Canadian
Mathematics
Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2008 Hypatia Contest

Wednesday, April 16, 2008

Solutions

1. (a) By definition, $3\nabla 2 = 2(3) + 2^2 + 3(2) = 6 + 4 + 6 = 16$.

   (b) We have

$$\begin{aligned} x\nabla(-1) &= 8 \\ 2x + (-1)^2 + x(-1) &= 8 \\ x + 1 &= 8 \\ x &= 7 \end{aligned}$$

   so $x = 7$.

   (c) We have

$$\begin{aligned} 4\nabla y &= 20 \\ 2(4) + y^2 + 4y &= 20 \\ y^2 + 4y + 8 &= 20 \\ y^2 + 4y - 12 &= 0 \\ (y+6)(y-2) &= 0 \end{aligned}$$

   so $y = -6$ or $y = 2$.

   (d) We have

$$\begin{aligned} (w-2)\nabla w &= 14 \\ 2(w-2) + w^2 + (w-2)w &= 14 \\ 2w - 4 + w^2 + w^2 - 2w &= 14 \\ 2w^2 - 4 &= 14 \\ 2w^2 &= 18 \\ w^2 &= 9 \end{aligned}$$

   so $w = 3$ or $w = -3$.

2. (a) The slope of the line through $A(7,8)$ and $B(9,0)$ is $\dfrac{8-0}{7-9} = \dfrac{8}{-2} = -4$.
   Therefore, the line has equation $y = -4x + b$ for some b.
   Since $B(9,0)$ lies on this line, then $0 = -4(9) + b$ so $b = 36$.
   Thus, the equation of the line is $y = -4x + 36$.

   (b) We want to determine the point of intersection between the lines having equations $y = -4x + 36$ and $y = 2x - 10$.
   Equating values of y, we obtain $-4x + 36 = 2x - 10$ or $46 = 6x$ so $x = \frac{23}{3}$.
   We substitute this value of x into the equation $y = 2x - 10$ to determine the value of y, obtaining $y = 2(\frac{23}{3}) - 10 = \frac{46}{3} - \frac{30}{3} = \frac{16}{3}$.
   Thus, the coordinates of P are $(\frac{23}{3}, \frac{16}{3})$.

   (c) *Solution 1*
   The x-coordinate of A is 7 and the x-coordinate of B is 9.
   The average of these x-coordinates is $\frac{1}{2}(7 + 9) = 8$.
   Since the x-coordinate of P is $\frac{23}{3} < 8$, then the x-coordinate of P is closer to that of A than that of B.
   Since the points P, A and B lie on a straight line, then P is closer to A than to B.

Solution 2
The y-coordinate of A is 8 and the y-coordinate of B is 0.
The average of these y-coordinates is $\frac{1}{2}(8+0)=4$.
Since the y-coordinate of P is $\frac{16}{3}>4$, then the y-coordinate of P is closer to that of A than that of B.
Since the points P, A and B lie on a straight line, then P is closer to A than to B.

Solution 3
The coordinates of A are $(7,8)$, of B are $(9,0)$, and of P are $(\frac{23}{3},\frac{16}{3})$, then

$$PA=\sqrt{(7-\tfrac{23}{3})^2+(8-\tfrac{16}{3})^2}=\sqrt{(-\tfrac{2}{3})^2+(\tfrac{8}{3})^2}=\sqrt{\tfrac{68}{9}}$$

and

$$PB=\sqrt{(9-\tfrac{23}{3})^2+(0-\tfrac{16}{3})^2}=\sqrt{(\tfrac{4}{3})^2+(-\tfrac{16}{3})^2}=\sqrt{\tfrac{272}{9}}$$

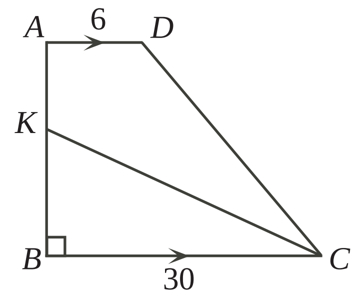
Thus, $PB>PA$, so P is closer to A than to B.

3. (a) *Solution 1*
Trapezoid $ABCD$ has bases $AD=6$ and $BC=30$, and height $AB=20$. (AB is a height since it is perpendicular to BC.)
Therefore, the area of $ABCD$ is $\frac{1}{2}(6+30)(20)=360$.

Solution 2
Join B to D.
Since AB is perpendicular to BC and AD is parallel to BC, then AB is perpendicular to AD.
Thus, $\triangle DAB$ is right-angled at A, and so has area $\frac{1}{2}(6)(20)=60$.
Also, $\triangle BDC$ can be considered as having base $BC=30$ and height equal to the length of BA (that is, height equal to 20), and so has area $\frac{1}{2}(30)(20)=300$.
The area of trapezoid $ABCD$ is the sum of the areas of $\triangle DAB$ and $\triangle BDC$, so equals $60+300=360$.

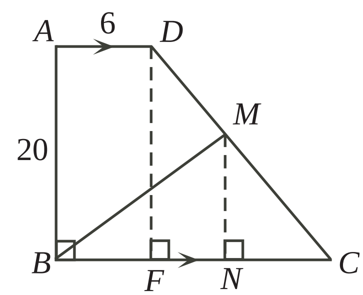
Solution 3
Since AB is perpendicular to BC and AD is parallel to BC, then AB is perpendicular to AD.
Drop a perpendicular from D to F on BC.
Then $ADFB$ is a rectangle that is 6 by 20, and so has area $6(20)=120$.
Also, $FC=BC-BF=30-AD=30-6=24$.
Thus, $\triangle DFC$ is right-angled at F, has height $DF=AB=20$ and base $FC=24$. Hence, the area of $\triangle DFC$ is $\frac{1}{2}(20)(24)=240$.
The area of trapezoid $ABCD$ is the sum of the areas of rectangle $ADFB$ and $\triangle DFC$, so equals $120+240$ or 360.

(b) First, we note that if K is on AB, then $\triangle KBC$ and quadrilateral $KADC$ cover the entire area of trapezoid $ABCD$.



Thus, if the areas of $\triangle KBC$ and quadrilateral $KADC$ are equal, then each equals one-half of the area of trapezoid $ABCD$, or $\frac{1}{2}(360) = 180$.
Suppose that $BK = h$.
Then $\triangle KBC$ has base $BC = 30$ and height $BK = h$, and so $\frac{1}{2}(30)h = 180$, or $h = 12$.
Therefore, $BK = 12$.

(c) *Solution 1*
As in (b), we want the area of $\triangle MBC$ to equal 180. Drop a perpendicular from M to N on BC.
Again as in (b), since the base BC of $\triangle MBC$ has length 30, then the height MN of $\triangle MBC$ needs to be 12 for its area to be 180.
Drop a perpendicular from D to F on BC.
Since DF is perpendicular to BC, then $ADFB$ is a rectangle, so $BF = 6$, which gives $FC = BC - BF = 30 - 6 = 24$.
Also, $DF = AB = 20$.



By the Pythagorean Theorem, $DC = \sqrt{20^2 + 24^2} = \sqrt{400 + 576} = \sqrt{976} = 4\sqrt{61}$.
Lastly, we must calculate the length of MC.

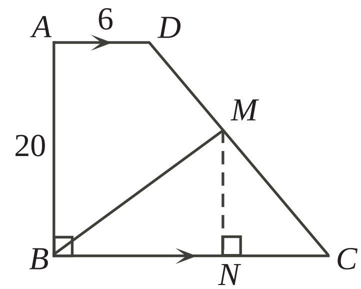
<u>Method 1</u>
We know that $\sin(\angle DCF) = \dfrac{DF}{DC} = \dfrac{20}{4\sqrt{61}} = \dfrac{5}{\sqrt{61}}$.
Since $MN = 12$, then $MC = \dfrac{MN}{\sin(\angle DCF)} = \dfrac{12}{5/\sqrt{61}}$, or $MC = \frac{12}{5}\sqrt{61}$.

<u>Method 2</u>
We know that $\triangle DFC$ is similar to $\triangle MNC$, since each is right-angled and they have a common angle at C.
Therefore, $\dfrac{MC}{MN} = \dfrac{DC}{DF}$ so $MC = \frac{12}{20}(4\sqrt{61}) = \frac{12}{5}\sqrt{61}$.

<u>Method 3</u>
Since $\triangle DFC$ is similar to $\triangle MNC$, then $\dfrac{NC}{MN} = \dfrac{FC}{DF}$ so $NC = \frac{12(24)}{20} = \frac{72}{5}$.
By the Pythagorean Theorem,

$$MC = \sqrt{MN^2 + NC^2} = \sqrt{12^2 + \left(\tfrac{72}{5}\right)^2} = \tfrac{12}{5}\sqrt{6^2 + 5^2} = \tfrac{12}{5}\sqrt{61}$$

Solution 2
As in (b), we want the area of $\triangle MBC$ to equal 180.



Drop a perpendicular from M to N on BC.
Again as in (b), since the base BC of $\triangle MBC$ has length 30, then the height MN of $\triangle MBC$ needs to be 12 for its area to be 180.
We place the diagram on a coordinate grid, with B at the origin, A on the positive y-axis, and C on the positive x-axis.
Thus, the coordinates of B are $(0,0)$, the coordinates of A are $(0,20)$, the coordinates of D are $(6,20)$, and the coordinates of C are $(30,0)$.
Since the length of MN is 12, then the coordinates of M are $(s,12)$ for some real number s.
But M lies on DC, so the slope of DC equals the slope of MC, or $\dfrac{0-20}{30-6}=\dfrac{0-12}{30-s}$ so $-20(30-s)=24(-12)$ or $20s-600=-288$ or $20s=312$ or $s=\frac{78}{5}$.
Using the coordinates of M and of C, we have

$$MC=\sqrt{\left(30-\tfrac{78}{5}\right)^2+(0-12)^2}=\sqrt{\left(\tfrac{72}{5}\right)^2+12^2}=\tfrac{12}{5}\sqrt{6^2+5^2}=\tfrac{12}{5}\sqrt{61}$$

4. (a) Taking all possible products of pairs,

$$\begin{aligned} 2(3)+2x+2(2x)+3x+3(2x)+x(2x) &= -7 \\ 6+15x+2x^2 &= -7 \\ 2x^2+15x+13 &= 0 \\ (2x+13)(x+1) &= 0 \end{aligned}$$

so $x=-1$ or $x=-\frac{13}{2}$.

(b) Since each term equals 1, -1 or 2, then the pairs of the terms whose products is 1 are those coming from 1×1 and $(-1)\times(-1)$.
We know that there are m terms equal to 1. How many pairs can these terms form?
To form a pair, there are m choices for the first entry and then $m-1$ choices for the first entry (all but the first term chosen). This gives $m(m-1)$ pairs.
But we have counted each pair twice here (as we have counted both a and b as well as b and a, while we only want to count one of these), so we divide by 2 to obtain $\frac{1}{2}m(m-1)$ pairs.
(Alternatively, we could have said that there were $\dbinom{m}{2}=\dfrac{m(m-1)}{2}$ pairs.)
We also know that there are n terms equal to -1.
In a similar way, these will form $\frac{1}{2}n(n-1)$ pairs.
Therefore, in total, there are $\frac{1}{2}m(m-1)+\frac{1}{2}n(n-1)$ pairs of distinct terms whose product is 1.

(c) Suppose that the sequence contains m terms equal to 2 and so $n=100-m$ terms equal to -1.
The terms equal to 2 form $\frac{1}{2}m(m-1)$ pairs, each contributing $2\times 2=4$ to the peizi-sum.

The terms equal to -1 form $\frac{1}{2}n(n-1) = \frac{1}{2}(100-m)(99-m)$ pairs each equal to $(-1) \times (-1) = 1$.
Since there are m terms equal to 2 and $100-m$ terms equal to -1, then there are $m(100-m)$ pairs formed by choosing one 2 and one -1, and so $m(100-m)$ pairs of terms, each contributing $2 \times (-1) = -2$ to the peizi-sum.
These are all of the possible types of pairs, so the peizi-sum, S, is

$$S = 4\left(\tfrac{1}{2}m(m-1)\right) + 1\left(\tfrac{1}{2}(100-m)(99-m)\right) + (-2)\left(m(100-m)\right)$$

or

$$S = 2m^2 - 2m + 50(99) - \tfrac{199}{2}m + \tfrac{1}{2}m^2 - 200m + 2m^2$$

or

$$S = \tfrac{9}{2}m^2 - \tfrac{603}{2}m + 4950$$

The equation $S = \frac{9}{2}m^2 - \frac{603}{2}m + 4950$ is a quadratic equation in m (forming a parabola opening upwards), and so is minimized at its vertex, which occurs at

$$m = -\frac{-\frac{603}{2}}{2\left(\frac{9}{2}\right)} = \tfrac{67}{2} = 33\tfrac{1}{2}$$

However, this value of m is not an integer, so is not the value of m that solves our problem.
The parabola formed by this equation is symmetric about its vertex and increases in either direction from its vertex, so the minimum value at an integer value of m occurs at both $m = 33$ and $m = 34$ (since these are the closest integers to $33\frac{1}{2}$ and are the same distance from $33\frac{1}{2}$).
We can substitute either value to determine the minimum possible peizi-sum, which is $\frac{9}{2}(33)^2 - \frac{603}{2}(33) + 4950 = -99$.