The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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2012 Fermat Contest

(Grade 11)

Thursday, February 23, 2012
(in North America and South America)

Friday, February 24, 2012
(outside of North America and South America)

Solutions

1. Since $\frac{60}{8} = 60 \div 8 = 7.5$, then this choice is not equal to a whole number.
Note as well that $\frac{60}{12} = 5$, $\frac{60}{5} = 12$, $\frac{60}{4} = 15$, and $\frac{60}{3} = 20$ are all whole numbers.

ANSWER: (B)

2. Simplifying the left side of the equation, we obtain $5 = 6 - x$.
Therefore, $x = 6 - 5 = 1$.

ANSWER: (C)

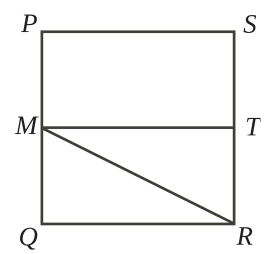
3. Since JFG is a straight line, then $\angle HFG = 180^\circ - \angle HFJ = 180^\circ - 110^\circ = 70^\circ$.
Since $\triangle FGH$ is isosceles with $HF = HG$, then $\angle HGF = \angle HFG = 70^\circ$.
Since the sum of the angles in $\triangle FGH$ is 180°, then $70^\circ + 70^\circ + x^\circ = 180^\circ$, and so $140 + x = 180$ or $x = 40$.

ANSWER: (E)

4. Simplifying inside the brackets first, $(1 + \frac{1}{3})(1 + \frac{1}{4}) = (\frac{4}{3})(\frac{5}{4}) = \frac{20}{12} = \frac{5}{3}$.

ANSWER: (A)

5. *Solution 1*
Draw a line from M to T on SR so that MT is parallel to QR.
Then $MTRQ$ is a rectangle. This means that the area of $\triangle MQR$ is half of the area of rectangle $MTRQ$.
Thus, the area of $MTRQ$ is $2 \times 100 = 200$.
Since M is the midpoint of PQ and $PQRS$ is a square, then T is the midpoint of SR.
This means that the area of $MTRQ$ is half of the area of $PQRS$.
Therefore, the area of $PQRS$ is $2 \times 200 = 400$.



Solution 2
Suppose that the side length of square $PQRS$ is $2x$.
Since M is the midpoint of PQ, then $MQ = \frac{1}{2}(2x) = x$.
Since $PQRS$ is a square, then $\triangle MQR$ is right-angled at Q.
Therefore, the area of $\triangle MQR$ is $\frac{1}{2}(MQ)(QR) = \frac{1}{2}(x)(2x) = x^2$.
Since the area of $\triangle MQR$ is 100, then $x^2 = 100$, and so $x = 10$, since $x > 0$.
Thus, the side length of square $PQRS$ is $2x = 20$ and so the area of square $PQRS$ is $20^2 = 400$.

ANSWER: (D)

6. Suppose that John ate x peanuts on the fourth night.
Since he ate 6 more peanuts each night than on the previous night, then he ate $x - 6$ peanuts on the third night, $(x - 6) - 6 = x - 12$ peanuts on the second night, and $(x - 12) - 6 = x - 18$ peanuts on the first night.
Since John ate 120 peanuts in total, then $x + (x - 6) + (x - 12) + (x - 18) = 120$, and so $4x - 36 = 120$ or $4x = 156$ or $x = 39$.
Therefore, John ate 39 peanuts on the fourth night.

ANSWER: (B)

7. Suppose that the side length of each of the five identical squares is x.
Then $PS = QR = x$ and $PQ = SR = 5x$.
Since the perimeter of rectangle $PQRS$ is 48, then $5x + x + 5x + x = 48$ or $12x = 48$ or $x = 4$.
Therefore, $PS = QR = 4$ and $PQ = SR = 5 \cdot 4 = 20$, and so the area of rectangle $PQRS$ is $20 \cdot 4 = 80$.

ANSWER: (C)

8. Since $v = 3x$ and $x = 2$, then $v = 3 \cdot 2 = 6$.
Therefore, $(2v - 5) - (2x - 5) = (2 \cdot 6 - 5) - (2 \cdot 2 - 5) = 7 - (-1) = 8$.

Answer: (B)

9. Suppose that Sally's original height was s cm.
Since Sally grew 20% taller, her new height is $1.2s$ cm.
Since Sally is now 180 cm tall, then $1.2s = 180$ or $s = \frac{180}{1.2} = 150$.
Thus, Sally grew $180 - 150 = 30$ cm.
Since Mary grew half as many centimetres as Sally grew, then Mary grew $\frac{1}{2} \cdot 30 = 15$ cm.
Since Mary and Sally were originally the same height, then Mary was originally 150 cm tall, and so is now $150 + 15 = 165$ cm tall.

Answer: (B)

10. Since $(2^a)(2^b) = 64$, then $2^{a+b} = 64$, using an exponent law.
Since $64 = 2^6$, then $2^{a+b} = 2^6$ and so $a + b = 6$.
Therefore, the average of a and b is $\frac{1}{2}(a + b) = 3$.

Answer: (D)

11. If N is divisible by both 5 and 11, then N is divisible by $5 \times 11 = 55$.
This is because 5 and 11 have no common divisor larger than 1.
Therefore, we are looking for a multiple of 55 between 400 and 600 that is odd.
One way to find such a multiple is to start with a known multiple of 55, such as 550.
We can add or subtract 55 from this multiple and still obtain multiples of 55.
Note that $550 + 55 = 605$, which is too large.
Now $550 - 55 = 495$ which is in the correct range and is odd.
Since we are told that there is only such such integer, then it must be the case that $N = 495$.
The sum of the digits of N is $4 + 9 + 5 = 18$.

Answer: (E)

12. Since $\triangle QUR$ and $\triangle SUR$ are equilateral, then $\angle QUR = \angle SUR = 60^\circ$.
Since $QU = PU = TU = SU$ and $QP = PT = TS$, then $\triangle QUP$, $\triangle PUT$ and $\triangle TUS$ are congruent.
Thus, $\angle QUP = \angle PUT = \angle TUS$.
The angles around point U add to 360°.
Thus, $\angle SUR + \angle QUR + \angle QUP + \angle PUT + \angle TUS = 360^\circ$ and so $60^\circ + 60^\circ + 3\angle TUS = 360^\circ$ or $3\angle TUS = 240^\circ$ or $\angle TUS = 80^\circ$.
Since $\triangle TUS$ is isosceles with $TU = SU$, then $\angle UST = \angle UTS$.
Since the angles in $\triangle TUS$ add to 180°, then $\angle TUS + \angle UST + \angle UTS = 180^\circ$.
Therefore, $80^\circ + 2\angle UST = 180^\circ$ and so $2\angle UST = 100^\circ$ or $\angle UST = 50^\circ$.

Answer: (A)

13. The quilt consists of 25 identical squares.
Of the 25 squares, 4 are entirely shaded, 8 are shaded with a single triangle that covers half of the square, and 4 are shaded with two triangles that each cover a quarter of the square.
Therefore, the shading is equivalent to the area of $4 + 8 \times \frac{1}{2} + 4 \times 2 \times \frac{1}{4} = 10$ squares.
As a percentage, the shading is $\frac{10}{25} \times 100\% = 40\%$ of the total area of the quilt.

Answer: (B)

14. *Solution 1*

Since the two terms have a common factor, then we factor and obtain $(x-2)((x-4)+(x-6)) = 0$.
This gives $(x-2)(2x-10) = 0$.
Therefore, $x - 2 = 0$ (which gives $x = 2$) or $2x - 10 = 0$ (which gives $x = 5$).
Therefore, the two roots of the equation are $x = 2$ and $x = 5$. Their product is 10.

Solution 2

We expand and then simplify the left side:

$$\begin{aligned}(x-4)(x-2)+(x-2)(x-6) &= 0\\ (x^2-6x+8)+(x^2-8x+12) &= 0\\ 2x^2-14x+20 &= 0\end{aligned}$$

Since the product of the roots of a quadratic equation of the form $ax^2 + bx + c = 0$ with $a \neq 0$ is $\dfrac{c}{a}$, then the product of the roots of the equation $2x^2 - 14x + 20 = 0$ is $\dfrac{20}{2} = 10$.

Answer: (C)

15. Because of the way in which the oranges are stacked, each layer is a rectangle whose length is 1 orange less and whose width is 1 orange less than the layer below.
The bottom layer is 5 by 7 and so contains 35 oranges.
The next layer is 4 by 6 and so contains 24 oranges.
The next layer is 3 by 5 and so contains 15 oranges.
The next layer is 2 by 4 and so contains 8 oranges.
The next layer is 1 by 3 and so contains 3 oranges. This is the last layer, as it consists of a single row of oranges.
The total number of oranges in the stack is thus $35 + 24 + 15 + 8 + 3 = 85$.

Answer: (D)

16. Since there are 30 people in a room and 60% of them are men, then there are $\frac{6}{10} \times 30 = 18$ men in the room and 12 women.
Since no men enter or leave the room, then these 18 men represent 40% of the final number in the room.
Thus, 9 men represent 20% of the the final number in the room, and so the final number of people is $5 \times 9 = 45$.
Since 18 of these are men and 12 of these are the women originally in the room, then $45 - 18 - 12 = 15$ women entered the room.

Answer: (E)

17. Since $3^{2011} = 3^1 \cdot 3^{2010} = 3 \cdot 3^{2010}$ and $3^{2012} = 3^2 \cdot 3^{2010} = 9 \cdot 3^{2010}$, then

$$\frac{3^{2011}+3^{2011}}{3^{2010}+3^{2012}} = \frac{3\cdot 3^{2010}+3\cdot 3^{2010}}{3^{2010}+9\cdot 3^{2010}} = \frac{3^{2010}(3+3)}{3^{2010}(1+9)} = \frac{3+3}{1+9} = \frac{6}{10} = \frac{3}{5}$$

Answer: (A)

18. In order to find N, which is the smallest possible integer whose digits have a fixed product, we must first find the minimum possible number of digits with this product. (This is because if the integer a has more digits than the integer b, then $a > b$.)
Once we have determined the digits that form N, then the integer N itself is formed by writing the digits in increasing order. (Given a fixed set of digits, the leading digit of N will contribute

to the largest place value, and so should be the smallest digit. The next largest place value should get the next smallest digit, and so on.)
Note that the digits of N cannot include 0, else the product of its digits would be 0.
Also, the digits of N cannot include 1, otherwise we could remove the 1s and obtain an integer with fewer digits (thus, a smaller integer) with the same product of digits.

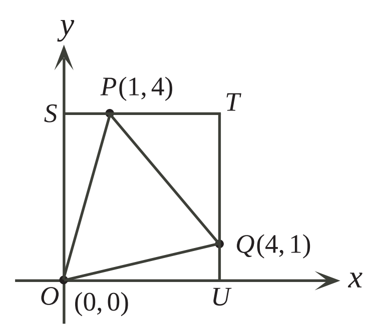
Since the product of the digits of N is 1728, we find the prime factorization of 1728 to help us determine what the digits are:

$$1728 = 9 \times 192 = 3^2 \times 3 \times 64 = 3^3 \times 2^6$$

We must try to find a combination of the smallest number of possible digits whose product is 1728.
Note that we cannot have 3 digits with a product of 1728 since the maximum possible product of 3 digits is $9 \times 9 \times 9 = 729$.
Let us suppose that we can have 4 digits with a product of 1728.
In order for N to be as small as possible, its leading digit (that is, its thousands digit) must be as small as possible.
From above, this digit cannot be 1.
This digit also cannot be 2, since otherwise the product of the remaining 3 digits would be 864, which is larger than the product of 3 digits can be.
Can the thousands digit be 3? If so, the remaining 3 digits have a product of 576.
Can 3 digits have a product of 576?
If one of these 3 digits were 7 or less, then the product of the 3 digits would be at most $7 \times 9 \times 9 = 567$, which is too small.
Therefore, if we have 3 digits with a product of 576, then each digit is 8 or 9.
Since the product is even, then at least one of the digits would have to be 8, leaving the remaining two digits to have a product of $576 \div 8 = 72$.
These two digits would then have to be 8 and 9.
Thus, we can have 3 digits with a product of 576, and so we can have 4 digits with a product of 1728 with smallest digit 3.
Therefore, the digits of N must be 3, 8, 8, 9. The smallest possible number formed by these digits is when the digits are placed in increasing order, and so $N = 3889$.
The sum of the digits of N is $3 + 8 + 8 + 9 = 28$.

Answer: (A)

19. We label the three points as $O(0,0)$, $P(1,4)$ and $Q(4,1)$.
There are three possible locations for the fourth vertex R of the parallelogram – between O and P (in the second quadrant), between P and Q (in the first quadrant), and between Q and O (in the fourth quadrant).
In each of these cases, $\triangle OPQ$ will make up half of the parallelogram, and so the area of the parallelogram is twice the area of $\triangle OPQ$.
There are many ways to calculate the area of $\triangle OPQ$.
We proceed by "completing the rectangle" which includes the x-axis, the y-axis, the line $y = 4$, and the line $x = 4$.
We label the point $(0,4)$ as S, the point $(4,4)$ as T, and the point $(4,0)$ as U.
(Note that rectangle $OSTU$ is in fact a square, so we have "completed the square"!)
The area of $\triangle OPQ$ equals the area of rectangle $OSTU$ minus the combined areas of $\triangle OSP$, $\triangle PTQ$, and $\triangle QUO$.



The area of rectangle $OSTU$ is $4 \cdot 4 = 16$, since it is a square with side length 4.
Consider $\triangle OSP$. It is right-angled at S, with $OS = 4$ and $SP = 1$.
Thus, its area is $\frac{1}{2}(OS)(SP) = \frac{1}{2}(4)(1) = 2$.
Consider $\triangle PTQ$. It is right-angled at T, with $PT = TQ = 3$.
Thus, its area is $\frac{1}{2}(PT)(TQ) = \frac{1}{2}(3)(3) = \frac{9}{2}$.
Consider $\triangle QUO$. It is right-angled at U, with $OU = 4$ and $UQ = 1$.
Thus, its area is $\frac{1}{2}(OU)(UQ) = \frac{1}{2}(4)(1) = 2$.
Therefore, the area of $\triangle OPQ$ is $16 - 2 - \frac{9}{2} - 2 = \frac{15}{2}$.
Thus, the area of the parallelogram is $2 \cdot \frac{15}{2} = 15$.

Answer: (A)

20. In the first race, Katie ran 100 m in the same time that Sarah ran 95 m.
This means that the ratio of their speeds is $100 : 95 = 20 : 19$.
In other words, in the time that Sarah runs 1 m, Katie runs $\frac{20}{19} \approx 1.053$ m.
Put another way, in the time that Katie runs 1 m, Sarah runs $\frac{19}{20} = 0.95$ m.
In the second race, Katie must run 105 m and Sarah must run 100 m.
If Sarah finishes first, then Katie must not have completed 105 m in the time that it takes Sarah to complete 100 m.
But Katie runs 1.053 m for 1 m that Sarah runs, so Katie will in fact run more than 105 m in the time that Sarah runs 100 m.
Therefore, Katie must finish first.
In the time that Katie runs 105 m, Sarah will run $105 \times \frac{19}{20} = \frac{1995}{20} = \frac{399}{4} = 99\frac{3}{4}$.
Thus, Sarah was $100 - 99\frac{3}{4} = \frac{1}{4} = 0.25$ m behind.
Therefore, when Katie crossed the finish line, Sarah was 0.25 m behind.

Answer: (B)

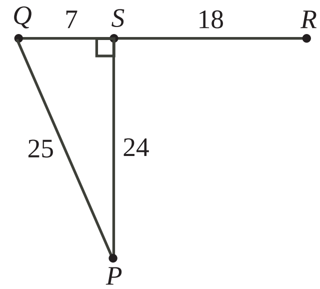
21. Since $x^2 = 8x + y$ and $y^2 = x + 8y$, then $x^2 - y^2 = (8x + y) - (x + 8y) = 7x - 7y$.
Factoring both sides, we obtain $(x + y)(x - y) = 7(x - y)$.
Since $x \neq y$, then $x - y \neq 0$, so we can divide both sides by $x - y$ to obtain $x + y = 7$.
Since $x^2 = 8x + y$ and $y^2 = x + 8y$, then

$$x^2 + y^2 = (8x + y) + (x + 8y) = 9x + 9y = 9(x + y) = 9 \cdot 7 = 63$$

Answer: (C)

22. From the chart, we see that $QR = 25$, $QS = 7$ and $SR = 18$.
Since $QR = QS + SR$ and QR is the largest of these three lengths, then S must be a point on line segment QR.
This gives the following configuration so far:

We have not yet used the fact that $PQ = 25$ or that $PS = 24$.
Note that $7^2 + 24^2 = 49 + 576 = 625 = 25^2$, so $QS^2 + PS^2 = PQ^2$.
Since these lengths satisfy this property, then the points P, S and Q form a triangle that is right-angled at S.
This gives the following configuration so far:



(We could have drawn P "above" QR.)
Since $\angle PSQ = 90^\circ$, then $\angle PSR = 90^\circ$.
Therefore, $PR^2 = PS^2 + SR^2 = 24^2 + 18^2 = 576 + 324 = 900$.
Since $PR > 0$, then $PR = \sqrt{900} = 30$.
Thus, the distance between cities P and R is 30.

ANSWER: (A)

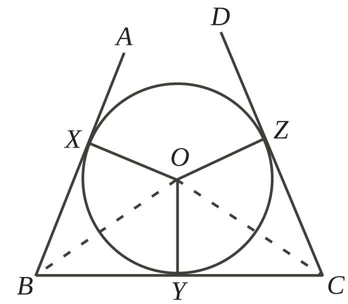
23. Initially, the bowl contains 320 g of white sugar and 0 g of brown sugar.
Mixture Y contains $(320 - x)$ g of white sugar and x g of brown sugar.
When Mixture Z (the final mixture) is formed, there is still 320 g of sugar in the bowl.
Since we are told that the ratio of the mass of white sugar to the mass of brown sugar is 49 : 15, then the mass of white sugar in Mixture Z is $\frac{49}{49+15} \cdot 320 = \frac{49}{64} \cdot 320 = 49 \cdot 5 = 245$ g and the mass of brown sugar in Mixture Z is $320 - 245 = 75$ g.
In order to determine the value of x (and hence determine the values of w and b), we need to determine the mass of each kind of sugar in Mixture Z in terms of x.
Recall that Mixture Y consists of $(320 - x)$ g of white sugar and x g of brown sugar, which are thoroughly mixed together.
Because Mixture Y is thoroughly mixed, then each gram of Mixture Y consists of $\dfrac{320 - x}{320}$ g of white sugar and $\dfrac{x}{320}$ g of brown sugar.
To form Mixture Z, x g of Mixture Y are removed.
This amount of Mixture Y that is removed contains $x \cdot \dfrac{x}{320} = \dfrac{x^2}{320}$ g of brown sugar.
Mixture Z is made by removing x g of Mixture Y (which contains $\dfrac{x^2}{320}$ g of brown sugar),then adding x g of brown sugar.
Thus the mass of brown sugar, in g, in Mixture Z is $x - \dfrac{x^2}{320} + x$.
Since Mixture Z includes 75 g of brown sugar, then

$$\begin{aligned} 2x - \frac{x^2}{320} &= 75 \\ 0 &= x^2 - 2(320)x + 75(320) \\ 0 &= x^2 - 640x + 24000 \\ 0 &= (x - 40)(x - 600) \end{aligned}$$

Therefore, $x = 40$ or $x = 600$.
Since the initial mixture consists of 320 g of sugar, then $x < 320$, so $x = 40$.
This tells us that Mixture Y consists of $320 - 40 = 280$ g of white sugar and 40 g of brown sugar. The ratio of these masses is 280 : 40, which equals 7 : 1 in lowest terms. Thus, $w = 7$ and $b = 1$.
Therefore, $x + w + b = 40 + 7 + 1 = 48$.

ANSWER: (A)

24. We use without proof the fact that if a circle with centre O and radius r touches (that is, is tangent to) line segments AB, BC and CD at X, Y and Z, respectively, then $\angle OBX = \angle OBY = \frac{1}{2}(\angle ABC)$ and $\angle OCY = \angle OCZ = \frac{1}{2}(\angle BCD)$.



Suppose that $\angle ABC = \theta$ and $\angle BCD = \alpha$.

Since OY is perpendicular to BC, then $\tan(\angle OBY) = \dfrac{OY}{BY}$ and $\tan(\angle OCY) = \dfrac{OY}{YC}$.

Thus, $BY = \dfrac{OY}{\tan(\angle OBY)} = \dfrac{r}{\tan(\theta/2)}$ and $YC = \dfrac{OY}{\tan(\angle OCY)} = \dfrac{r}{\tan(\alpha/2)}$.

Since $BC = BY + YC$, then

$$\begin{aligned} BC &= \frac{r}{\tan(\theta/2)} + \frac{r}{\tan(\alpha/2)} \\ BC &= r\left(\frac{1}{\tan(\theta/2)} + \frac{1}{\tan(\alpha/2)}\right) \\ BC &= r\left(\frac{\tan(\theta/2)+\tan(\alpha/2)}{\tan(\theta/2)\tan(\alpha/2)}\right) \\ r &= \frac{BC\tan(\theta/2)\tan(\alpha/2)}{\tan(\theta/2)+\tan(\alpha/2)} \end{aligned}$$

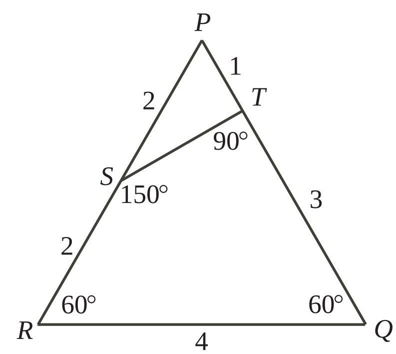
Consider now the given quadrilateral $QRST$. We know that $TQ = 3$. Since $\triangle PQR$ is equilateral, then $QR = PQ$ and so $QR = PT + TQ = 1 + 3 = 4$.

Since $PR = QR = 4$ and S is the midpoint of PR, then $RS = 2$.

Since $\triangle PST$ has $PT = 1$ and $PS = \frac{1}{2}PR = 2$ and $\angle SPT = 60^\circ$, then it is a 30°-60°-90° triangle, so we have $ST = \sqrt{3}$.

Also, we have $\angle PTS = 90^\circ$ and $\angle PST = 30^\circ$, which give $\angle STQ = 90^\circ$ and $\angle RST = 150^\circ$.

To summarize, quadrilateral $QRST$ has $QR = 4$, $RS = 2$, $ST = \sqrt{3}$, $TQ = 3$, $\angle TQR = 60^\circ$, $\angle QRS = 60^\circ$, $\angle RST = 150^\circ$, and $\angle STQ = 90^\circ$.



Next, we determine the radius of a circle tangent to each set of three consecutive sides, ignoring the fact that the fourth side might restrict the size of the circle. After determining these radii, we examine how the fourth side comes into play.

Suppose that we have a circle tangent to line segments TQ, QR and RS.

By the formula above, the radius of this circle would equal

$$\frac{4\tan(60^\circ/2)\tan(60^\circ/2)}{\tan(60^\circ/2)+\tan(60^\circ/2)} = \frac{4\tan(30^\circ)\tan(30^\circ)}{\tan(30^\circ)+\tan(30^\circ)} \approx 1.1547$$

Suppose that we have a circle tangent to line segments QR, RS and ST.

By the formula above, the radius of this circle would equal

$$\frac{2\tan(60^\circ/2)\tan(150^\circ/2)}{\tan(60^\circ/2)+\tan(150^\circ/2)} = \frac{2\tan(30^\circ)\tan(75^\circ)}{\tan(30^\circ)+\tan(75^\circ)} = 1$$

Suppose that we have a circle tangent to line segments RS, ST and TQ.
By the formula above, the radius of this circle would equal

$$\frac{\sqrt{3}\tan(150^\circ/2)\tan(90^\circ/2)}{\tan(150^\circ/2)+\tan(90^\circ/2)} = \frac{\sqrt{3}\tan(75^\circ)\tan(45^\circ)}{\tan(75^\circ)+\tan(45^\circ)} \approx 1.3660$$

Suppose that we have a circle tangent to line segments ST, TQ and QR.
By the formula above, the radius of this circle would equal

$$\frac{3\tan(90^\circ/2)\tan(60^\circ/2)}{\tan(90^\circ/2)+\tan(60^\circ/2)} = \frac{3\tan(45^\circ)\tan(30^\circ)}{\tan(45^\circ)+\tan(30^\circ)} \approx 1.0981$$

We need to determine the radius of the largest circle that can be drawn inside quadrilateral $QRST$.
The largest such circle will be touching at least two adjacent sides of $QRST$. Why is this? If a circle were touching zero sides or one side of $QRST$, it could be slid until it was touching two consecutive sides and then expanded a little bit and so cannot be the largest such circle. If it were touching two opposite sides of $QRST$, then it could be slid to touch a side adjacent to one of these two sides, perhaps losing contact with one of the two opposite sides.
Consider now a circle touching two adjacent sides of $QRST$. Such a circle can be expanded while maintaining contact to these two sides until it touches a third side. Once it touches the third side, it can't be expanded any further because its radius is fixed by the calculation that we did at the beginning of the solution.
Therefore, the largest circle will be touching three of the sides of $QRST$.

In order to complete the solution, we need to determine which of the circles that touch three sides actually lie completely inside $QRST$.
We do this by examining each of the four pairs of consecutive sides, and determining what the largest circle is that can be drawn touching these sides.

- We consider a circle tangent to ST and TQ. We expand the circle, keeping it tangent to ST and TQ, until it touches either QR or RS. From the previous calculations, the circle that also touches QR has radius about 1.0981, and the circle that also touches RS has radius about 1.3660. Thus, the circle will first touch QR. In this case, the largest circle that is completely inside the quadrilateral has radius 1.0981.
- We consider a circle tangent to TQ and QR. We expand the circle, keeping it tangent to TQ and QR, until it touches either ST or RS. From the previous calculations, the circle that also touches ST has radius about 1.0981, and the circle that also touches RS has radius about 1.1547. Thus, the circle will first touch ST. In this case, the largest circle that is completely inside the quadrilateral has radius 1.0981.
- We consider a circle tangent to QR and RS. We expand the circle, keeping it tangent to QR and RS, until it touches either ST or TQ. From the previous calculations, the circle that also touches ST has radius 1, and the circle that also touches TQ has radius about 1.1547. Thus, the circle will first touch ST. In this case, the largest circle that is completely inside the quadrilateral has radius 1.
- We consider a circle tangent to RS and ST. We expand the circle, keeping it tangent to RS and ST, until it touches either QR or TQ. From the previous calculations, the circle that also touches QR has radius about 1, and the circle that also touches TQ has radius about 1.3660. Thus, the circle will first touch QR. In this case, the largest circle that is completely inside the quadrilateral has radius 1.

Finally, comparing the four cases, we see that the largest circle that we can obtain has radius about 1.0981, which is closest to 1.10.

Answer: (B)

25. Let N be an arbitrary positive integer with the desired properties.
Let $S(N)$ represent the sum of the digits of N and let $S(2N)$ represent the sum of the digits of $2N$. In the table below, we make a claim about how each digit of N contributes to $S(2N)$. We use the data in the table to answer the question, following which we justify the data in the table:

Digit in N	$2\times$Digit	Contribution to $S(2N)$
3	6	6
4	8	8
5	10	$1+0=1$
6	12	$1+2=3$

Suppose that the digits of N include w 3s, x 4s, y 5s, and z 6s. Note that $w,x,y,z\geq 1$.

Step 1: Using information about $S(N)$ and $S(2N)$

Since $S(N)=900$, then $3w+4x+5y+6z=900$.
Since each 3 in N contributes 6 to $S(2N)$, each 4 in N contributes 8 to $S(2N)$, each 5 in N contributes 1 to $S(2N)$, and each 6 in N contributes 3 to $S(2N)$, then $S(2N)=900$ tells us that $6w+8x+y+3z=900$.

Step 2: Understanding which values of N will be largest possible and smallest possible

The largest possible value of N will be the integer N^+ that satisfies the given properties, has the largest number of digits (that is, the largest value of $w+x+y+z$), has the largest actual digits given this fixed number of digits, and has its digits in decreasing order from left to right (since the larger digits will correspond to the largest place values).
The smallest possible value of N will be the integer N^- that satisfies the given properties, has the smallest number of digits (that is, the smallest value of $w+x+y+z$), has the smallest actual digits given this fixed number of digits, and has its digits in increasing order from left to right.
Since we want to determine the number of digits in the product N^+N^-, we mostly care only about the number of digits in N^+ and N^- (that is, the maximum and minimum values of $w+x+y+z$), and perhaps as well about the leading digits of each.

Step 3: Simplifying equations

We have that w,x,y,z are positive integers that satisfy both $3w+4x+5y+6z=900$ and $6w+8x+y+3z=900$.
Multiplying the first of these equations by 2, we obtain $6w+8x+10y+12z=1800$.
When we subtract the second equation from this, we obtain $9y+9z=900$, or $y+z=100$.
Since $y+z=100$, then $3w+4x+5y+6z=900$ becomes $3w+4x+5(y+z)+z=900$ or $3w+4x+500+z=900$ or $3w+4x+z=400$.
Also since $y+z=100$, we obtain $w+x+y+z=w+x+100$, so to minimize and maximize $w+x+y+z$, we have to minimize and maximize $w+x$.

Step 4: Restating part of goal

We want to find the maximum and minimum possible values of $w + x + y + z$ subject to the conditions that w, x, y, z are positive integers that satisfy $3w + 4x + 5y + 6z = 900$ and $6w + 8x + y + 3z = 900$.
From Step 3, these equations are true if and only if $y + z = 100$ and $3w + 4x + z = 400$. (This is because we can use each pair of equations to obtain the other pair.)
Therefore, we want to find the maximum and minimum possible values of $w+x+y+z$ subject to the conditions $y + z = 100$ and $3w + 4x + z = 400$.
Since $y + z$ is fixed, this is the same as finding the maximum and minimum possible values of $w + x$ subject to the conditions $y + z = 100$ and $3w + 4x + z = 400$.

Step 5: Determining information about N^+

We determine the maximum possible value of $w + x$.
We rewrite $3w + 4x + z = 400$ as $3(w + x) = 400 - x - z$.
To make $w + x$ as large as possible, we want the right side of this equation to be as large as possible, and so we want x and z to be as small as possible.
Note that $x \geq 1$ and $z \geq 1$ and so $400 - x - z \leq 398$.
Also, since the left side of $3(w + x) = 400 - x - z$ is divisible by 3, then the right side must be divisible by 3.
The largest multiple of 3 less than or equal to 398 is 396.
Therefore, $3(w + x) \leq 396$ and so $w + x \leq 132$.
Thus, the maximum possible value of $w + x + y + z$ is $132 + 100 = 232$.
To achieve this maximum, we need $400 - x - z = 396$ (that is, $x + z = 4$). The values $w = 129$, $x = 3$, $y = 99$, $z = 1$ achieve this maximum (and satisfy $y + z = 100$ and $3w + 4x + z = 400$).
Therefore, N^+, the largest possible value of N, consists of 232 digits (all 3s, 4s, 5s and 6s including at least one of each) arranged in descending order.
Thus, N^+ satisfies $6 \times 10^{231} < N^+ < 7 \times 10^{231}$.
(As it turns out, we will not actually have to determine the actual digits of N^+.)

Step 6: Determining information about N^-

We determine the minimum possible value of $w + x$.
We rewrite $3w + 4x + z = 400$ as $4(w + x) = 400 + w - z$.
To make $w + x$ as small as possible, we want the right side of this equation to be as small as possible, and so we want w to be as small as possible and z to be as large as possible.
Now $w \geq 1$ and since $y + z = 100$ and $y \geq 1$, then $z \leq 99$.
Thus, $400 + w - z \geq 302$.
Since the left side of the equation $4(w + x) = 400 + w - z$ is divisible by 4, then the right side must be divisible by 4.
The smallest multiple of 4 greater than or equal to 302 is 304.
Therefore, $4(w + x) \geq 304$ and so $w + x \geq 76$.
Thus, the minimum possible value of $w + x + y + z$ is $76 + 100 = 176$.
To achieve this minimum, we need $400 + w - z = 304$ (that is, $z - w = 96$). The values $w = 3$, $x = 73$, $y = 1$, $z = 99$ achieve this minimum (and satisfy $y + z = 100$ and $3w + 4x + z = 400$).
Therefore, N^- consists of 176 digits (all 3s, 4s, 5s and 6s including at least one of each) arranged in ascending order.
Thus, N^- satisfies $3 \times 10^{175} < N^- < 4 \times 10^{175}$, since this value of N begins with a 3 and has 176 digits.
(Again, we will not actually have to determine the actual digits of N^-.)

<u>Step 7: Determining the number of digits in $N^- \cdot N^+$</u>
Finally, since $6 \times 10^{231} < N^+ < 7 \times 10^{231}$ and $3 \times 10^{175} < N^- < 4 \times 10^{175}$, then

$$18 \times 10^{406} = (3 \times 10^{175}) \cdot (6 \times 10^{231}) < N^- \cdot N^+ < (4 \times 10^{175}) \cdot (7 \times 10^{231}) = 28 \times 10^{406}$$

Therefore, $N^- \cdot N^+$ has 408 digits.

<u>Justification of data in table</u>
We must still justify the data in the table above.
Suppose that N ends with the digits $abcd$. That is, $N = \cdots dcba$.
Then we can write $N = \cdots + 1000d + 100c + 10b + a$.
Then $2N = \cdots + 1000(2d) + 100(2c) + 10(2b) + (2a)$. The difficulty in determining the digits of $2N$ is that each of $2a$, $2b$, $2c$ and $2d$ may not be a single digit.
We use the notation $u(2a)$ and $t(2a)$ to represent the units digit and tens digit of $2a$, respectively.
Note that $u(2a)$ is one of 0, 2, 4, 6, or 8, and $t(2a)$ is 0 or 1.
We define $u(2b), t(2b), u(2c), t(2c), u(2d), t(2d)$ similarly.
Note that $2a = 10 \cdot t(2a) + u(2a)$ and $2b = 10 \cdot t(2b) + u(2b)$ and $2c = 10 \cdot t(2c) + u(2c)$ and $2d = 10 \cdot t(2d) + u(2d)$.
Thus,

$$\begin{aligned} 2N &= \cdots + 1000(10 \cdot t(2d) + u(2d)) + 100(10 \cdot t(2c) + u(2c)) \\ &\qquad + 10(10 \cdot t(2b) + u(2b)) + (10 \cdot t(2a) + u(2a)) \\ &= \cdots + 1000(u(2d) + t(2c)) + 100(u(2c) + t(2b)) + 10(u(2b) + t(2a)) + u(2a) \end{aligned}$$

Since $u(2a), u(2b), u(2c), u(2d) \leq 8$ and $t(2a), t(2b), t(2c), t(2d) \leq 1$, then each of $u(2d) + t(2c)$ and $u(2c) + t(2b)$ and $u(2b) + t(2a)$ and $u(2a)$ is a single digit, so these are the thousands, hundreds, tens and units digits, respectively, of $2N$.
Thus, the sum of the digits of $2N$ is

$$\begin{aligned} u(2a) + (u(2b) + t(2a)) + (u(2c) + t(2b)) + (u(2d) + t(2c)) + \cdots &= \\ (t(2a) + u(2a)) + (t(2b) + u(2b)) + (t(2c) + u(2c)) + \cdots & \end{aligned}$$

The above argument extends to the left for the remaining digits of N.
In other words, if m is a digit in N, then its contribution to the sum of the digits of $2N$ is the sum of the tens and units digits of $2m$.
Therefore, the digits of N contribute to the sum of the digits of $2N$ as outlined in the table above.

ANSWER: (A)