Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2010 Galois Contest

Friday, April 9, 2010

Solutions

1. (a) Emily's new showerhead uses 13 L of water per minute.
At this rate, it will take Emily $\frac{260}{13} = 20$ minutes of showering to use 260 L of water.

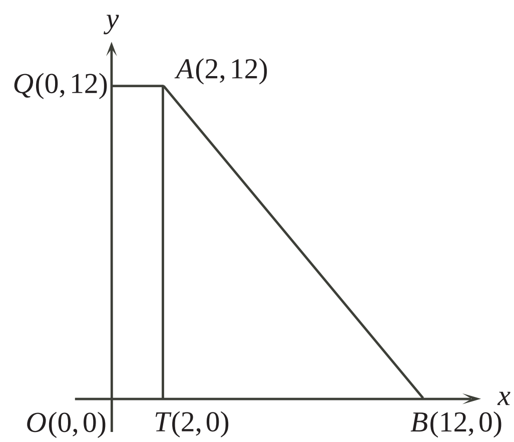
(b) By using the new showerhead, Emily is using $18 - 13 = 5$ L of water per minute less than when she used the old showerhead.
Thus for a 10 minute shower using the new showerhead, Emily uses $10 \times 5 = 50$ L less water.

(c) From part (b), we know that Emily saves 5 L of water per minute by using the new showerhead.
For a 15 minute shower, Emily saves $15 \times 5 = 75$ L of water.
Since Emily is charged 8 cents per 100 L of water, she will save $\frac{8}{100} \times 75 = 6$ cents in water costs for a 15 minute shower.

(d) *Solution 1*
Emily is charged 8 cents per 100 L of water and is saving 5 L of water per minute.
Thus, she is saving $\frac{8}{100} \times 5 = \frac{8}{20} = \frac{2}{5}$ of a cent per minute by using the new showerhead.
To save \$30 or 3000 cents, it will take Emily $3000 \div \frac{2}{5} = 3000 \times \frac{5}{2} = 7500$ minutes of showering.

Solution 2
From part (c), Emily saves 6 cents in 15 minutes of showering. Since $3000 \div 6 = 500$, it takes $15 \times 500 = 7500$ minutes to save \$30.

2. (a) *Solution 1*
If point T is placed at $(2, 0)$, then T is on OB and AT is perpendicular to OB.
Since QO is perpendicular to OB, then QO is parallel to AT.
Both QA and OT are horizontal, so then QA is parallel to OT.
Therefore, $QATO$ is a rectangle.
The area of rectangle $QATO$ is $QA \times QO$ or $(2 - 0) \times (12 - 0) = 24$.
Since AT is perpendicular to TB, we can treat AT as the height of $\triangle ATB$ and TB as the base.
The area of $\triangle ATB$ is $\frac{1}{2} \times TB \times AT$ or $\frac{1}{2} \times (12 - 2) \times (12 - 0) = \frac{1}{2} \times 10 \times 12 = 60$.
The area of $QABO$ is the sum of the areas of rectangle $QATO$ and $\triangle ATB$, or $24 + 60 = 84$.



Solution 2
Both QA and OB are horizontal, so then QA is parallel to OB.
Thus, $QABO$ is a trapezoid.
Since QO is perpendicular to OB, we can treat QO as the height of the trapezoid.
Then, $QABO$ has area $\frac{1}{2} \times QO \times (QA + OB) = \frac{1}{2} \times 12 \times (2 + 12) = \frac{1}{2} \times 12 \times 14 = 84$.

(b) Since CO is perpendicular to OB, we can treat CO as the height of $\triangle COB$ and OB as the base. The area of $\triangle COB$ is $\frac{1}{2} \times OB \times CO$ or $\frac{1}{2} \times (12 - 0) \times (p - 0) = \frac{1}{2} \times 12 \times p = 6p$.

(c) Since QA is perpendicular to QC, we can treat QC as the height of $\triangle QCA$ and QA as the base. The area of $\triangle QCA$ is $\frac{1}{2} \times QA \times QC$ or $\frac{1}{2} \times (2 - 0) \times (12 - p) = \frac{1}{2} \times 2 \times (12 - p) = 12 - p$.

(d) The area of $\triangle ABC$ can be found by subtracting the area of $\triangle COB$ and the area of $\triangle QCA$ from the area of quadrilateral $QABO$.
From parts (a), (b) and (c), the area of $\triangle ABC$ is thus $84 - 6p - (12 - p) = 72 - 5p$.
Since the area of $\triangle ABC$ is 27, then $72 - 5p = 27$ or $5p = 45$, so $p = 9$.

3. (a) We solve the system of equations by the method of elimination.
Adding the first equation to the second, we get $2x = 52$, and so $x = 26$.
Substituting $x = 26$ into the first equation, we get $26 + y = 42$, and so $y = 16$.
The solution to the system of equations is $(x, y) = (26, 16)$.

(b) *Solution 1*
We proceed as in part (a) by solving the system of equations by the method of elimination.
Adding the first equation to the second, we get $2x = p + q$, and so $x = \frac{p+q}{2}$.
We are given that p is an even integer and that q is an odd integer.
The sum of an even integer and an odd integer is always an odd integer.
Thus, $\frac{p+q}{2}$ is an odd integer divided by two, which is never an integer.
Therefore, the given system of equations has no positive integer solutions.

Solution 2
We proceed as in part (a) by solving the system of equations by the method of elimination.
Adding the first equation to the second, we get $2x = p + q$. Since the sum of an even integer and an odd integer is always an odd integer, the right side of the equation $2x = p + q$ is always odd. However, the left side of this equation is always even for any integer x.
Therefore, the given system of equations has no positive integer solutions.

(c) The left side of the equation, $x^2 - y^2$, is a difference of squares.
Factoring $x^2 - y^2$, then the equation $x^2 - y^2 = 420$ becomes $(x + y)(x - y) = 420$.
Since x and y are positive integers, then $x + y$ is a positive integer.
Since $(x + y)(x - y) = 420$ and $x + y$ is a positive integer, then $x - y$ is a positive integer.
Since x and y are positive integers, then $x + y > x - y$.
Thus, we are searching for pairs of positive integers whose product is equal to 420.
We list all of the possibilities below where $x + y > x - y$:

$x+y$	$x-y$	$(x+y)(x-y)$
420	1	420
210	2	420
140	3	420
105	4	420
84	5	420
70	6	420
60	7	420
42	10	420
35	12	420
30	14	420
28	15	420
21	20	420

Each of the pairs of factors listed above determines a system of equations.
For example the first pair, 420 and 1, gives the system of equations:

$$\begin{aligned} x + y &= 420 \\ x - y &= 1 \end{aligned}$$

From part (b), we know that for positive integer solutions (x, y) of this system of equations to exist, one of the factors cannot be odd if the other is even.
Thus, we may eliminate the pairs of factors that have different parity (one factor is odd and the other factor is even) from our table above.
The following possibilities remain:

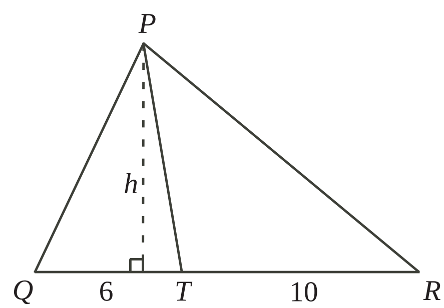
$x+y$	$x-y$	$(x+y)(x-y)$
210	2	420
70	6	420
42	10	420
30	14	420

We also know from part (b), that to determine x for each of the 4 systems of equations generated by the table above, we add the two factors and then divide the sum by 2.
The value of y is then determined by substituting x back into either equation.
We complete the solutions in the table below:

$x+y$	$x-y$	$(x+y)(x-y)$	$2x$	x	y
210	2	420	212	106	104
70	6	420	76	38	32
42	10	420	52	26	16
30	14	420	44	22	8

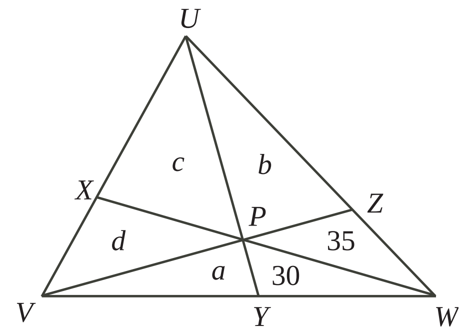
Therefore, the pairs of positive integers (x, y) that satisfy $x^2 - y^2 = 420$ are $(106, 104), (38, 32), (26, 16)$, and $(22, 8)$.

4. (a) Construct the altitude of $\triangle PQT$ from P to QT.
Let the length of the altitude be h.
Note that this altitude of $\triangle PQT$ is also the altitude of $\triangle PTR$.
The ratio of the area of $\triangle PQT$ to the area of $\triangle PTR$ is $\dfrac{\frac{1}{2} \times QT \times h}{\frac{1}{2} \times TR \times h} = \dfrac{QT}{TR} = \dfrac{6}{10} = \dfrac{3}{5}$.



(b) From part (a), we can generalize the fact that if two triangles have their bases along the same straight line and they share a common vertex that is not on this line, then the ratio of their areas is equal to the ratio of the lengths of their bases.
This generalization will be used throughout the solutions to parts (b) and (c).
We will also adopt the notation $|\triangle XYZ|$ to represent the area of $\triangle XYZ$.
Using the fact above, $\dfrac{|\triangle AEF|}{|\triangle DEF|} = \dfrac{AF}{FD} = \dfrac{3}{1}$. Thus, $|\triangle AEF| = 3 \times |\triangle DEF| = 3(17) = 51$.
Then, $|\triangle AED| = |\triangle AEF| + |\triangle DEF| = 51 + 17 = 68$.
Also, $\dfrac{|\triangle ECD|}{|\triangle AED|} = \dfrac{EC}{AE} = \dfrac{2}{1}$. Thus, $|\triangle ECD| = 2 \times |\triangle AED| = 2(68) = 136$.
Then, $|\triangle DCA| = |\triangle ECD| + |\triangle AED| = 136 + 68 = 204$.
Since D is the midpoint of BC, $\dfrac{BD}{DC} = \dfrac{1}{1}$, and $\dfrac{|\triangle BDA|}{|\triangle DCA|} = \dfrac{BD}{DC} = \dfrac{1}{1}$.
Then, $|\triangle BDA| = |\triangle DCA| = 204$ and $|\triangle ABC| = |\triangle BDA| + |\triangle DCA| = 204 + 204 = 408$.

(c) Let the area of $\triangle PYV$, $\triangle PZU$, $\triangle UXP$, and $\triangle XVP$, be a, b, c, and d, respectively.



Since $\dfrac{|\triangle PYV|}{|\triangle PYW|} = \dfrac{VY}{YW} = \dfrac{4}{3}$,
then $|\triangle PYV| = \frac{4}{3} \times |\triangle PYW| = \frac{4}{3}(30) = 40$.
Thus, $a = 40$.
Also, $\dfrac{|\triangle VZW|}{|\triangle VZU|} = \dfrac{ZW}{ZU} = \dfrac{|\triangle PZW|}{|\triangle PZU|}$ or $|\triangle VZW| \times |\triangle PZU| = |\triangle PZW| \times |\triangle VZU|$.

Thus, $\dfrac{|\triangle VZU|}{|\triangle PZU|} = \dfrac{|\triangle VZW|}{|\triangle PZW|} = \dfrac{35+30+40}{35} = \dfrac{105}{35} = \dfrac{3}{1}$.
Therefore, $\dfrac{|\triangle VZU|}{|\triangle PZU|} = \dfrac{3}{1}$, or $\dfrac{b+c+d}{b} = \dfrac{3}{1}$ or $b+c+d = 3b$ and $c+d = 2b$.
Next, $\dfrac{|\triangle UVY|}{|\triangle UYW|} = \dfrac{VY}{YW} = \dfrac{4}{3}$, so $\dfrac{40+c+d}{30+35+b} = \dfrac{4}{3}$.
Since $c+d = 2b$, we have $3(40+2b) = 4(65+b)$, so $120+6b = 260+4b$,
then $2b = 140$ and $b = 70$.
Next, $\dfrac{|\triangle UXW|}{|\triangle XVW|} = \dfrac{UX}{XV} = \dfrac{|\triangle UXP|}{|\triangle XVP|}$, or $\dfrac{35+b+c}{30+a+d} = \dfrac{c}{d}$.
Since $b = 70$ and $a = 40$, $\dfrac{105+c}{70+d} = \dfrac{c}{d}$, or $d(105+c) = c(70+d)$.
Thus, $105d + cd = 70c + cd$ or $105d = 70c$, and $\dfrac{d}{c} = \dfrac{70}{105} = \dfrac{2}{3}$ or $d = \dfrac{2}{3}c$.
Since $c+d = 2b = 2(70) = 140$, we have $c+d = c + \frac{2}{3}c = \frac{5}{3}c = 140$, or $c = \frac{3}{5}(140) = 84$.
Therefore, the area of $\triangle UXP$ is 84.