Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest

Tuesday, April 7, 2009

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Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 3 marks each. **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.

Instructions for SHORT ANSWER parts:

1. **SHORT ANSWER** parts are indicated like this: .
2. **Enter the answer in the appropriate box in the answer booklet.** For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

Instructions for FULL SOLUTION parts:

1. **FULL SOLUTION** parts are indicated like this: .
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

NOTE: At the completion of the Contest, insert the information sheet inside the answer booklet.

The names of some top-scoring students will be published in the Euclid Results on our Web site, http://www.cemc.uwaterloo.ca.

NOTES:
1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked " ", full marks will be given for a correct answer placed in the appropriate box in the answer booklet. **If an incorrect answer is given, marks may be given for work shown.** Students are strongly encouraged to show their work.
4. All calculations and answers should be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., rather than as $12.566\ldots$ or $4.646\ldots$, except where otherwise indicated.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth, grade and sex, on the Student Information Form, and that you have answered the question about eligibility.

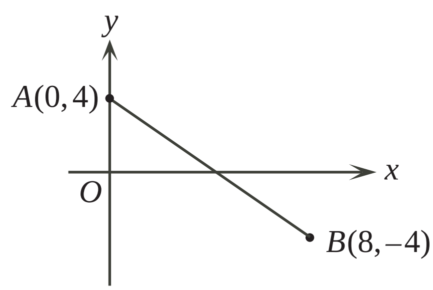
A Note about Writing Solutions

For each problem marked " ", a full solution is required. The solutions that you provide in the answer booklet should be well organized and contain mathematical statements and words of explanation when appropriate. Working out some of the details in rough on a separate piece of paper before writing your finished solution is a good idea. Your final solution should be written so that the marker can understand your approach to the problem and all of the mathematical steps of your solution.

1. (a) A line has equation $6x + 3y - 21 = 0$. What is the slope of the line?

   (b) A line with a slope of 3 passes through the points $(1, 0)$ and $(5, c)$. What is the value of c?

   (c) The point (k, k) lies on the line segment AB shown in the diagram. Determine the value of k.

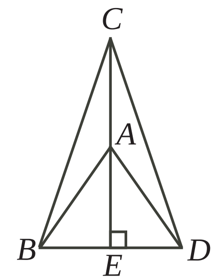


2. (a) What is the sum of the two numbers that satisfy the equation $x^2 - 6x - 7 = 0$?

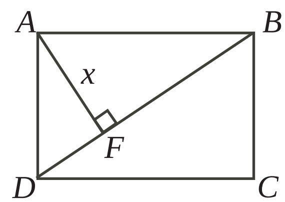
   (b) What is the product of the two numbers that satisfy the equation $5x^2 - 20 = 0$?

   (c) Determine the average of the numbers that satisfy the equation $x^3 - 6x^2 + 5x = 0$.

3. (a) In the diagram, $AB = AC = AD = BD$ and CAE is a straight line segment that is perpendicular to BD. What is the measure of $\angle CDB$?



(b) In rectangle $ABCD$, F is on diagonal BD so that AF is perpendicular to BD. Also, $BC = 30$, $CD = 40$ and $AF = x$. Determine the value of x.

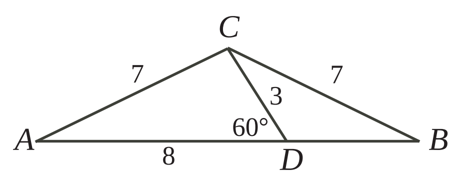


4. (a) In an arithmetic sequence, the first term is 1 and the last term is 19. The sum of all the terms in the sequence is 70. How many terms does the sequence have?

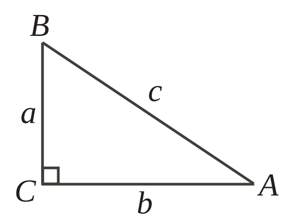
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9 is an arithmetic sequence with four terms.)

(b) Suppose that $a(x + b(x + 3)) = 2(x + 6)$ for *all* values of x. Determine a and b.

5. (a) In the diagram, $\triangle ABC$ is isosceles with $AC = BC = 7$. Point D is on AB with $\angle CDA = 60^\circ$, $AD = 8$, and $CD = 3$. Determine the length of BD.

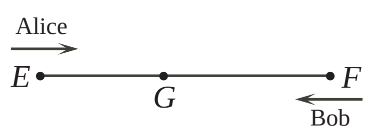


(b) In the diagram, $\triangle ABC$ is right-angled at C. Also, $2\sin B = 3\tan A$. Determine the measure of angle A.



6. (a) An integer n, with $100 \le n \le 999$, is chosen at random. What is the probability that the sum of the digits of n is 24?

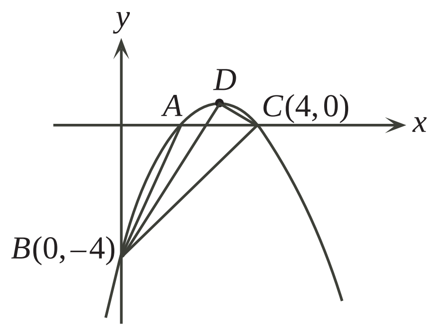
(b) Alice drove from town E to town F at a constant speed of 60 km/h. Bob drove from F to E along the same road also at a constant speed. They started their journeys at the same time and passed each other at point G.



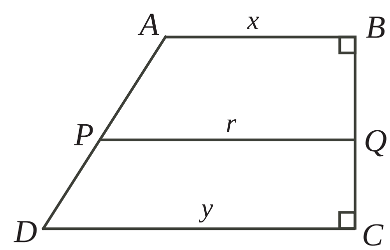
Alice drove from G to F in 45 minutes. Bob drove from G to E in 20 minutes. Determine Bob's constant speed.

7. (a) The parabola $y = x^2 - 2x + 4$ is translated p units to the right and q units down. The x-intercepts of the resulting parabola are 3 and 5. What are the values of p and q?

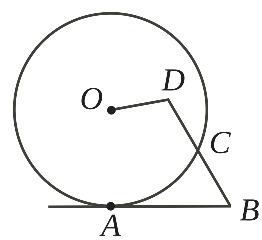
(b) In the diagram, D is the vertex of a parabola. The parabola cuts the x-axis at A and at $C(4,0)$. The parabola cuts the y-axis at $B(0,-4)$. The area of $\triangle ABC$ is 4. Determine the area of $\triangle DBC$.



8. (a) $ABCD$ is a trapezoid with parallel sides AB and DC. Also, BC is perpendicular to AB and to DC. The line PQ is parallel to AB and divides the trapezoid into two regions of equal area. If $AB = x$, $DC = y$, and $PQ = r$, prove that $x^2 + y^2 = 2r^2$.



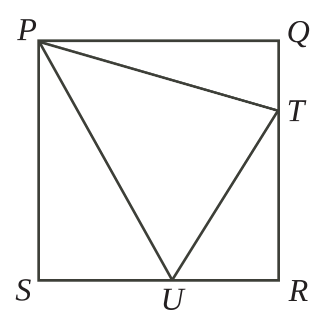
(b) In the diagram, AB is tangent to the circle with centre O and radius r. The length of AB is p. Point C is on the circle and D is inside the circle so that BCD is a straight line, as shown. If $BC = CD = DO = q$, prove that $q^2 + r^2 = p^2$.



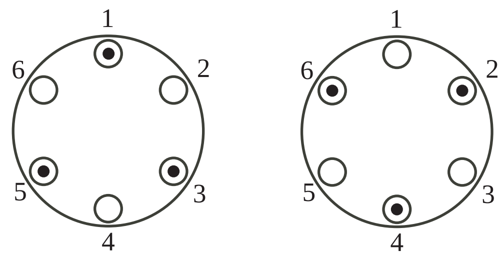
9. (a) If $\log_2 x$, $(1 + \log_4 x)$, and $\log_8 4x$ are consecutive terms of a geometric sequence, determine the possible values of x.

(A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a constant. For example, 3, 6, 12 is a geometric sequence with three terms.)

(b) In the diagram, $PQRS$ is a square with sides of length 4. Points T and U are on sides QR and RS respectively such that $\angle UPT = 45^\circ$. Determine the maximum possible perimeter of $\triangle RUT$.



10. Suppose there are n plates equally spaced around a circular table. Ross wishes to place an identical gift on each of k plates, so that no two neighbouring plates have gifts. Let $f(n, k)$ represent the number of ways in which he can place the gifts. For example $f(6, 3) = 2$, as shown below.



(a) Determine the value of $f(7, 3)$.

(b) Prove that $f(n, k) = f(n-1, k) + f(n-2, k-1)$ for all integers $n \geq 3$ and $k \geq 2$.

(c) Determine the smallest possible value of $n + k$ among all possible ordered pairs of integers (n, k) for which $f(n, k)$ is a positive multiple of 2009, where $n \geq 3$ and $k \geq 2$.

Canadian Mathematics Competition

For students...

Thank you for writing the 2009 Euclid Contest!
In 2008, more than 14 000 students from around the world registered to write the Euclid Contest.

If you are graduating from secondary school, good luck in your future endeavours!
If you will be returning to secondary school next year, encourage your teacher to register you for the 2009 Sun Life Financial Canadian Open Mathematics Challenge, which will be written in late November.
Visit our website

www.cemc.uwaterloo.ca

to find

- More information about the Sun Life Financial Canadian Open Mathematics Challenge
- Free copies of past contests
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