The CENTRE for EDUCATION in
MATHEMATICS and COMPUTING

2011 Hypatia Contest

Wednesday, April 13, 2011

Solutions

1. (a) Since D is the midpoint of AB, it has coordinates $(\frac{1}{2}(0+0), \frac{1}{2}(0+6)) = (0,3)$.
The line passing through C and D has slope $\frac{3-0}{0-8}$ or $-\frac{3}{8}$.
The y-intercept of this line is the y-coordinate of point D, or 3.
Therefore, the equation of the line passing through points C and D is $y = -\frac{3}{8}x + 3$.

(b) Since E is the midpoint of BC, it has coordinates $(\frac{1}{2}(8+0), \frac{1}{2}(0+0)) = (4,0)$.
Next, we find the equation of the line passing through the points A and E.
This line has slope $\frac{6-0}{0-4}$ or $-\frac{6}{4}$ or $-\frac{3}{2}$.
The y-intercept of this line is the y-coordinate of point A, or 6.
Therefore, the equation of the line passing through points A and E is $y = -\frac{3}{2}x + 6$.
Point F is the intersection point of the lines with equation $y = -\frac{3}{8}x+3$ and $y = -\frac{3}{2}x+6$.
To find the coordinates of point F we solve the system of equations by equating y:

$$\begin{aligned} -\tfrac{3}{8}x + 3 &= -\tfrac{3}{2}x + 6 \\ 8(-\tfrac{3}{8}x + 3) &= 8(-\tfrac{3}{2}x + 6) \\ -3x + 24 &= -12x + 48 \\ 9x &= 24 \end{aligned}$$

So the x-coordinate of point F is $\frac{24}{9}$ or $x = \frac{8}{3}$.
Substituting $x = \frac{8}{3}$ into $y = -\frac{3}{2}x + 6$, we find that $y = -\frac{3}{2} \times \frac{8}{3} + 6$ or $y = 2$.
The coordinates of point F are $(\frac{8}{3}, 2)$.

(c) Triangle DBC has base BC of length 8, and height BD of length 3.
Therefore, the area of $\triangle DBC$ is $\frac{1}{2} \times 8 \times 3$ or 12.

(d) The area of quadrilateral $DBEF$ is the area of $\triangle DBC$ minus the area of $\triangle FEC$.
Triangle FEC has base EC.
Note that $EC = BC - BE = 8 - 4$ or 4.
The height of $\triangle FEC$ is equal to the vertical distance from point F to the x-axis.
That is, the height of $\triangle FEC$ is equal to the y-coordinate of point F, or 2.
Therefore, the area of $\triangle FEC$ is $\frac{1}{2} \times 4 \times 2$ or 4.
Thus, the area of quadrilateral $DBEF$ is $12 - 4$ or 8.

2. (a) Since the tens digit to be used is 3, we must consider the number of possibilities for the ones digit.
Given that the digits 0 and 9 cannot be used, there are 8 choices remaining for the ones digit.
However if the ones digit is a 3, then the number formed, 33, is a multiple of 11 which is not permitted.
Since 33 is the only multiple of 11 with tens digit 3, each of the other 7 choices for the ones digit is possible.
There are 7 numbers in S whose tens digit is a 3.
In fact, the argument above can be repeated to show that there are 7 numbers in S for each of the possible tens digits 1 through to 8.
This will be useful information for part (d) to follow.

(b) Since the ones digit to be used is 8, we must consider the number of possibilities for the tens digit.
Given that the digits 0 and 9 cannot be used, there are 8 choices remaining for the tens digit.
However if the tens digit is 8, then the number formed, 88, is a multiple of 11 which is not

permitted.
Since 88 is the only multiple of 11 with ones digit 8, each of the other 7 choices for the tens digit is possible.
There are 7 numbers in S whose ones digit is 8.
In fact, the argument above can be repeated to show that there are 7 numbers in S for each of the possible ones digits 1 through to 8.
This will also be useful information for part (d) to follow.

(c) *Solution 1*
Ignoring the second restriction that no number in S be a multiple of 11, there are 8 choices for the ones digit and 8 choices for the tens digit.
For each of the 8 choices for the tens digit, there are 8 choices for the units digit ignoring multiples of 11.
Thus, there would be 8×8 or 64 numbers in S, ignoring the second restriction.
Included in these 64 numbers are the numbers $11, 22, 33, 44, 55, 66, 77, 88$, or 8 multiples of 11.
These are the only multiples of 11 in our 64 possibilities.
Removing these from the number of possibilities, there are $64 - 8$ or 56 numbers in S.

Solution 2
Given that the digits 0 and 9 cannot be used, there are 8 choices for the tens digit.
For each choice of a tens digit, choosing the ones digit to be equal to that tens digit gives the only number that is a multiple of 11.
That is, for each possible choice of a tens digit, the ones digit cannot equal the tens digit.
Since the ones digit cannot equal $0, 9$ or the tens digit, there are 7 possible choices of a ones digit for each choice of a tens digit.
Thus, there are $8 \times 7 = 56$ numbers in S.

(d) Our work in part (a) shows that for each of the possible tens digits, 1 through 8, there are 7 numbers in S that have that tens digit.
That is, of the 56 numbers in S there are 7 whose tens digit is 1, 7 whose tens digit is 2, and so on to include 7 whose tens digit is 8.
Similarly, our work in part (b) shows that for each of the possible ones digits, 1 through 8, there are 7 numbers in S that have that ones digit.
That is, of the 56 numbers in S there are 7 whose ones digit is 1, 7 whose ones digit is 2, and so on to include 7 whose ones digit is 8.
We may determine the sum of the 56 numbers in S by considering the sum of their tens digits separately from the sum of their ones digits.
First, consider the sum of the ones digits of all of the numbers in S.
Each of the numbers 1 through 8 appear 7 times in the ones digit.
The sum of the numbers from 1 to 8 is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \frac{(8)(9)}{2} = 36$.
Since each of these occur 7 times, then the sum of the ones digits for all numbers in S is 7×36 or 252.
Next, consider the sum of the tens digits of all of the numbers in S.
Again, each of the numbers 1 through 8 appear 7 times in the tens digit.
The sum of the numbers from 1 to 8 is 36.
Since each of these occur 7 times, the sum of the tens digits for all numbers in S is 7×36 or 252.
Since these are tens digits, they add 10×252 or 2520 to the total sum.
Thus, the sum of all of the numbers in S is the combined sum of all 56 ones digits, 252, and all 56 tens digits, which add 2520 to the sum, for a total of 2772.

3. (a) *Solution 1*

Since $3x = 5y$, then $y = \frac{3}{5}x$.
In the given Trenti-triple, the value of x is 50.
Thus, $y = \frac{3}{5}(50)$ or $y = 30$.
Since $3x = 2z$, then $z = \frac{3}{2}x$.
Since $x = 50$, then $z = \frac{3}{2}(50)$ or $z = 75$.
The Trenti-triple is $(50, 30, 75)$.

Solution 2

Let $3x = 5y = 2z = k$.
Since x, y, z are positive integers, then k is a positive integer that is divisible by 3, 5 and 2.
Any positive integer that is divisible by 3, 5 and 2 must be divisible by their least common multiple, which is $3 \times 5 \times 2$ or 30.
Since k is divisible by 30, then $k = 30m$ for some positive integer m.
That is, $3x = 5y = 2z = 30m$ and so $x = 10m$, $y = 6m$ and $z = 15m$.
Since $x = 50$, then $50 = 10m$ or $m = 5$.
Therefore, $y = 6 \times 5$ or $y = 30$ and $z = 15 \times 5$ or $z = 75$.
The Trenti-triple is $(50, 30, 75)$.

(b) *Solution 1*

Since $3x = 5y$, then $x = \frac{5}{3}y$.
Since x is a positive integer, then $\frac{5}{3}y$ is a positive integer and so y is divisible by 3, since 5 is not.
Since $5y = 2z$, then $z = \frac{5}{2}y$.
Since z is a positive integer, then $\frac{5}{2}y$ is a positive integer and so y is divisible by 2, since 5 is not.
Thus, y is divisible by both 2 and 3 and so y is divisible by the least common multiple of 2 and 3.
Therefore, in every Trenti-triple, y is divisible by 6.

Solution 2

From our work in part (a) Solution 2, it follows that since $y = 6m$ for some positive integer m, then y is divisible by 6 for every Trenti-triple.

(c) *Solution 1*

From part (b) Solution 1, we know that y is divisible by 6 for every Trenti-triple.
We can similarly show that x is divisible by 10 for every Trenti-triple.
Since $3x = 5y$, then $y = \frac{3}{5}x$.
Since y is a positive integer, then $\frac{3}{5}x$ is a positive integer and so x is divisible by 5, since 3 is not.
Since $3x = 2z$, then $z = \frac{3}{2}x$.
Since z is a positive integer, then $\frac{3}{2}x$ is a positive integer and so x is divisible by 2, since 3 is not.
Thus, x is divisible by both 5 and 2 and so x is divisible by the least common multiple of 5 and 2.
Therefore, in every Trenti-triple, x is divisible by 10.
We can similarly show that z is divisible by 15 for every Trenti-triple.
Since $3x = 2z$, then $x = \frac{2}{3}z$.
Since x is a positive integer, then $\frac{2}{3}z$ is a positive integer and so z is divisible by 3, since 2 is not.
Since $5y = 2z$, then $y = \frac{2}{5}z$.

Since y is a positive integer, then $\frac{2}{5}z$ is a positive integer and so z is divisible by 5, since 2 is not.
Thus, z is divisible by both 3 and 5 and so z is divisible by the least common multiple of 3 and 5.
Therefore, in every Trenti-triple, z is divisible by 15.
Since in every Trenti-triple, y is divisible by 6, x is divisible by 10, and z is divisible by 15, then their product xyz is divisible by $6 \times 10 \times 15$ or 900.

Solution 2
From our work in part (a) Solution 2, we have that $x = 10m$, $y = 6m$ and $z = 15m$ for some positive integer m.
Therefore, the product xyz is $(10m)(6m)(15m)$ or $900m^3$, and thus is divisible by 900 for every Trenti-triple.

4. (a) $F(8) = 6$ since

$$\begin{aligned} 8 &= 1+1+1+1+1+1+1+1 \\ &= 1+1+1+1+1+3 \\ &= 1+1+3+3 \\ &= 1+1+1+5 \\ &= 3+5 \\ &= 1+7 \end{aligned}$$

(b) Let us first begin by defining each of the ways that a positive integer n can be written as the sum of positive odd integers as a *representation* of n.
There are $F(n)$ representations of n.
To each possible representation of n, we may add a 1 to create a representation of $n+1$.
For example, $3+3+1$ is a representation of 7; adding a 1, $3+3+1+1$, creates a representation of 8.
Since every representation of n can be used to create a representation of $n+1$ in this way, then there are at least as many representations of $n+1$ as there are of n, so $F(n+1) \geq F(n)$.
Next, we will show that in fact $F(n+1) \geq F(n)+1$ by finding one additional representation of $n+1$ not described above.
We will do this by considering the cases when $n+1$ is odd and when $n+1$ is even.

<u>Case 1: $n+1$ is odd</u>
Since $n+1$ is odd, then $n+1$ is a representation of itself.
Since this representation of $n+1$ does not include a 1 ($n > 3$), then it must be a new representation not created by adding a 1 to a representation of n as described above.
Therefore, if $n+1$ is odd then $F(n+1) \geq F(n)+1$ and so $F(n+1) > F(n)$.

<u>Case 2: $n+1$ is even</u>
Since $n+1$ is even, then n is odd.
Since n is odd and $n > 3$, then $n \geq 5$.
Since n is odd and $n \geq 5$, then $n-2$ is odd and $n-2 \geq 3$.
Since $n+1 = (n-2)+3$ and $n-2 \geq 3$, then $(n-2)+3$ is a representation of $n+1$ that does not include a 1.
Since this representation of $n+1$ does not include a 1, then it must be a new representation not created by adding a 1 to a representation of n as described above.
Therefore, if $n+1$ is even then $F(n+1) \geq F(n)+1$ and so $F(n+1) > F(n)$.

Thus for all integers $n > 3$, $F(n+1) > F(n)$.

(c) Let a_n be the representation of n as the sum of n 1s.
As an example from part (a), $a_8 = 1+1+1+1+1+1+1+1$.
Let b_n be the representation of n as $(n-1)+1$ if n is even and as $(n-2)+1+1$ if n is odd.
Therefore, $b_8 = 7+1$ since 8 is even.
Let S_n be the list of the remaining representations of n.
From part (a), list S_8 consists of the following representations:

$$\begin{aligned} 8 &= 1+1+1+1+1+3 \\ &= 1+1+3+3 \\ &= 1+1+1+5 \\ &= 3+5 \end{aligned}$$

Since each of a_n and b_n are single representations of n, there are $F(n)-2$ representations in S_n.
Note that when $n=4$, S_n has no representations for n.
Consider the representations $a_n + S_n$ of $2n$.
These representations of $2n$ are n 1s added to each of the representations of S_n.
Again using our work from part (a) as an example, the representations of 16 given by $a_8 + S_8$ are:

$$\begin{aligned} a_8 + S_8 &= (1+1+1+1+1+1+1+1)+(1+1+1+1+1+3) \\ &= (1+1+1+1+1+1+1+1)+(1+1+3+3) \\ &= (1+1+1+1+1+1+1+1)+(1+1+1+5) \\ &= (1+1+1+1+1+1+1+1)+(3+5) \end{aligned}$$

In general, consider the following representations of $2n$:

- $a_n + S_n$ (there are $F(n)-2$ representations here and when $n=4$ there are none)
- $b_n + S_n$ (there are $F(n)-2$ representations here and when $n=4$ there are none)
- $a_n + a_n$
- $a_n + b_n$
- $b_n + b_n$
- $(2n-1)+1$
- $(2n-3)+3$

There are $2 \times [F(n)-2] + 5 = 2F(n)+1$ representations in this list.
If these are all distinct, then $F(2n) \geq 2F(n)+1 > 2F(n)$, as required.
Since $n > 3$, then $n-3 > 0$ or $2n-3 > n$ and thus $2n-1 > n$ also.
Since both $2n-3$ and $2n-1$ are greater than n, then there can be no overlap between the last two lists of representations and the first five lists of representations in the above list.
There is no overlap between any of the third, fourth or fifth lists of representations by the definitions of a_n and b_n.
Similarly, there can be no overlap between the first two lists of representations and the third, fourth and fifth lists of representations by the definitions of a_n, b_n and S_n.
This leaves us to consider the possibility of overlap between the first two lists of representations only.
Suppose that there is a representation of $2n$ that is included in both $a_n + S_n$ and in $b_n + S_n$.
Since this representation is included in $a_n + S_n$, then part of it looks like a_n, so the representation includes n 1s.

Since this representation is included in $b_n + S_n$, then part of it looks like b_n, so the representation includes either $n-1$ or $n-2$ depending on whether n is even or odd. Because the representation already includes some 1s (from the a_n portion), we cannot automatically include the 1 or $1+1$ from b_n.
So this representation includes n 1s and either $n-1$ or $n-2$.
These parts add to either $2n-1$ or $2n-2$.
The only way to complete the representation is then either with a 1 or with a $1+1$.
But this means that the representation then is n 1s plus $(n-1)+1$ if n is even or is n 1s plus $(n-2)+1+1$ if n is odd.
This means that this representation must be $a_n + b_n$.
Since neither a_n or b_n is in S_n, then this representation cannot actually be in $a_n + S_n$ or in $b_n + S_n$, so there cannot be any overlap between these two collections of representations.
Therefore, $F(2n) \geq 2F(n) + 1 > 2F(n)$, as required.