The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

2011 Pascal Contest

(Grade 9)

Thursday, February 24, 2011

Solutions

1. Calculating, $6 \times (5-2) + 4 = 6 \times 3 + 4 = 18 + 4 = 22$.

   Answer: (B)

2. Converting to a numerical expression, we obtain $943 - 87$ which equals 856.

   Answer: (E)

3. Since $2011^2 = 4044121$ and $\sqrt{2011} \approx 44.8$, then the list of numbers in increasing order is $\sqrt{2011}, 2011, 2011^2$.
   (If n is a positive integer with $n > 1$, then $n^2 > n$ and $\sqrt{n} < n$, so the list $\sqrt{n}, n, n^2$ is always in increasing order.)

   Answer: (C)

4. From the graph, the mass of fats is 32 g and the mass of carbohydrates is 48 g.
   Therefore, the ratio of the mass of fats to the mass of carbohydrates is $32 : 48$.
   Since each of 32 and 48 is divisible by 16, we can reduce the ratio by dividing both parts by 16 to obtain the simplified ratio $2 : 3$.

   Answer: (B)

5. When $x = -2$, we have $(x+1)^3 = (-2+1)^3 = (-1)^3 = -1$.

   Answer: (A)

6. After Peyton has added 15 L of oil, the new mixture contains $30 + 15 = 45$ L of oil and 15 L of vinegar.
   Thus, the total volume of the new mixture is $45 + 15 = 60$ L.
   Of this, the percentage that is oil is $\frac{45}{60} \times 100\% = \frac{3}{4} \times 100\% = 75\%$.

   Answer: (A)

7. When three $1 \times 1 \times 1$ cubes are joined together as in the diagram, the resulting prism is $3 \times 1 \times 1$.
   This prism has four rectangular faces that are 3×1 and two rectangular faces that are 1×1.
   Therefore, the surface area is $4 \times (3 \times 1) + 2 \times (1 \times 1) = 4 \times 3 + 2 \times 1 = 12 + 2 = 14$.

   Answer: (B)

8. Since the 17th day of the month is a Saturday and there are 7 days in a week, then the previous Saturday was the $17 - 7 = 10$th day of the month and the Saturday before that was the $10 - 7 = 3$rd day of the month.
   Since the 3rd day of the month was a Saturday, then the 2nd day was a Friday and the 1st day of the month was a Thursday.

   Answer: (D)

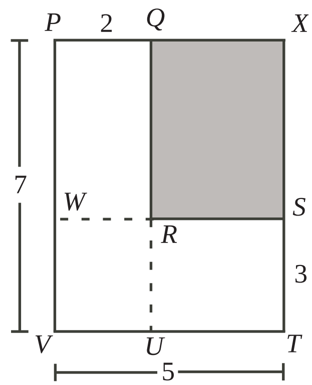
9. *Solution 1*
   Since $PQUV$ and $WSTV$ are rectangles that share a common right angle at V, then PQ, WS and VT are parallel, as are PV, QU, and ST. This tells us that all of the angles in the diagram are right angles.
   Since $PQUV$ is a rectangle, then $VU = PQ = 2$.
   Since $VT = 5$ and $VU = 2$, then $UT = VT - VU = 5 - 2 = 3$.
   Note that $RSTU$ is a rectangle, since it has four right angles.
   Therefore, the area of $PQRSTV$ equals the sum of the areas of rectangles $PQUV$ and $RSTU$, or $2 \times 7 + 3 \times 3 = 23$.

(We could also consider the area of $PQRSTV$ to be the sum of the areas of rectangle $PQRW$ and rectangle $WSTV$.)

Solution 2

Since $PQUV$ and $WSTV$ are rectangles that share a common right angle at V, then PQ, WS and VT are parallel, as are PV, QU, and ST. This tells us that all of the angles in the diagram are right angles.

We can consider $PQRSTV$ to be a large rectangle $PXTV$ with a smaller rectangle $QXSR$ removed.



The area of rectangle $PXTV$ is $7 \times 5 = 35$.
Since $PQUV$ is a rectangle, then $QU = PV = 7$.
Since PV is parallel to QU and ST, then $RU = ST = 3$.
Thus, $QR = QU - RU = 7 - 3 = 4$.
Since $WSTV$ is a rectangle, then $WS = VT = 5$.
Since VT is parallel to WS and PQ, then $WR = PQ = 2$.
Thus, $RS = WS - WR = 5 - 2 = 3$.
Therefore, rectangle $QXSR$ is 4 by 3, and so has area 12.
Therefore, the area of $PQRSTV$ is $35 - 12 = 23$.

Solution 3

Since $PQUV$ and $WSTV$ are rectangles that share a common right angle at V, then PQ, WS and VT are parallel, as are PV, QU, and ST. This tells us that all of the angles in the diagram are right angles.

If we add up the areas of rectangle $PQUV$ and $WSTV$, we get exactly the region $PQRSTV$, but have added the area of $WRUV$ twice. Thus, the area of $PQRSTV$ equals the area of $PQUV$ plus the area of $WSTV$ minus the area of $WRUV$.

We note that rectangle $PQUV$ is 2 by 7, rectangle $WSTV$ is 3 by 5, and rectangle $WRUV$ is 2 by 3 (since $WR = PQ = 2$ and $RU = ST = 3$).

Therefore, the area of $PQRSTV$ equals $2 \times 7 + 3 \times 5 - 2 \times 3 = 14 + 15 - 6 = 23$.

Answer: (E)

10. John first writes the integers from 1 to 20 in increasing order.
When he erases the first half of the numbers, he erases the numbers from 1 to 10 and rewrites these at the end of the original list.
Therefore, the number 1 has 10 numbers to its left. (These numbers are $11, 12, \ldots, 20$.)
Thus, the number 2 has 11 numbers to its left, and so the number 3 has 12 numbers to its left.
(We could write out the new list to verify this.)

Answer: (C)

11. When we convert each of the possible answers to a decimal, we obtain 1.1, 1.11, 1.101, 1.111, and 1.011.
Since the last of these is the only one greater than 1 and less than 1.1, it is closest to 1.

Answer: (E)

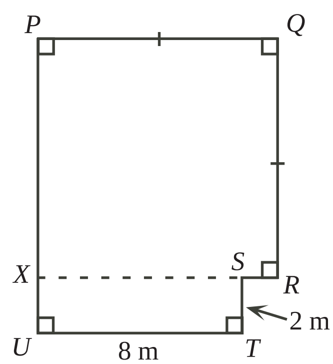
12. We note that $\frac{17}{4} = 4\frac{1}{4}$ and $\frac{35}{2} = 17\frac{1}{2}$.
Therefore, the integers between these two numbers are the integers from 5 to 17, inclusive.
The odd integers in this range are $5, 7, 9, 11, 13, 15$, and 17, of which there are 7.

Answer: (D)

13. The first four terms of the sequence are $1, 4, 2, 3$.
Since each term starting with the fifth is the sum of the previous four terms, then the fifth term is $1 + 4 + 2 + 3 = 10$.
Also, the sixth term is $4 + 2 + 3 + 10 = 19$, the seventh term is $2 + 3 + 10 + 19 = 34$, and the eighth term is $3 + 10 + 19 + 34 = 66$.

Answer: (A)

14. We extend the short horizontal side, RS, to the left until it reaches the long vertical side.



Since $PQRX$ has three right angles, then it must have a fourth right angle and so must be a rectangle.
Since $PQ = QR$, then $PQRX$ is in fact a square.
Since the exterior angle at S is a right angle, then $XSTU$ is also a rectangle.
Since $XSTU$ is 2 m by 8 m, then its area is $2 \times 8 = 16$ m^2.
Since the area of the whole garden is 97 m^2, then the area of $PQRX$ is $97 - 16 = 81$ m^2.
Since $PQRX$ is a square, then its side length is $\sqrt{81} = 9$ m.
Therefore, $PQ = QR = RX = XP = 9$ m.
Since $XSTU$ is a rectangle, then $XS = UT = 8$ m and $XU = ST = 2$ m.
Therefore, $PU = PX + XU = 9 + 2 = 11$ m and $SR = XR - XS = 9 - 8 = 1$ m.
Finally, we determine the perimeter of the garden by starting at P and proceeding clockwise.
The perimeter is $9 + 9 + 1 + 2 + 8 + 11 = 40$ m.

Answer: (C)

15. Since each of five friends paid an extra \$3 to cover Luxmi's portion of the bill, then Luxmi's share was $5 \times \$3 = \$15$.
Since each of the six friends had an equal share, then the total bill is $6 \times \$15 = \$90$.

Answer: (A)

16. The set S contains 25 multiples of 2 (that is, even numbers).
When these are removed, the set S is left with only the odd integers from 1 to 49.
At this point, there are $50 - 25 = 25$ integers in S.
We still need to remove the multiples of 3 from S.
Since S only contains odd integers at this point, then we must remove the odd multiples of 3 between 1 and 49.
These are 3, 9, 15, 21, 27, 33, 39, 45, of which there are 8.
Therefore, the number of integers remaining in the set S is $25 - 8 = 17$.

Answer: (D)

17. *Solution 1*
We work from right to left as we would if doing this calculation by hand.
In the units column, we have $L - 4$ giving 1. Thus, $L = 5$. (There is no borrowing required.)
This gives

$$\begin{array}{rcccc} & 6 & K & 0 & 5 \\ - & M & 9 & N & 4 \\ \hline & 2 & 0 & 1 & 1 \end{array}$$

In the tens column, we have $0 - N$ giving 1.
Since 1 is larger than 0, we must borrow from the hundreds column. Thus, $10 - N$ gives 1, which means $N = 9$. This gives

$$\begin{array}{rcccc} & & {\scriptstyle K-1} & {\scriptstyle 10} & \\ & 6 & \cancel{K} & \cancel{0} & 5 \\ - & M & 9 & 9 & 4 \\ \hline & 2 & 0 & 1 & 1 \end{array}$$

In the hundreds column, we have $K - 9$ but we have already borrowed 1 from K, so we have $(K - 1) - 9$ giving 0.
Therefore, we must be subtracting 9 from 9, which means that K should be 10, which is not possible.
We can conclude, though, that $K = 0$ and that we have borrowed from the 6. This gives

$$\begin{array}{rcccc} & {\scriptstyle 5} & & & \\ & \cancel{6} & 0 & 0 & 5 \\ - & M & 9 & 9 & 4 \\ \hline & 2 & 0 & 1 & 1 \end{array}$$

In the thousands column, we have $5 - M = 2$ or $M = 3$.
This gives $6005 - 3994 = 2011$, which is correct.
Finally, $K + L + M + N = 0 + 5 + 3 + 9 = 17$.

Solution 2
Since $6K0L - M9N4 = 2011$, then $M9N4 + 2011 = 6K0L$.
We start from the units column and work towards the left.
Considering the units column, the sum $4 + 1$ has a units digit of L. Thus, $L = 5$. (There is no carry to the tens column.) This gives

$$\begin{array}{cccc} & M & 9 & N & 4 \\ + & 2 & 0 & 1 & 1 \\ \hline & 6 & K & 0 & 5 \end{array}$$

Considering the tens column, the sum $N + 1$ has a units digit of 0. Thus, $N = 9$. (There is a carry of 1 to the hundreds column.) This gives

$$\begin{array}{ccccc} & & ^{1} & & \\ & M & 9 & 9 & 4 \\ + & 2 & 0 & 1 & 1 \\ \hline & 6 & K & 0 & 5 \end{array}$$

Considering the hundreds column, the sum $9 + 0$ plus the carry of 1 from the tens column has a units digit of K. Since $9 + 0 + 1 = 10$, then $K = 0$. There is a carry of 1 from the hundreds column to the thousands column. This gives

$$\begin{array}{ccccc} & ^{1} & & & \\ & M & 9 & 9 & 4 \\ + & 2 & 0 & 1 & 1 \\ \hline & 6 & 0 & 0 & 5 \end{array}$$

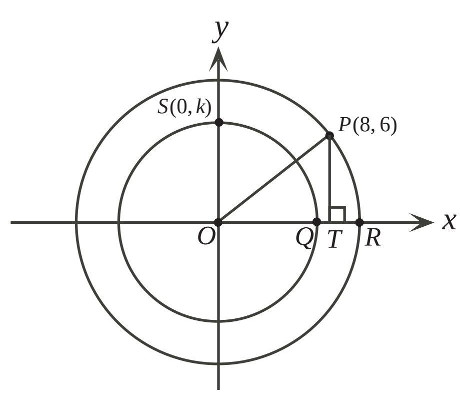
Considering the thousands column, the sum $M+2$ plus the carry of 1 from the hundreds column equals 6. Therefore, $M + 2 + 1 = 6$ or $M = 3$.
This gives $3994 + 2011 = 6005$ or $6005 - 3994 = 2011$, which is correct.
Finally, $K + L + M + N = 0 + 5 + 3 + 9 = 17$.

Answer: (A)

18. The difference between $\frac{1}{6}$ and $\frac{1}{12}$ is $\frac{1}{6} - \frac{1}{12} = \frac{2}{12} - \frac{1}{12} = \frac{1}{12}$, so $LP = \frac{1}{12}$.
Since LP is divided into three equal parts, then this distance is divided into three equal parts, each equal to $\frac{1}{12} \div 3 = \frac{1}{12} \times \frac{1}{3} = \frac{1}{36}$.
Therefore, M is located $\frac{1}{36}$ to the right of L.
Thus, the value at M is $\frac{1}{12} + \frac{1}{36} = \frac{3}{36} + \frac{1}{36} = \frac{4}{36} = \frac{1}{9}$.

Answer: (C)

19. We can determine the distance from O to P by dropping a perpendicular from P to T on the x-axis.



We have $OT = 8$ and $PT = 6$, so by the Pythagorean Theorem,

$$OP^2 = OT^2 + PT^2 = 8^2 + 6^2 = 64 + 36 = 100$$

Since $OP > 0$, then $OP = \sqrt{100} = 10$.
Therefore, the radius of the larger circle is 10.
Thus, $OR = 10$.
Since $QR = 3$, then $OQ = OR - QR = 10 - 3 = 7$.
Therefore, the radius of the smaller circle is 7.
Since S is on the positive y-axis and is 7 units from the origin, then the coordinates of S are $(0, 7)$, which means that $k = 7$.

Answer: (E)

20. *Solution 1*
Consider $\triangle UPV$.
Since $PU = PV$, then $\triangle UPV$ is isosceles, with

$$\angle PUV = \angle PVU = \tfrac{1}{2}(180^\circ - \angle UPV) = \tfrac{1}{2}(180^\circ - 24^\circ) = \tfrac{1}{2}(156^\circ) = 78^\circ$$

Since PVS is a straight line, then $\angle QVS = 180^\circ - \angle PVU = 180^\circ - 78^\circ = 102^\circ$.
Consider $\triangle QVS$.
The sum of the angles in this triangle is 180°, and so $102^\circ + x^\circ + y^\circ = 180^\circ$.
Therefore, $x + y = 180 - 102 = 78$.

Solution 2
Consider $\triangle UPV$.
Since $PU = PV$, then $\triangle UPV$ is isosceles, with

$$\angle PUV = \angle PVU = \tfrac{1}{2}(180^\circ - \angle UPV) = \tfrac{1}{2}(180^\circ - 24^\circ) = \tfrac{1}{2}(156^\circ) = 78^\circ$$

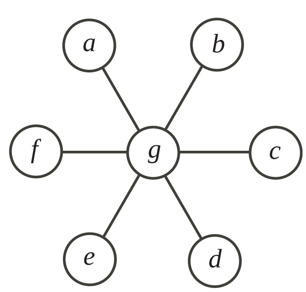
Since $\angle PVU$ is an exterior angle to $\triangle QVS$, then $\angle PVU = \angle VQS + \angle VSQ$.
Therefore, $78^\circ = y^\circ + x^\circ$ or $x + y = 78$.

Answer: (D)

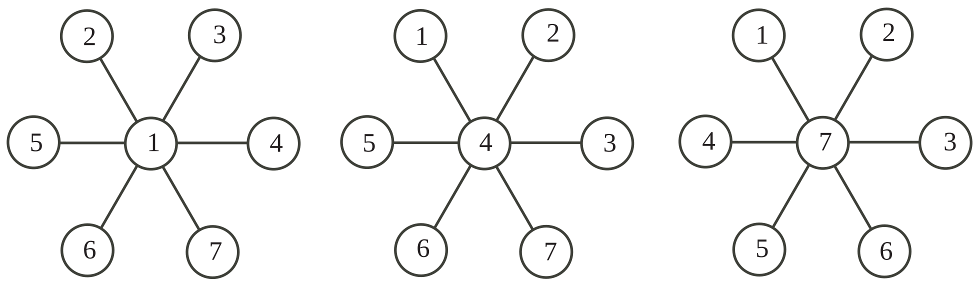
21. Since level C contains the same number of dots as level B and level D contains twice as many dots as level C, then level D contains twice as many dots as level B.
Similarly, level F contains twice as many dots as level D, level H contains twice as many dots as level F, and so on.
Put another way, the number of dots doubles from level B to level D, from level D to level F, from level F to level H, and so on.
Since there are 26 levels, then there are 24 levels after level B.
Thus, the number of dots doubles $24 \div 2 = 12$ times from level B to level Z.
Therefore, the number of dots on level Z is $2 \times 2^{12} = 2^{13} = 8192$.

Answer: (D)

22. We label the circles from a to g, as shown:



Let S be sum of the integers in any straight line.
Therefore, $S = a + g + d = b + g + e = c + g + f$.
Thus, $3S = (a + g + d) + (b + g + e) + (c + g + f) = a + b + c + d + e + f + 3g$.
Since the variables a to g are to be replaced by the integers from 1 to 7, in some order, then $a + b + c + d + e + f + g = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$.
Thus, $3S = (a + b + c + d + e + f + g) + 2g = 28 + 2g$, and so $3S = 28 + 2g$.
Since $3S$ is an integer divisible by 3, then $28 + 2g$ should also be divisible by 3.
Since g must be an integer between 1 and 7, we can try the seven possibilities and see that the only values of g for which $28 + 2g$ is divisible by 3 are 1, 4 and 7.
We must verify that we can actually complete the diagram for each of these values:



Therefore, there are 3 possibilities for the number in the centre circle.

Answer: (C)

23. First, we count the number of quadruples (p, q, r, s) of non-negative integer solutions to the equation $2p+q+r+s = 4$. Then, we determine which of these satisfies $p+q+r+s = 3$. This will allow us to calculate the desired probability.
Since each of p, q, r, and s is a non-negative integer and $2p + q + r + s = 4$, then there are three possible values for p: $p = 2$, $p = 1$, and $p = 0$.
Note that, in each case, $q + r + s = 4 - 2p$.

Case 1: $p = 2$
Here, $q + r + s = 4 - 2(2) = 0$.
Since each of q, r and s is non-negative, then $q = r = s = 0$, so $(p, q, r, s) = (2, 0, 0, 0)$.
There is 1 solution in this case.

Case 2: $p = 1$
Here, $q + r + s = 4 - 2(1) = 2$.
Since each of q, r and s is non-negative, then the three numbers q, r and s must be 0, 0 and 2 in some order, or 1, 1 and 0 in some order.
There are three ways to arrange a list of three numbers, two of which are the same. (With a, a, b, the arrangements are aab and aba and baa.)
Therefore, the possible quadruples here are

$$(p, q, r, s) = (1, 2, 0, 0), (1, 0, 2, 0), (1, 0, 0, 2), (1, 1, 1, 0), (1, 1, 0, 1), (1, 0, 1, 1)$$

There are 6 solutions in this case.

Case 3: $p = 0$
Here, $q + r + s = 4$.
We will look for non-negative integer solutions to this equation with $q \geq r \geq s$. Once we have found these solutions, all solutions can be found be re-arranging these initial solutions.

If $q = 4$, then $r + s = 0$, so $r = s = 0$.
If $q = 3$, then $r + s = 1$, so $r = 1$ and $s = 0$.
If $q = 2$, then $r + s = 2$, so $r = 2$ and $s = 0$, or $r = s = 1$.
The value of q cannot be 1 or 0, because if it was, then $r + s$ would be at least 3 and so r or s would be at least 2. (We are assuming that $r \leq q$ so this cannot be the case.)
Therefore, the solutions to $q + r + s = 4$ must be the three numbers 4, 0 and 0 in some order, 3, 1 and 0 in some order, 2, 2 and 0 in some order, or 2, 1 and 1 in some order.
In Case 2, we saw that there are three ways to arrange three numbers, two of which are equal. In addition, there are six ways to arrange a list of three different numbers. (With a, b, c, the arrangements are abc, acb, bac, bca, cab, cba.)
The solution $(p, q, r, s) = (0, 4, 0, 0)$ has 3 arrangements.
The solution $(p, q, r, s) = (0, 3, 1, 0)$ has 6 arrangements.
The solution $(p, q, r, s) = (0, 2, 2, 0)$ has 3 arrangements.
The solution $(p, q, r, s) = (0, 2, 1, 1)$ has 3 arrangements.
(In each of these cases, we know that $p = 0$ so the different arrangements come from switching q, r and s.)
There are 15 solutions in this case.

Overall, there are $1 + 6 + 15 = 22$ solutions to $2p + q + r + s = 4$.
We can go through each of these quadruples to check which satisfy $p + q + r + s = 3$.
The quadruples that satisfy this equation are exactly those from Case 2.
We could also note that $2p + q + r + s = 4$ and $p + q + r + s = 3$ means that

$$p = (2p + q + r + s) - (p + q + r + s) = 4 - 3 = 1$$

Therefore, of the 22 solutions to $2p + q + r + s = 4$, there are 6 that satisfy $p + q + r + s = 3$, so the desired probability is $\frac{6}{22} = \frac{3}{11}$.

ANSWER: (B)

24. The largest integer with exactly 100 digits is the integer that consists of 100 copies of the digit 9.
This integer is equal to $10^{100} - 1$.
Therefore, we want to determine the largest integer n for which $14n \leq 10^{100} - 1$.
This is the same as trying to determine the largest integer n for which $14n < 10^{100}$, since $14n$ is an integer.
We want to find the largest integer n for which $n < \dfrac{10^{100}}{14} = \dfrac{10}{14} \times 10^{99} = \dfrac{5}{7} \times 10^{99}$.
This is equivalent to calculating the number $\dfrac{5}{7} \times 10^{99}$ and rounding down to the nearest integer.
Put another way, this is the same as calculating $\dfrac{5}{7} \times 10^{99}$ and truncating the number at the decimal point.
The decimal expansion of $\dfrac{5}{7}$ is $0.\overline{714285}$. (We can see this either using a calculator or by doing long division.)
Therefore, the integer that we are looking for is the integer obtained by multiplying $0.\overline{714285}$ by 10^{99} and truncating at the decimal point.
In other words, we are looking for the integer obtained by shifting the decimal point in $0.\overline{714285}$ by 99 places to the right, and then ignoring everything after the new decimal point.
Since the digits in the decimal expansion repeat with period 6, then the integer consists of 16 copies of the digits 714285 followed by 714. (This has $16 \times 6 + 3 = 99$ digits.)
In other words, the integer looks like $714285\ 714285\ \cdots\ 714285\ 714$.

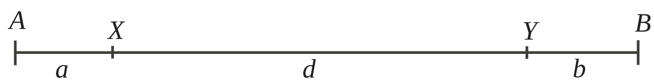
We must determine the digit that is the 68th digit from the right.
If we start listing groups from the right, we first have 714 (3 digits) followed by 11 copies of 714285 (66 more digits). This is 69 digits in total.
Therefore, the "7" that we have arrived at is the 69th digit from the right.
Moving one digit back towards the right tells us that the 68th digit from the right is 1.

ANSWER: (A)

25. First, we note that the three people are interchangeable in this problem, so it does not matter who rides and who walks at any given moment. We abbreviate the three people as D, M and P.
We call their starting point A and their ending point B.
Here is a strategy where all three people are moving at all times and all three arrive at B at the same time:

D and M get on the motorcycle while P walks.
D and M ride the motorcycle to a point Y before B.
D drops off M and rides back while P and M walk toward B.
D meets P at point X.
D picks up P and they drive back to B meeting M at B.
Point Y is chosen so that D, M and P arrive at B at the same time.

Suppose that the distance from A to X is a km, from X to Y is d km, and the distance from Y to B is b km.



In the time that it takes P to walk from A to X at 6 km/h, D rides from A to Y and back to X at 90 km/h.
The distance from A to X is a km.
The distance from A to Y and back to X is $a + d + d = a + 2d$ km.
Since the time taken by P and by D is equal, then $\dfrac{a}{6} = \dfrac{a+2d}{90}$ or $15a = a + 2d$ or $7a = d$.
In the time that it takes M to walk from Y to B at 6 km/h, D rides from Y to X and back to B at 90 km/h.
The distance from Y to B is b km, and the distance from Y to X and back to B is $d+d+b = b+2d$ km.
Since the time taken by M and by D is equal, then $\dfrac{b}{6} = \dfrac{b+2d}{90}$ or $15b = b + 2d$ or $7b = d$.
Therefore, $d = 7a = 7b$, and so we can write $d = 7a$ and $b = a$.
Thus, the total distance from A to B is $a + d + b = a + 7a + a = 9a$ km.
However, we know that this total distance is 135 km, so $9a = 135$ or $a = 15$.
Finally, D rides from A to Y to X to B, a total distance of $(a + 7a) + 7a + (7a + a) = 23a$ km.
Since $a = 15$ km and D rides at 90 km/h, then the total time taken for this strategy is $\frac{23\times 15}{90} = \frac{23}{6} \approx 3.83$ h.
Since we have a strategy that takes 3.83 h, then the smallest possible time is no more than 3.83 h. Can you explain why this is actually the smallest possible time?

If we didn't think of this strategy, another strategy that we might try would be:

D and M get on the motorcycle while P walks.
D and M ride the motorcycle to B.
D drops off M at B and rides back to meet P, who is still walking.
D picks up P and they drive back to B. (M rests at B.)

This strategy actually takes 4.125 h, which is longer than the strategy shown above, since M is actually sitting still for some of the time.

Answer: (A)