removed a total of 27.
Removing one 8, we have removed a total of 35.
We can now obtain a total removed of 38 by removing a 3.
Here, we have removed 5 digits.
We cannot remove 4 or fewer digits with a total of 38, as the maximum sum of 4 or fewer digits in theory is $4(9) = 36$.
Therefore, the minimum number of digits that we can remove is 5.

(c) First, we enumerate the digits of y:

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	5	15	15	15	15	6	5	5	5	5

We can determine the sum of the digits by adding directly as above, or by noticing that each digit occurs at least 5 times, and grouping as follows:

$$5(0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) + 10(1 + 2 + 3 + 4) + 5 = 330$$

We want to remove digits from y whose sum is $330 - 210 = 120$ and in such a way that, among the digits that remain, at most one of the digits 0 to 9 occurs an odd number of times.
We use the strategy of removing digits in such a way that each of 0 to 9 occurs an even number of times, and the digits sum to at most 210, and then perhaps add back in a single

digit (giving the final palindrome an odd number of digits) to make the sum equal to 210.
We have to make each digit occur an even number of times, so first we remove 1 each of 0, 1, 2, 3, 4, 6, 7, 8, 9, to obtain:

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	4	14	14	14	14	6	4	4	4	4

whose sum is 290.
We now want to remove digits in pairs, removing the smallest *number* of digits that makes the total sum less than or equal to 210.
We do this by removing the largest possible digits first. Removing four 8's and four 9's gives

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	4	14	14	14	14	6	4	4	0	0

whose sum is 222.
We need to remove at least two more digits to obtain a sum of less than or equal to 210. (We cannot remove a single digit equal to at least 12.)
We could do this for example by removing two 6's, obtaining

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	4	14	14	14	14	6	2	4	0	0

We could then add back in a single 0 to maintain the sum of 210.
In total, we have removed $9 + 4 + 4 + 2 - 1 = 18$ digits.
This is the minimum number of digits because we have removed the minimum number of digits to form a palindrome with an even number of digits, and then added one digit back in.