The CENTRE for EDUCATION
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2014 Galois Contest

Wednesday, April 16, 2014
(in North America and South America)

Thursday, April 17, 2014
(outside of North America and South America)

Solutions

1. (a) The three angles shown in the pie chart are $(2x)^\circ$, $(3x)^\circ$ and 90°.
Since these three angles form a complete circle, then $(2x)^\circ+(3x)^\circ+90^\circ = 360^\circ$, or $5x = 270$ and so $x = 54$.

(b) The ratio of the number of bronze medals to the number of silver medals to the number of gold medals is equal to the ratio of the sector angles, $(2x)^\circ$ to $(3x)^\circ$ to 90°, respectively.
Since $x = 54$, then the required ratio is $2(54) : 3(54) : 90$ or $108 : 162 : 90$.
Dividing each term by 18 the ratio becomes $6 : 9 : 5$, which is written in lowest terms.

(c) Since the ratio of the number of bronze to silver to gold medals is $6 : 9 : 5$, let the number of bronze, silver and gold medals in the trophy case be $6x$, $9x$ and $5x$ respectively.
Since the total number of medals in the trophy case is 80, then $6x + 9x + 5x = 80$ or $20x = 80$ and so $x = 4$.
Thus, there are $6 \times 4 = 24$ bronze medals, $9 \times 4 = 36$ silver medals, and $5 \times 4 = 20$ gold medals in the trophy case.

(d) The trophy case begins with 24, 36 and 20 bronze, silver and gold medals, respectively.
Recall that the number of medals is in the ratio $6 : 9 : 5$.
For the ratio of the final number of medals to remain unchanged, we claim that the number of medals added by the teacher must also be in the ratio $6 : 9 : 5$.
(We will prove this claim is true at the end of the solution.)
Since $6 : 9 : 5$ is in lowest terms, the smallest number of medals that the teacher could have added is 6 bronze, 9 silver and 5 gold.
Therefore, the smallest number of medals that could now be in the trophy case is $80 + 6 + 9 + 5$ or 100 medals.
We note that the number of bronze, silver and gold medals is now 30, 45 and 25, which is still in the ratio $6 : 9 : 5$.

Proof of Claim: Let the number of bronze, silver and gold medals added be b, s and g respectively. When these are added to the existing medals, the number of bronze, silver and gold medals becomes $(24 + b)$, $(36 + s)$ and $(20 + g)$. The claim is that for the new ratio, $(24 + b) : (36 + s) : (20 + g)$, to remain unchanged (that is, to equal $24 : 36 : 20$), then $b : s : g$ must equal $6 : 9 : 5$. If $(24 + b) : (36 + s) : (20 + g) = 24 : 36 : 20$, then $\frac{36+s}{24+b} = \frac{36}{24}$ and $\frac{20+g}{36+s} = \frac{20}{36}$. From the first equation, $24(36) + 24s = 36(24) + 36b$ and so $24s = 36b$ or $\frac{s}{b} = \frac{36}{24} = \frac{9}{6}$. Similarly, from the second equation we can show that $\frac{g}{s} = \frac{5}{9}$. Thus, $b : s : g = 6 : 9 : 5$ as claimed.

2. (a) *Solution 1*
Each of the 200 passengers who checked exactly one bag is charged \$20 to do so.
Each of the 45 passengers who checked exactly two bags is charged \$20 for the first bag plus \$7 for the second bag, or \$27 in total for the two bags.
Thus, the total charge for all checked bags is $(200 \times \$20) + (45 \times \$27)$ or \$5215.

Solution 2
All 245 passengers checked at least one bag.
They were each charged \$20 to check this first bag.
The 45 passengers who checked a second bag were each charged an additional \$7 to do so.
Thus, the total charge for all checked bags is $(245 \times \$20) + (45 \times \$7)$ or \$5215.

(b) *Solution 1*
Since each of the 245 passengers checked at least one bag, then the total baggage fees collected for the first bag is $245 \times \$20 = \$4900$.
A total of $\$5173 - \$4900 = \$273$ in baggage fees remains to be collected.
Since all passengers checked exactly one or exactly two bags, then the remaining \$273 in baggage fees is collected from the passengers who checked a second bag.
The cost to check a second bag is \$7.
Thus, the number of passengers who checked exactly two bags is $\frac{273}{7} = 39$.

Solution 2
Let the number of passengers who checked exactly one bag be n.
Since there were 245 passengers on board, and each checked exactly one bag or exactly two bags, then the remaining $(245 - n)$ passengers checked exactly two bags.
Each of the n passengers who checked exactly one bag is charged \$20 to do so.
Each of the $(245 - n)$ passengers who checked exactly two bags is charged \$20 for the first bag plus \$7 for the second bag, or \$27 in total for the two bags.
Since the total charge for all checked bags is \$5173, then $(n \times 20) + ((245 - n) \times 27) = 5173$.
Solving, $20n + 6615 - 27n = 5173$ or $1442 = 7n$, and so $n = 206$.
That is, $245 - n = 245 - 206 = 39$ passengers checked exactly two bags.

Solution 3
All 245 passengers checked at least one bag.
They were each charged \$20 to check this first bag.
Let the number of passengers who checked exactly two bags be m.
The m passengers who checked a second bag were each charged an additional \$7 to do so.
Thus, the total charge for all checked bags is $(245 \times \$20) + (m \times \$7)$, so $4900 + 7m = 5173$ or $7m = 273$ and $m = 39$. Therefore, 39 passengers checked exactly two bags.

(c) Assume that each of the 245 passengers checked at most two bags.
The charge to check exactly two bags is \$27, so in this case the total baggage fees collected could not have exceeded $245 \times \$27 = \$6615$.
Since \$6825 (which is greater than \$6615) was collected in baggage fees on this third flight, then at least one passenger must have checked at least three bags.
(It is possible to have baggage fees total \$6825 if 215 passengers check exactly 2 bags, and 30 passengers check exactly 3 bags.
Here, the total baggage fees collected would be $(215 \times \$27) + (30 \times \$34) = \$6825$.)

(d) Assume that each passenger (of which there are at most 245), checked at most two bags.
Let the number of passengers who checked exactly one bag be a and the number of passengers who checked exactly two bags be b.
While it may be the case that there are passengers who checked no bags, they don't contribute to the \$142 collected and so we may ignore them.
Each of the a passengers who checked exactly one bag is charged \$20, while each of the b passengers who checked exactly two bags is charged \$27.
Since the total fees collected was \$142, then $20a + 27b = 142$.
Solving for a we get, $a = \frac{142 - 27b}{20}$ and since both a and b must be non-negative integers, we systematically try values for b in the table below to see if any gives a non-negative integer value for a. Since $27b$ is at most 142, but $27(5) = 135$ and $27(6) = 162$, then b is at most 5 ($27b$ is larger than 162 when b is larger than 6).

Value of b	Calculation of a
0	$a = \frac{142-27(0)}{20} = 7.1$
1	$a = \frac{142-27(1)}{20} = 5.75$
2	$a = \frac{142-27(2)}{20} = 4.4$
3	$a = \frac{142-27(3)}{20} = 3.05$
4	$a = \frac{142-27(4)}{20} = 1.7$
5	$a = \frac{142-27(5)}{20} = 0.35$

Each of the values of a calculated above is not a non-negative integer.
Thus, there are no non-negative integers a and b that make $20a + 27b = 142$.
Therefore, there is no combination of passengers who check at most two bags such that the baggage fees collected total \$142.
Therefore, there must be at least one passenger who checked at least 3 bags.
(It is possible to have baggage fees total \$142 if 4 passengers check exactly 2 bags, and 1 passenger checks exactly 3 bags. Here, the total baggage fees collected would be $(4 \times \$27) + (1 \times \$34) = \$142$.)

3. (a) *Solution 1*
The cards numbered 1 and 7 are in Emily's set and since $1 + 7 = 8$, then we have found one pair that she can select.
To maintain a sum of 8, we must decrease 7 by 1 when increasing 1 by 1.
That is, the cards numbered 2 and 6 have a sum of 8 and both are in Emily's set so we have found a second pair that she can select.
Repeating the process again, we get the third pair of cards numbered 3 and 5.
The 3 pairs that Emily can select from her set, each having a sum of 8, are $(1, 7), (2, 6)$ and $(3, 5)$.
(Note that an attempt to repeat the process one more time gives $(4, 4)$, however there is only 1 card numbered 4 in Emily's set.)

Solution 2
If we let the smaller card be numbered a and the larger card be numbered b, then $a+b = 8$ or $b = 8 - a$.
Since $a < b$, then $a < 8 - a$ or $2a < 8$ and so $a < 4$.
Since $a \geq 1$, then the only possible values for a are $1, 2, 3$.
Thus, the three pairs having a sum of 8 are $(1, 7)$, $(2, 6)$ and $(3, 5)$.

(b) *Solution 1*
As in part (a), we first attempt to use the card numbered 1 (the smallest numbered card in the set) to form a pair whose sum is 13.
However, the largest number that we can select to pair 1 with is 10, and this gives a sum of $1 + 10 = 11$ which is less than the required sum of 13.

In a similar manner, begin by first selecting the largest numbered card in the set, 10.
When paired with the card numbered 10, the card numbered 3 gives a sum of 13.
Thus, $(3, 10)$ is one pair that Silas may select.
As in part (a), if we increase the lower numbered card by 1 and decrease the higher numbered card by 1, then we maintain a constant sum, 13.
This gives the pairs $(4, 9), (5, 8), (6, 7)$.
We can not continue this process once we reach $(6, 7)$ since the lower numbered card would then become the higher (we would have the pair $(7, 6)$) and we need pairs (a, b) where $a < b$.
Thus, there are exactly 4 pairs, $(3, 10), (4, 9), (5, 8), (6, 7)$, that Silas may select.

Solution 2
If we let the smaller card be numbered a and the larger card be numbered b, then $a+b = 13$ or $b = 13 - a$.
Since $a < b$, then $a < 13 - a$ or $2a < 13$ and so $a < 6.5$.
Also, since $b \leq 10$, then $13 - a \leq 10$ or $3 \leq a$.
Since $3 \leq a < 6.5$, then the only possible values for a are $3, 4, 5, 6$.
Thus, there are exactly 4 pairs, $(3, 10), (4, 9), (5, 8), (6, 7)$, that Silas may select.

(c) If $k \leq 50$, then the maximum sum of any pair is $49 + 50 = 99$.
Therefore to achieve a sum of 100, it must be the case that $k > 50$.
If $k = 51$, then the pair $(49, 51)$ has sum 100.
However, this is the only pair having sum 100.
If $k = 52$, then the pairs $(49, 51)$ and $(48, 52)$ both have sum 100, but these are the only 2 pairs that sum to 100.
Each time we increase k by 1 starting from 51, we obtain one additional pair whose sum is 100, because there is an additional value of b (the larger numbered card in the pair) that can be used.
If $k = 51 + 9 = 60$, then we have the following ten pairs whose sum is 100:
$(49, 51)$, $(48, 52)$, $(47, 53)$, $(46, 54)$, $(45, 55)$, $(44, 56)$, $(43, 57)$, $(42, 58)$, $(41, 59)$, $(40, 60)$.
If we increase k again to $k = 61$, then an additional pair, $(39, 61)$, increases the number of pairs whose sum is 100 to 11.
Thus, Daniel must have a set of $k = 60$ cards numbered consecutively from 1 to 60.

(d) We show that the possible values of S are $S = 67, 68, 84, 85$.

Suppose that S is odd; that is, $S = 2k + 1$ for some integer $k \geq 0$.
The pairs of positive integers (a, b) with $a < b$ and $a + b = S$ are

$$(1, 2k), (2, 2k - 1), (3, 2k - 2), \ldots, (k - 1, k + 2), (k, k + 1)$$

(Since $a < b$, then a is less than half of S (or $k + \frac{1}{2}$) so the possible values of a are 1 to k.)
These pairs satisfy all of the requirements, except possibly the fact that $a \leq 75$ and $b \leq 75$.
Since $a < b$, then we only need to consider whether or not $b \leq 75$.
If $2k \leq 75$, then each of these pairs is an allowable pair, and there are k such pairs.
For there to be 33 such pairs, we have $k = 33$, which gives $S = 2(33) + 1 = 67$.
If $2k > 75$, then not all of these pairs are allowable pairs, as some have b values which are too large.
Counting from the left, the first pair with an allowable b value has $b = 75$, which gives $a = S - 75 = (2k + 1) - 75 = 2k - 74$.
This means that the allowable pairs are

$$(2k - 74, 75), (2k - 73, 74), \ldots, (k - 1, k + 2), (k, k + 1)$$

There are $k - (2k - 74) + 1 = 75 - k$ such pairs.
For there to be 33 such pairs, we have $k = 42$, which gives $S = 2(42) + 1 = 85$.
To summarize the case where $S = 2k + 1$ is odd, there are k allowable pairs when $2k \le 75$ and $75 - k$ allowable pairs when $2k > 75$, giving possible values of S of 67 and 85.

Suppose that S is even; that is, $S = 2k$ for some integer $k \ge 1$.
The pairs of positive integers (a, b) with $a < b$ and $a + b = S$ are

$$(1, 2k - 1), (2, 2k - 2), (3, 2k - 3), \ldots, (k - 2, k + 2), (k - 1, k + 1)$$

(Since $a < b$, then a is less than half of S (or k) so the possible values of a are 1 to $k - 1$.)
If $2k - 1 \le 75$, then each of these pairs is an allowable pair, and there are $k - 1$ such pairs.
For there to be 33 such pairs, we have $k = 34$, which gives $S = 2(34) = 68$.
If $2k - 1 > 75$, then not all of these pairs are allowable pairs, as some have b values which are too large.
Counting from the left, the first pair with an allowable b value has $b = 75$, which gives $a = S - 75 = 2k - 75$.
This means that the allowable pairs are

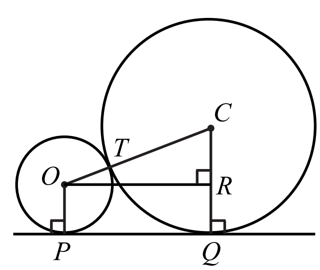
$$(2k - 75, 75), (2k - 74, 74), \ldots, (k - 2, k + 2), (k - 1, k + 1)$$

There are $(k - 1) - (2k - 75) + 1 = 75 - k$ such pairs.
For there to be 33 such pairs, we have $k = 42$, which gives $S = 2(42) = 84$.
To summarize the case where $S = 2k$ is even, there are $k - 1$ allowable pairs when $2k - 1 \le 75$ and $75 - k$ allowable pairs when $2k - 1 > 75$, giving possible values of S of 68 and 84.

Overall, the possible values of S are 67, 68, 84, and 85.

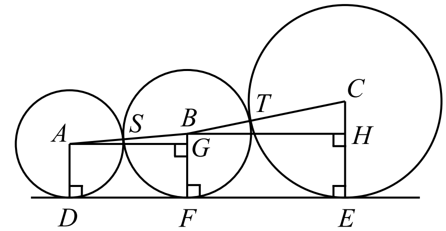
When $S = 67$, the 33 pairs are: $(1, 66), (2, 65), (3, 64), \ldots, (31, 36), (32, 35), (33, 34)$.
When $S = 68$, the 33 pairs are: $(1, 67), (2, 66), (3, 65), \ldots, (31, 37), (32, 36), (33, 35)$.
When $S = 84$, the 33 pairs are: $(9, 75), (10, 74), (11, 73), \ldots, (39, 45), (40, 44), (41, 43)$.
When $S = 85$, the 33 pairs are: $(10, 75), (11, 74), (12, 73), \ldots, (40, 45), (41, 44), (42, 43)$.

4. (a) As suggested, we begin by constructing the segment from O, parallel to PQ, meeting CQ at R.
Both OP and CQ are perpendicular to PQ and since OR is parallel to PQ, then OR is also perpendicular to OP and CQ.
That is, $ORQP$ is a rectangle (it has 4 right angles).
The radius of the small circle is 2 and so $OP = OT = 2$ (since both are radii).
The radius of the large circle is 5 and so $CQ = CT = 5$ (since both are radii).



Since O, T, C are collinear with $OT = 2$ and $CT = 5$, then $OC = OT + CT = 2 + 5 = 7$.
In rectangle $ORQP$, $RQ = OP = 2$.
Therefore, $CR = CQ - RQ = 5 - 2 = 3$.
In right-angled $\triangle OCR$, we have $OC^2 = CR^2 + OR^2$ by the Pythagorean Theorem.
Thus, $OR^2 = OC^2 - CR^2 = 7^2 - 3^2 = 40$, and so $OR = \sqrt{40} = 2\sqrt{10}$ (since $OR > 0$).
Finally, $PQ = OR = 2\sqrt{10}$ (since $ORQP$ is a rectangle).

(b) *Solution 1*



Let the centres of the circles be A, B, C, as shown.
Let F be the point of tangency between the third circle and the horizontal line.
As in part (a), we construct line segments AG and BH parallel to DE and line segments AD, BF, CE perpendicular to DE.
Label S and T, the points of tangency, so then A, S, B are collinear as are B, T, C collinear.
Let the radius of the third circle be r so that $BF = BS = BT = r$.
The radius of the small circle is 4, so $AS = AD = 4$.
The radius of the large circle is 9, so $CT = CE = 9$.
Let $DF = y$. Then $FE = DE - DF = 24 - y$.
As in part (a), $GF = AD = 4$ and $DF = AG = y$ (since $AGFD$ is a rectangle).
Similarly, $HE = BF = r$ and $BH = FE = 24 - y$ (since $BHEF$ is a rectangle).
In right-angled $\triangle ABG$, $AB = AS + BS = 4 + r$ and $BG = BF - GF = r - 4$.
By the Pythagorean Theorem, $AB^2 = AG^2 + BG^2$ or $(4 + r)^2 = y^2 + (r - 4)^2$.
(Note that in the diagram we have assumed that $r > 4$, however if $r < 4$, then G would be placed on AD such that $AG = AD - GD = 4 - r$. In this case, we get $AB^2 = AG^2 + BG^2$ or $(4+r)^2 = (4-r)^2 + y^2$. Since $(4-r)^2 = (r-4)^2$, the equation given by the Pythagorean Theorem is not dependent on which of these two circles has a larger radius.)
In right-angled $\triangle BCH$, $BC = BT + CT = r + 9$ and $CH = CE - HE = 9 - r$.
By the Pythagorean Theorem, $BC^2 = BH^2 + CH^2$ or $(r + 9)^2 = (24 - y)^2 + (9 - r)^2$.
(Note that in the diagram we have assumed that $r < 9$, however if $r > 9$, then H would be placed on BF such that $BH = BF - HF = r - 9$. In this case, we get $BC^2 = BH^2 + CH^2$ or $(r + 9)^2 = (9 - r)^2 + (24 - y)^2$. Since $(9 - r)^2 = (r - 9)^2$, the equation given by the Pythagorean Theorem is not dependent on which of these two circles has a larger radius.)

Next, we solve the system of equations

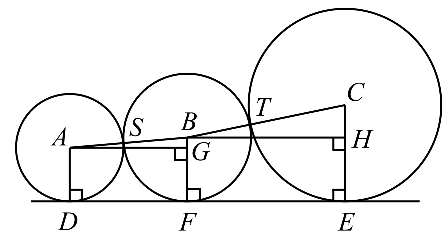
$$(4 + r)^2 = y^2 + (r - 4)^2 \quad (1)$$
$$(r + 9)^2 = (24 - y)^2 + (9 - r)^2 \quad (2)$$

Equation (1) becomes $y^2 = (4 + r)^2 - (r - 4)^2$.
Expanding and simplifying we get $y^2 = 16 + 8r + r^2 - r^2 + 8r - 16$ or $y^2 = 16r$.
Equation (2) becomes $(24 - y)^2 = (r + 9)^2 - (9 - r)^2$.
Instead of expanding, we can factor the right side as a difference of squares, so that $(24 - y)^2 = (r + 9 + 9 - r)(r + 9 - 9 + r) = (18)(2r) = 36r$.
Thus the system of equations simplifies to

$$y^2 = 16r \quad (3)$$
$$(24 - y)^2 = 36r \quad (4)$$

Since $y^2 = 16r = \frac{4}{9}(36r) = \frac{4}{9}(24 - y)^2$, then $y = \pm\frac{2}{3}(24 - y)$.
Solving these two equations, $y = \frac{2}{3}(24 - y)$ and $y = -\frac{2}{3}(24 - y)$, gives $y = \frac{48}{5}$ or $y = -48$.
Since $y > 0$, then $y = \frac{48}{5}$.
Finally, we substitute $y = \frac{48}{5}$ into (3) to get $16r = \left(\frac{48}{5}\right)^2$, so then $r = \frac{48^2}{5^2} \times \frac{1}{16} = \frac{144}{25}$.
The radius of the third circle is $\frac{144}{25}$.

Solution 2



Let the centres of the circles be A, B, C, as shown.
Let F be the point of tangency between the third circle and the horizontal line.
As in part (a), we construct line segments AG and BH parallel to DE and line segments AD, BF, CE perpendicular to DE.
Label S and T, the points of tangency, so then A, S, B are collinear as are B, T, C collinear.
We begin by considering the more general case in which we let the radius of the circle with centre A be r_1, the radius of the circle with centre B be r_2, and the radius of the circle with centre C be r_3. (As was discussed in Solution 1, we may assume that $r_1 < r_2 < r_3$.)
We then have $AD = AS = r_1$, $BS = BF = BT = r_2$, and $CT = CE = r_3$.
As in part (a), $GF = AD = r_1$ and $DF = AG$ (since $AGFD$ is a rectangle).
Similarly, $HE = BF = r_2$ and $BH = FE$ (since $BHEF$ is a rectangle).
In right-angled $\triangle ABG$, $AB = r_1 + r_2$ and $BG = BF - GF = r_2 - r_1$.
By the Pythagorean Theorem, $AG^2 = AB^2 - BG^2 = (r_1 + r_2)^2 - (r_2 - r_1)^2$.
Expanding and simplifying, we get

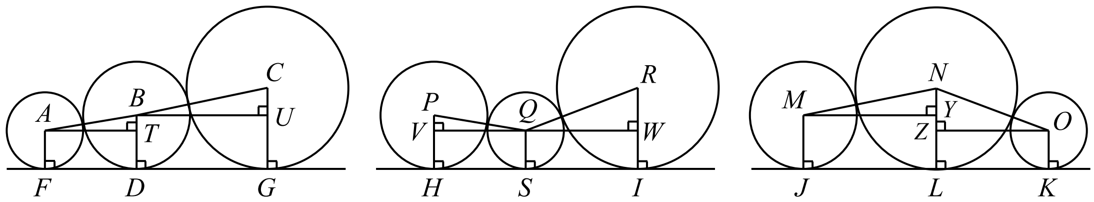
$$\begin{aligned} AG^2 &= r_1^2 + 2r_1r_2 + r_2^2 - r_2^2 + 2r_1r_2 - r_1^2 \\ &= 4r_1r_2 \\ \therefore AG &= 2\sqrt{r_1r_2} \text{ since } AG > 0 \end{aligned}$$

Similarly, in right-angled $\triangle BCH$, $BC = r_2 + r_3$ and $CH = CE - HE = r_3 - r_2$.
By the Pythagorean Theorem, $BH^2 = BC^2 - CH^2 = (r_2 + r_3)^2 - (r_3 - r_2)^2$.
Factoring the right side as a difference of squares, we get

$$\begin{aligned} BH^2 &= (r_2 + r_3 + r_3 - r_2)(r_2 + r_3 - r_3 + r_2) \\ &= (2r_3)(2r_2) \\ &= 4r_2r_3 \\ \therefore BH &= 2\sqrt{r_2r_3} \text{ since } BH > 0 \end{aligned}$$

Thus $DE = DF + FE = AG + BH = 2\sqrt{r_1r_2} + 2\sqrt{r_2r_3}$.
Given that $DE = 24$, $r_1 = 4$ and $r_3 = 9$, we substitute to get $24 = 2\sqrt{4r_2} + 2\sqrt{9r_2}$.
Simplifying, we get $12 = \sqrt{4r_2} + \sqrt{9r_2}$ or $12 = 2\sqrt{r_2} + 3\sqrt{r_2}$ or $12 = 5\sqrt{r_2}$ and so $\sqrt{r_2} = \frac{12}{5}$.
Finally, we square both sides to get $r_2 = \frac{12^2}{5^2} = \frac{144}{25}$.
Therefore, the radius of the third circle is $\frac{144}{25}$.

(c) We begin by constructing line segments as in part (b), and label the diagram as shown.



As we did in part (b) Solution 2, we can show that

$$FG = FD + DG = AT + BU = 2\sqrt{r_1r_2} + 2\sqrt{r_2r_3}$$
$$HI = HS + SI = VQ + QW = 2\sqrt{r_1r_2} + 2\sqrt{r_1r_3}$$
$$JK = JL + LK = MY + ZO = 2\sqrt{r_2r_3} + 2\sqrt{r_1r_3}$$

(Note that we could show each of these results algebraically as we did in part (b), or we could notice that the circles with centres P and Q have the same radii as those with centres B and A respectively, and so $PQ = AB$ or more importantly, $VQ = AT$.)

Since $r_2 < r_3$, then $r_1r_2 < r_1r_3$ and so $2\sqrt{r_1r_2} < 2\sqrt{r_1r_3}$.
Since $r_1 < r_2$, then $r_1r_3 < r_2r_3$ and so $2\sqrt{r_1r_3} < 2\sqrt{r_2r_3}$.
Let $x = \sqrt{r_1r_2}, y = \sqrt{r_1r_3}$ and $z = \sqrt{r_2r_3}$.
Then $x < y < z$.
Since $y < z$, then $x + y < x + z$ so $HI < FG$.
Since $x < y$, then $x + z < y + z$ so $FG < JK$.
Since the lengths of FG, HI, JK are $18, 20, 22$ in some order and $HI < FG < JK$, then $HI = 18, FG = 20$ and $JK = 22$.
Thus, our 3 equations become

$$2x + 2y = 18 \qquad\qquad x + y = 9 \qquad (1)$$
$$2x + 2z = 20 \qquad \text{or} \qquad x + z = 10 \qquad (2)$$
$$2z + 2y = 22 \qquad\qquad z + y = 11 \qquad (3)$$

Adding equations (1), (2), (3) we get $2(x + y + z) = 30$, and so $x + y + z = 15$ (4).
Subtracting equation (1) from equation (4) gives $z = (x + y + z) - (x + y) = 15 - 9 = 6$.
Similarly, subtracting each of the equations (2) and (3) from equation (4) in turn, we get $y = 5$ and $x = 4$.
Since $z = 6$, then $\sqrt{r_2r_3} = 6$ or $r_2r_3 = 6^2$.
Since $y = 5$, then $\sqrt{r_1r_3} = 5$ or $r_1r_3 = 5^2$.
Since $x = 4$, then $\sqrt{r_1r_2} = 4$ or $r_1r_2 = 4^2$.
Multiplying these 3 equations together gives $r_1^2r_2^2r_3^2 = 4^2 \cdot 5^2 \cdot 6^2$ or $(r_1r_2r_3)^2 = (4 \cdot 5 \cdot 6)^2$ and so $r_1r_2r_3 = 4 \cdot 5 \cdot 6 = 120$ (since $r_1, r_2, r_3 > 0$).
Finally, dividing this equation by $r_2r_3 = 6^2$ gives $r_1 = \frac{r_1r_2r_3}{r_2r_3} = \frac{120}{6^2} = \frac{10}{3}$.
Similarly, we get $r_2 = \frac{r_1r_2r_3}{r_1r_3} = \frac{120}{5^2} = \frac{24}{5}$ and $r_3 = \frac{r_1r_2r_3}{r_1r_2} = \frac{120}{4^2} = \frac{15}{2}$.