Canadian
Mathematics
Competition
An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2008 Galois Contest

Wednesday, April 16, 2008

Solutions

1. (a) Since $(36, 25, x)$ is an O'Hara triple, then $\sqrt{36} + \sqrt{25} = x$, or $x = 6 + 5 = 11$.

   (b) Since $(a, 9, 5)$ is an O'Hara triple, then $\sqrt{a} + \sqrt{9} = 5$, or $\sqrt{a} + 3 = 5$, so $\sqrt{a} = 2$ or $a = 4$.

   (c) We want to find integers a and b with $\sqrt{a} + \sqrt{b} = 6$.
   We can find five such pairs by trying

   - $\sqrt{a} = 5$ with $\sqrt{b} = 1$ so $a = 25$ and $b = 1$,
   - $\sqrt{a} = 4$ with $\sqrt{b} = 2$ so $a = 16$ and $b = 4$,
   - $\sqrt{a} = 3$ with $\sqrt{b} = 3$ so $a = 9$ and $b = 9$,
   - $\sqrt{a} = 2$ with $\sqrt{b} = 4$ so $a = 4$ and $b = 16$,
   - $\sqrt{a} = 1$ with $\sqrt{b} = 5$ so $a = 1$ and $b = 25$.

   Therefore, five O'Hara triples with $x = 6$ are $(25, 1, 6)$, $(16, 4, 6)$, $(9, 9, 6)$, $(4, 16, 6)$, $(1, 25, 6)$.
   (Note that we are not asked to prove that these are the only triples, only to find five of them.)

2. (a) The line has slope $\dfrac{9-5}{6-0} = \dfrac{4}{6} = \dfrac{2}{3}$.
   Since the line passes through $P(0, 5)$, then its y-intercept is 5.
   Thus, the equation of the line is $y = \frac{2}{3}x + 5$.

   (b) A line that is perpendicular to the line from (a) must have slope equal to the negative reciprocal of $\frac{2}{3}$. That is, its slope equals $-\dfrac{1}{\frac{2}{3}} = -\dfrac{3}{2}$.
   Thus, this line has equation $y = -\frac{3}{2}x + b$ for some real number b.
   Since the line passes through $Q(6, 9)$, then the point $(6, 9)$ satisfies the equation of the line, so $9 = -\frac{3}{2}(6) + b$ or $9 = -9 + b$ or $b = 18$.
   Therefore, the equation of the line is $y = -\frac{3}{2}x + 18$.

   (c) Since every point on the x-axis has y-coordinate equal to 0, we find the x-coordinate of R by using the equation of the line $y = -\frac{3}{2}x + 18$ and setting $y = 0$ to get $0 = -\frac{3}{2}x + 18$ or $\frac{3}{2}x = 18$ or $x = \frac{2}{3}(18) = 12$.
   Thus, the coordinates of R are $(12, 0)$.

   (d) *Solution 1*
   Triangle PQR is right-angled at Q, so its area is $\frac{1}{2}(PQ)(QR)$.
   Since the coordinates of P are $(0, 5)$, of Q are $(6, 9)$, and of R are $(12, 0)$, then

   $$PQ = \sqrt{(6-0)^2 + (9-5)^2} = \sqrt{6^2 + 4^2} = \sqrt{52} = 2\sqrt{13}$$

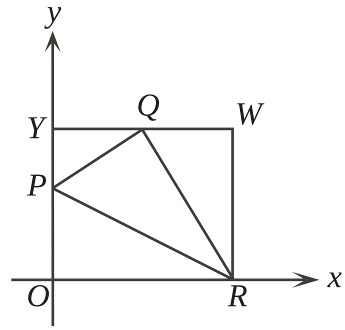
   and

   $$QR = \sqrt{(6-12)^2 + (9-0)^2} = \sqrt{6^2 + 9^2} = \sqrt{117} = 3\sqrt{13}$$

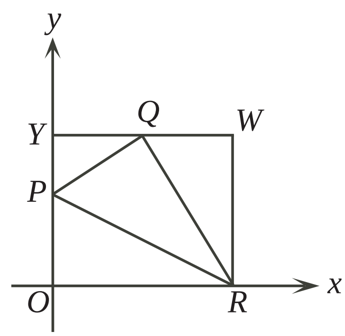
   Therefore, the area is $\frac{1}{2}(2\sqrt{13})(3\sqrt{13}) = 3(13) = 39$.

Solution 2

We "complete the rectangle" by drawing a horizontal line through Q meeting the y-axis at $Y(0,9)$ and a vertical line through R meeting the previous horizontal line at $W(12,9)$.

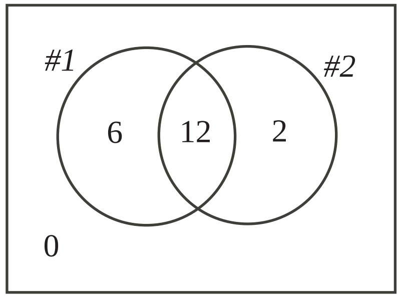


The rectangle $ORWY$ has width 12 and height 9 and so has area $12(9) = 108$.
The area of $\triangle PQR$ is the area of this rectangle minus the areas of $\triangle POR$, $\triangle PYQ$ and $\triangle QWR$.
Each of the these last three triangles is right-angled as it has two sides parallel to the axes.



From the diagram, the area of $\triangle POR$ is $\frac{1}{2}(5)(12) = 30$.
Also, the area of $\triangle PYQ$ is $\frac{1}{2}(4)(6) = 12$.
Lastly, the area of $\triangle QWR$ is $\frac{1}{2}(6)(9) = 27$.
From this information, the area of $\triangle PQR$ is $108 - 30 - 12 - 27 = 39$.

3. (a) The largest possible number who could have answered both questions correctly is 14. (This would happen if all who answered question 2 correctly also answered question 1 correctly.)
There cannot be more than 14 people who answered both questions correctly, because only 14 people answered question 2 correctly.
To find the smallest possible number who answered both questions correctly, we find the largest possible number who answered at least one question incorrectly. To do this, we try to ensure that different students answered each question incorrectly.
Here, 2 students answered question 1 incorrectly and 6 students answered question 2 incorrectly, so at most 8 students answered one question incorrectly. (There would be fewer than 8 students if there was overlap between those who answered each question incorrectly.)
Since at most 8 students answered one question incorrectly, then at least $20 - 8 = 12$ students answered both questions correctly.
We can actually achieve this number by arranging the students as seen in the Venn diagram, where the circles show the number of students who answered each question correctly.



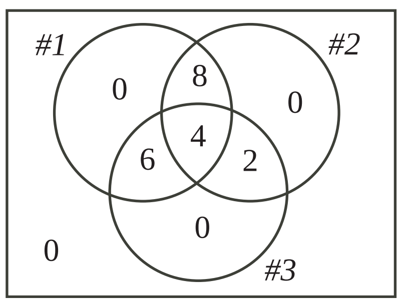
(b) The largest possible number who could have answered all three questions correctly is 12. (This would happen if all who answered question 3 correctly also answered questions 1 and 2 correctly.) There cannot be more than 12 people who answered all three questions correctly, because only 12 people answered question 3 correctly.

To find the smallest possible number who answered all three questions correctly, we find the largest possible number who answered at least one question incorrectly. To do this, we try to ensure that different students answered each question incorrectly.

Here, 2 students answered question 1 incorrectly, 6 students answered question 2 incorrectly, and 8 students answered question 3 incorrectly, so at most 16 students answered one question incorrectly.

Since at most 16 students answered one question incorrectly, then at least $20 - 16 = 4$ students answered all three questions correctly.

We can actually achieve this number by arranging the students as seen in the Venn diagram, where the circles show the number of students who answered each question correctly.



(c) We model our approach from (b).

To find the smallest possible number who answered all three questions correctly, we find the largest possible number who answered at least one question incorrectly. To do this, we try to ensure that different students answered each question incorrectly.

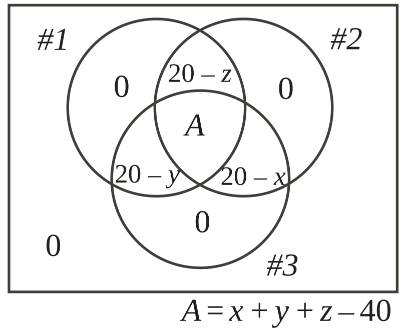
Here, $20-x$ students answered question 1 incorrectly, $20-y$ students answered question 2 incorrectly, and $20-z$ students answered question 3 incorrectly, so at most $60-x-y-z$ students answered one question incorrectly.

(Since $40 \leq x+y+z \leq 60$, then $0 \leq 60-x-y-z \leq 20$, so it makes sense to talk about $60-x-y-z$ students in this context.)

Since at most $60-x-y-z$ students answered one question incorrectly, then at least $20-(60-x-y-z) = x+y+z-40$ students answered all three questions correctly.

(Note that $0 \leq x+y+z-40 \leq 20$ since $40 \leq x+y+z \leq 60$, so $x+y+z-40$ is an admissible number of students.)

We can actually achieve this number by arranging the students as seen in the Venn diagram, where the circles show number of the students who answered each question correctly.



4. (a) The list starts as 1, 2, 3, 4, 5, 6.
If Carolyn removes 2, then Paul removes the remaining positive divisor of 2 (that is, 1) to leave the list 3, 4, 5, 6.
Carolyn must remove a number from this list that has at least one positive divisor other than itself remaining.
The only such number is 6, so Carolyn removes 6 and so Paul removes the remaining positive divisor of 6 (that is, 3), to leave the list 4, 5.
Carolyn cannot remove either of the remaining numbers as neither has a positive divisor other than itself remaining.
Thus, Paul removes 4 and 5.
In summary, Carolyn removes 2 and 6 for a sum of $2 + 6 = 8$ and Paul removes 1, 3, 4, and 5 for a sum of $1 + 3 + 4 + 5 = 13$.

(b) Since Carolyn removes a single number on each turn in such a way that Paul must be able to remove a number from the list, then Carolyn can remove at most half of the numbers of the list. In this case, Carolyn can remove at most five numbers.
The maximum possible five numbers that Carolyn could remove are the largest five numbers from the list (that is, 6, 7, 8, 9, 10), whose sum is 40.
This is the maximum possible without referring to all of the rules of the game. So is it possible for her to remove these five numbers?
In order to do so, she must remove them in an order which forces Paul to remove only one number on each of his turns.
If Carolyn removes 7 first, Paul removes only 1.
If Carolyn removes 9 next, Paul removes only 3.
If Carolyn removes 6 next, Paul removes only 2.
If Carolyn removes 8 next, Paul removes only 4.
If Carolyn removes 10 next, Paul removes only 5.
(Carolyn could have switched her last two turns.)
Therefore, Carolyn can indeed remove the five largest numbers, so her maximum possible final sum is 40.

(c) As in (b), Carolyn can remove at most half of the numbers from the list, so can remove at most 7 numbers.
Can she possibly remove 7 numbers?
If Carolyn removes 7 numbers, then Paul must also remove 7 numbers since he removes at least one number for each one that Carolyn removes and there are only 14 numbers.
Since Paul always removes numbers that are divisors of the number just removed by Carolyn, then Paul can never remove a number larger than $\frac{1}{2}n$. (For him to do so, Carolyn would have to have removed a number larger than $2 \times \frac{1}{2}n = n$. This is impossible.)
Therefore, if Carolyn actually removes 7 numbers, then she must remove the 7 numbers larger than $\frac{1}{2}n$ (that is, 8, 9, 10, 11, 12, 13, 14).

Whichever number Carolyn removes first, Paul will remove 1 on his first turn, as it is a positive divisor of every positive integer. (Paul might remove other numbers too.)
At this stage, at least one of 11 and 13 is left in the list (depending on whether Carolyn removed one of these on her first turn).
For the sake of argument, assume that 11 is still left in the list. (The argument is the same if 13 is left.)
Carolyn cannot now remove 11 from the list. This is because 11 is a prime number and its only positive divisors are 1 and 11, so 11 does not have a positive divisor other than itself left in the list, so by the last rule, Carolyn cannot remove 11.
Thus, Carolyn cannot remove all of the numbers from 8 to 14, so cannot remove 7 numbers from the list.