**Canadian Mathematics Competition**

***An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario***

# *2009 Fryer Contest*

## Wednesday, April 8, 2009

*Solutions*

1. (a) The total cost to make 100 cups of lemonade is $\$0.15 \times 100 + \$12.00 = \$15.00 + \$12.00 = \$27.00$.

   (b) The total money earned for selling 100 cups of lemonade is $\$0.75 \times 100 = \$75.00$.
   The profit is the total money earned minus the total cost, or $\$75.00 - \$27.00 = \$48.00$.

   (c) Let $x$ represent the number of cups that Emily must sell to break even.
   The total cost to make $x$ cups of lemonade is $\$0.15 \times x + \$12.00 = \$0.15x + \$12.00$.
   The total money earned by selling $x$ cups is $\$0.75 \times x$ or $\$0.75x$.
   For a profit of \$0, total cost must equal total money earned.
   Therefore, $\$0.15x + \$12.00 = \$0.75x$ or $\$12.00 = \$0.60x$ or $x = 20$.
   Emily must sell 20 cups of lemonade to break even.

   (d) Let $n$ represent the number of cups that Emily must sell to make a profit of exactly \$17.00.
   As in (c), the total cost to make $n$ cups of lemonade is $\$0.15n + \$12.00$.
   As in (c), the total money earned by selling $n$ cups is $\$0.75n$.
   For a profit of \$17.00, the total money earned minus the total cost must equal \$17.00.
   Thus, $\$0.75n-(\$0.15n+\$12.00) = \$17.00$ or $\$0.60n-\$12.00 = \$17.00$ or $\$0.60n = \$29.00$.
   This gives $n = \dfrac{29}{0.6} = \dfrac{290}{6} = 48\dfrac{1}{3}$.
   Since $n$ represents the number of cups that Emily sells, $n$ must be a non-negative integer and as such, cannot equal $48\frac{1}{3}$.
   Therefore, it is not possible for Emily to make a profit of exactly \$17.00.

2. (a) Evaluating, $2\nabla 5 = \dfrac{2+5}{1+2\times 5} = \dfrac{7}{11}$.

   (b) Evaluating the expression in brackets first,
   $(1\nabla 2)\nabla 3 = \left(\dfrac{1+2}{1+1\times 2}\right)\nabla 3 = \left(\dfrac{3}{3}\right)\nabla 3 = 1\nabla 3 = \dfrac{1+3}{1+1\times 3} = 1$.
   (Note that for any $b > 0$, $1\nabla b = \dfrac{1+b}{1+1\times b} = \dfrac{1+b}{1+b} = 1$.)

   (c) By definition, $2\nabla x = \dfrac{2+x}{1+2x}$. Thus,
$$\begin{aligned} \frac{2+x}{1+2x} &= \frac{5}{7} \\ 7(2+x) &= 5(1+2x) \\ 14+7x &= 5+10x \\ 9 &= 3x \end{aligned}$$
   so $x = 3$.

   (d) We have, $x\nabla y = \dfrac{x+y}{1+xy}$. Thus, from the given information, $\dfrac{x+y}{1+xy} = \dfrac{x+y}{17}$.
   The numerator, $x+y$, of each of these two fractions is non-zero since $x > 0$ and $y > 0$.
   Two equivalent fractions having equal, non-zero numerators have equal denominators.
   Thus, $1 + xy = 17$ or $xy = 16$. The possible ordered pairs of positive integers $(x, y)$, for which $xy = 16$, are $(1, 16)$, $(16, 1)$, $(2, 8)$, $(8, 2)$, and $(4, 4)$.

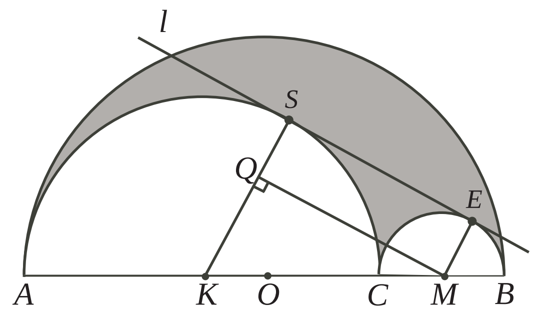
3. (a) We know that $OA$ and $OB$ are each radii of the semi-circle with centre $O$.
   Thus, $OA = OB = OC + CB = 32 + 36 = 68$.
   Therefore, $AC = AO + OC = 68 + 32 = 100$.

   (b) The semi-circle with centre $K$ has radius $AK = \frac{1}{2}(AC) = \frac{1}{2}(100) = 50$.
   Thus, this semi-circle has an area equal to $\frac{1}{2}\pi(AK)^2 = \frac{1}{2}\pi(50)^2 = 1250\pi$.

(c) The shaded area is equal to the area of the largest semi-circle with centre $O$, minus the combined areas of the two smaller unshaded semi-circles with centres $K$ and $M$.
The radius of the smaller unshaded circle is $MB = \frac{1}{2}(CB) = \frac{1}{2}(36) = 18$.
Therefore, the shaded area equals

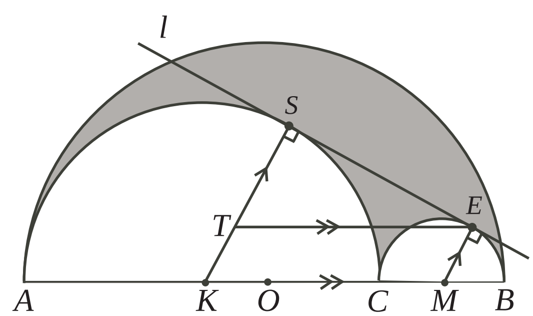
$$\begin{aligned} & \tfrac{1}{2}\pi(OB)^2 - \left(\tfrac{1}{2}\pi(AK)^2 + \tfrac{1}{2}\pi(MB)^2\right) \\ = \ & \tfrac{1}{2}\pi(68)^2 - \left(\tfrac{1}{2}\pi(50)^2 + \tfrac{1}{2}\pi(18)^2\right) \\ = \ & \tfrac{1}{2}\pi(68^2 - 50^2 - 18^2) \\ = \ & \tfrac{1}{2}\pi(4624 - 2500 - 324) \\ = \ & \tfrac{1}{2}\pi(1800) \\ = \ & 900\pi \end{aligned}$$

(d) *Solution 1*
Construct line segments $KS$ and $ME$ perpendicular to line $l$.
Position point $Q$ on $KS$ so that $MQ$ is perpendicular to $KS$, as shown.
In quadrilateral $MQSE$,
$\angle MQS = \angle QSE = \angle SEM = 90^\circ$.
Hence, quadrilateral $MQSE$ is a rectangle.




The larger unshaded semi-circle has radius 50, so $KC = KS = 50$.
The smaller unshaded semi-circle has radius 18, so $ME = MC = MB = 18$.
Thus, $MK = MC + KC = 18 + 50 = 68$.
The area of quadrilateral $KSEM$ is the sum of the areas of rectangle $MQSE$ and $\triangle MKQ$.
Since $QS = ME = 18$, then $KQ = KS - QS = 50 - 18 = 32$.
Using the Pythagorean Theorem in $\triangle MKQ$, $MK^2 = KQ^2 + QM^2$ or $68^2 = 32^2 + QM^2$ or $QM = \sqrt{68^2 - 32^2} = 60$ (since $QM > 0$).
The area of $\triangle MKQ$ is $\frac{1}{2}(KQ)(QM) = \frac{1}{2}(32)(60) = 960$.
The area of rectangle $MQSE$ is $(QM)(QS) = (60)(18) = 1080$.
Thus, the area of quadrilateral $KSEM$ is $960 + 1080 = 2040$.

*Solution 2*
Construct line segments $KS$ and $ME$ perpendicular to line $l$.
Since $KS$ and $ME$ are each perpendicular to line $l$, they are parallel to one another and thus, $KSEM$ is a trapezoid.
Position point $T$ on $KS$ so that $TE$ is parallel to $KM$, as shown.




The larger unshaded semi-circle has radius 50, so $KC = KS = 50$.
The smaller unshaded semi-circle has radius 18, so $ME = MC = MB = 18$.
Thus, $KM = KC + CM = 50 + 18 = 68$.
Since $KTEM$ is a parallelogram $TE = KM = 68$ and $KT = ME = 18$.
Thus, $TS = KS - KT = 50 - 18 = 32$.
Using the Pythagorean Theorem in $\triangle TSE$, $SE^2 = TE^2 - TS^2$ or $SE^2 = 68^2 - 32^2$ or $SE = \sqrt{68^2 - 32^2} = 60$ (since $SE > 0$).
The area of trapezoid $KSEM$ is $\dfrac{(SE)(ME + KS)}{2} = \dfrac{(60)(18 + 50)}{2} = (30)(68) = 2040$.

4. When we are calculating such a sum by hand, in each column starting at the right hand end, we add up the digits and add the carry to this to create a column total. We take this total, write down its unit digit and take the integer formed by the rest of the total to become the carry into the next column to the left.
For example, if a column total is 124, we write down the units digit 4 beneath this column and carry 12 into the next column to the left.

(a) The sum of the digits in the ones column is $2(101) = 202$. Thus, the ones digit $A$ is 2, and 20 is carried left to the tens column.

(b) The sum of the digits in the tens column is $2(100) = 200$ and the carry into the tens column is 20. Thus, the total in the tens column is $200 + 20 = 220$.
Therefore, the tens digit $B$ is 0 and 22 is carried left to the hundreds column.
The sum of the digits in the hundreds column is $2(99) = 198$, which when added to the carry of 22 gives a column total of $198 + 22 = 220$.
Thus, the hundreds digit $C$ is also 0.

(c) We show that the middle digit of the sum is 3 by using the following steps:

- Step 1: The sum has 101 digits
- Step 2: The total in the middle column is at least 113
- Step 3: The total in the middle column cannot be 114 or more

Steps 2 and 3 together will tell us that the total in the middle column is exactly 113, so the middle digit in the sum is the units digit of 113, which is 3. We will need to use the following Fact:

Fact: The carry from one column to the next is always at most 22.

We leave the explanation of this Fact to the end, but use it twice. We number the columns in the sum from left to right.

<u>Step 1: The sum has 101 digits</u>
The total in the 1st column is 2 plus the carry from the 2nd column. Thus, the total in the 1st column will have only 1 digit unless the carry from the 2nd column is at least 8.
For the carry from the 2nd column to be at least 8, then the total in the 2nd column must be at least 80. The sum of the digits in the 2nd column is 4, so for a total of at least 80, the carry from the 3rd column would have to be at least 76.
But every carry is at most 22 (see the Fact below), so this is impossible.
Thus, the total in the 1st column has only 1 digit.
This means that the sum has exactly the same number of digits as the largest number among the numbers being added, so has 101 digits.

<u>Step 2: The total in the middle column is at least 113</u>
Since the sum has 101 digits, then the middle digit (or column) is the 51st column from the left. (There will be 50 digits to the left of this column and 50 digits to the right, giving 101 digits in total.)
The sum of the digits in the 51st column is $2(51) = 102$.
The sum of the digits in the 52nd column is $2(52) = 104$.
The sum of the digits in the 53rd column is $2(53) = 106$.
Thus, the carry from the 53rd column to the 52nd column is at least 10, so the total of the 52nd column is at least $104 + 10 = 114$.
Thus, the carry from the 52nd column to the 51st column is at least 11, so the total of the 51st column is at least $102 + 11 = 113$.

Step 3: The total in the middle column cannot be 114 or more
If the total in the 51st column was 114 or more, then the carry from the 52nd column would have to be at least $114 - 102 = 12$.
For the carry from the 52nd column to be at least 12, then the total in the 52nd column would have to be at least 120. For this total to be at least 120, then the carry from the 53rd column would have to be at least $120 - 104 = 16$.
For the carry from the 53rd column to be at least 16, then the total in the 53rd column would have to be at least 160. For this total to be at least 160, then the carry from the 54th column would have to be at least $160 - 106 = 54$.
But by the Fact (see below), the carry cannot be more than 22, so this is impossible.
Therefore, the total in the middle column cannot be 114 or more, so must be exactly 113.

It remains to look at the Fact.

Fact: The carry from one column to the next is always at most 22
We start from the rightmost column.
The sum of the digits in the 101st column is $2(101) = 202$, so 20 is carried to the 100th column.
The total of the 100th column is thus $2(100) + 20 = 220$, so 22 is carried to the 99th column.
The total of the 99th column is thus $2(99)+22 = 220$, so 22 is carried to the 98th column.
To this point, none of the carries is larger than 22.
Suppose that at some point the carry is at least 23.
If this is the case, then starting from the right end, there must be a first time that the carry is at least 23.
Let's suppose that this first time is the carry from $n$th column.
We know that $n \leq 98$, since the carries from the first three columns are all at most 22.
Also, we know that the carry into the $n$th column is at most 22, since the carry from the $n$th column is the first time that the carry is more than 22.
Let's look at this $n$th column.
The $n$th column includes $n$ 2's, so the sum of the digits in this column is $2n$.
Since $n \leq 98$, then the sum of the digits in the $n$th column is at most $2(98) = 196$.
For the carry from the $n$th column to be at least 23, the total in the $n$th column must be at least 230, which means that the carry into the $n$th column is at least $230 - 196 = 34$.
But we already know that the carry into the $n$th column is at most 22, since the carry from the $n$th column is the first time that the carry is at least 22.
This means that we have a contradiction, since the carry into the $n$th column cannot be both at most 22 and at least 34.
Thus, the only thing that can be wrong is our assumption that at some point the carry is at least 23.
Therefore, the carry is always at most 22.
This completes the argument, and shows that the total of the middle column is exactly 113, so the middle digit of the sum is 3.