The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2023 Cayley Contest

(Grade 10)

Wednesday, February 22, 2023
(in North America and South America)

Thursday, February 23, 2023
(outside of North America and South America)

Solutions

1. Since each of the three fractions is equal to 1, then $\frac{1}{1} + \frac{2}{2} + \frac{3}{3} = 1 + 1 + 1 = 3$.

   Answer: (C)

2. Since $3n = 9 + 9 + 9 = 3 \times 9$, then $n = 9$.
   Alternatively, we could note that $9 + 9 + 9 = 27$ and so $3n = 27$ which gives $n = \frac{27}{3} = 9$.

   Answer: (D)

3. We add 25 minutes to 1 hour and 48 minutes in two steps.
   First, we add 12 minutes to 1 hour and 48 minutes to get 2 hours.
   Then we add $25 - 12 = 13$ minutes to 2 hours to get 2 hours and 13 minutes.
   Alternatively, we could note that 1 hour and 48 minutes is $60 + 48 = 108$ minutes, and so the time that is 25 minutes longer is 133 minutes, which is $120 + 13$ minutes or 2 hours and 13 minutes.

   Answer: (A)

4. On Day 1, Lucy sees 2 blue jays and 3 cardinals, and so sees 1 more cardinal than blue jay.
   On Day 2, Lucy sees 3 blue jays and 3 cardinals.
   On Day 3, Lucy sees 2 blue jays and 4 cardinals, and so sees 2 more cardinals than blue jays.
   Thus, over the three days, Lucy saw $1 + 0 + 2 = 3$ more cardinals than blue jays.

   Answer: (B)

5. When looking at  through a window from behind, the digits appear as mirror images and in reverse order, and so appear as .

   Answer: (C)

6. *Solution 1*
   Since $\angle BCD$ is a straight angle, then $\angle ACB = 180^\circ - \angle ACD = 180^\circ - 150^\circ = 30^\circ$.
   Since the measures of the angles in $\triangle ABC$ add to 180°, then $x^\circ + 90^\circ + 30^\circ = 180^\circ$ and so $x = 180 - 120 = 60$.

   Solution 2
   Since the measures of the angles in $\triangle ABC$ add to 180°, then

   $$\angle ACB = 180^\circ - \angle ABC - \angle BAC = 180^\circ - 90^\circ - x^\circ = 90^\circ - x^\circ$$

   Since $\angle BCD = 180^\circ$, then

   $$\begin{aligned} \angle ACB + \angle ACD &= 180^\circ \\ (90^\circ - x^\circ) + 150^\circ &= 180^\circ \\ 240 - x &= 180 \end{aligned}$$

   and so $x = 60$.

   Answer: (E)

7. A cube has six identical faces.
   If the surface area of a cube is 24, the area of each face is $\frac{24}{6} = 4$.
   Since each face of this cube is a square with area 4, the edge length of the cube is $\sqrt{4} = 2$.
   Thus, the volume of the cube is 2^3 which equals 8.

   Answer: (E)

8. If $\frac{4}{5}$ of the beads are yellow, then $\frac{1}{5}$ are green.
Since there are 4 green beads, the total number of beads must be $4 \times 5 = 20$.
Thus, Charlie needs to add $20 - 4 = 16$ yellow beads.

Answer: (A)

9. *Solution 1*
Suppose that the original number is 100.
When 100 is increased by 60%, the result is 160.
To return to the original value of 100, 160 must be decreased by 60.
This percentage is $\frac{60}{160} \times 100\% = \frac{3}{8} \times 100\% = 37.5\%$.

Solution 2
Suppose that the original number is x for some $x > 0$.
When x is increased by 60%, the result is $1.6x$.
To return to the original value of x, $1.6x$ must be decreased by $0.6x$.
This percentage is $\frac{0.6x}{1.6x} \times 100\% = \frac{3}{8} \times 100\% = 37.5\%$.

Answer: (E)

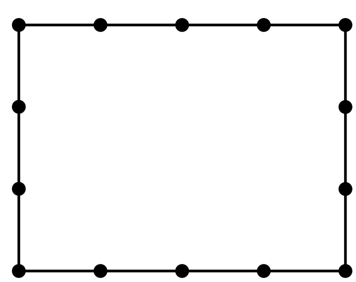
10. Since each door can be open or closed, there are 2 possible states for each door.
Since there are 5 doors, there are $2^5 = 32$ combinations of states for the 5 doors.
If the doors are labelled P, Q, R, S, T, the pairs of doors that can be opened are PQ, PR, PS, PT, QR, QS, QT, RS, RT, ST. There are 10 such pairs.
Therefore, if one of the 32 combinations of states is chosen at random, the probability that exactly two doors are open is $\frac{10}{32}$ which is equivalent to $\frac{5}{16}$.

Answer: (A)

11. After Karim eats n candies, he has $23 - n$ candies remaining.
Since he divides these candies equally among his three children, the integer $23 - n$ must be a multiple of 3.
If $n = 2, 5, 11, 14$, we obtain $23 - n = 21, 18, 12, 9$, each of which is a multiple of 3.
If $n = 9$, we obtain $23 - n = 14$, which is not a multiple of 3.
Therefore, n cannot equal 9.

Answer: (C)

12. A rectangle that is 6 m by 8 m has perimeter $2 \times (6 \text{ m} + 8 \text{ m}) = 28$ m.
If posts are put in every 2 m around the perimeter starting at a corner, then we would guess that it will take $\dfrac{28 \text{ m}}{2 \text{ m}} = 14$ posts.
The diagram below confirms this:

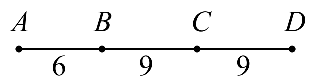


Answer: (B)

13. Since $2023 = 7 \times 17^2$, then any perfect square that is a multiple of 2023 must have prime factors of both 7 and 17.
Furthermore, the exponents of the prime factors of a perfect square must be all even.
Therefore, any perfect square that is a multiple of 2023 must be divisible by 7^2 and by 17^2, and so it is at least $7^2 \times 17^2$ which equals 7×2023.
Therefore, the smallest perfect square that is a multiple of 2023 is 7×2023.
We can check that 2023^2 is larger than 7×2023 and that none of 4×2023 and 17×2023 and $7 \times 17 \times 2023$ is a perfect square.

ANSWER: (C)

14. Since B is between A and D and $BD = 3AB$, then B splits AD in the ratio $1 : 3$.
Since $AD = 24$, then $AB = 6$ and $BD = 18$.
Since C is halfway between B and D, then $BC = \frac{1}{2}BD = 9$.



Thus, $AC = AB + BC = 6 + 9 = 15$.

ANSWER: (E)

15. Since $a = \dfrac{1}{n}$ where n is a positive integer with $n > 1$, then $0 < a < 1$ and $\dfrac{1}{a} = n > 1$.
Thus, $0 < a < 1 < \dfrac{1}{a}$, which eliminates choices (D) and (E).
Since $0 < a < 1$, then a^2 is positive and $a^2 < a$, which eliminates choices (A) and (C).
Thus, $0 < a^2 < a < 1 < \dfrac{1}{a}$, which tells us that (B) must be correct.

ANSWER: (B)

16. *Solution 1*
Since AB and ED are parallel, quadrilateral $ABDE$ is a trapezoid.
We know that $AB = 30$ cm.
Since $ABCF$ is a rectangle, then $FC = AB = 30$ cm.
Suppose that $DC = x$ cm.
Then $ED = FC - FE - DC = (30 \text{ cm}) - (5 \text{ cm}) - (x \text{ cm}) = (25 - x)$ cm.
The height of the trapezoid is the length of AF, which is 14 cm.
Since the area of the trapezoid is 266 cm^2, then

$$\begin{aligned}
266 \text{ cm}^2 &= \frac{30 \text{ cm} + (25 - x) \text{ cm}}{2} \times (14 \text{ cm}) \\
266 &= \frac{55 - x}{2} \times 14 \\
532 &= (55 - x) \times 14 \\
38 &= 55 - x \\
x &= 55 - 38
\end{aligned}$$

and so $DE = x$ cm $= 17$ cm.

Solution 2
Let $DC = x$ cm.
Rectangle $ABCF$ has $AB = 30$ cm and $AF = 14$ cm, and so the area of $ABCF$ is $(30 \text{ cm}) \times (14 \text{ cm}) = 420 \text{ cm}^2$.
The area of $\triangle AFE$, which is right-angled at F, is

$$\tfrac{1}{2} \times AF \times FE = \tfrac{1}{2} \times (14 \text{ cm}) \times (5 \text{ cm}) = 35 \text{ cm}^2$$

The area of quadrilateral $ABDE$ is 266 cm^2.
The area of $\triangle BCD$, which is right-angled at C, is

$$\tfrac{1}{2} \times BC \times DC = \tfrac{1}{2} \times (14 \text{ cm}) \times (x \text{ cm}) = 7x \text{ cm}^2$$

Comparing the area of rectangle $ABCF$ to the combined areas of the pieces, we obtain

$$\begin{aligned}(35 \text{ cm}^2) + (266 \text{ cm}^2) + (7x \text{ cm}^2) &= 420 \text{ cm}^2 \\ 301 + 7x &= 420 \\ 7x &= 119 \\ x &= 17\end{aligned}$$

Thus, the length of DC is 17 cm.

Answer: (A)

17. Megan's car travels 100 m at $\dfrac{5}{4}$ m/s, and so takes $\dfrac{100 \text{ m}}{5/4 \text{ m/s}} = \dfrac{400}{5} \text{ s} = 80$ s.
Hana's car completes the 100 m in 5 s fewer, and so takes 75 s.
Thus, the average speed of Hana's car was $\dfrac{100 \text{ m}}{75 \text{ s}} = \dfrac{100}{75} \text{ m/s} = \dfrac{4}{3}$ m/s.

Answer: (C)

18. The number of bars taken from the boxes is $1 + 2 + 4 + 8 + 16 = 31$.
If these bars all had mass 100 g, their total mass would be 3100 g.
Since their total mass is 2920 g, they are $3100 \text{ g} - 2920 \text{ g} = 180$ g lighter.
Since all of the bars have a mass of 100 g or of 90 g, then it must be the case that 18 of the bars are each 10 g lighter (that is, have a mass of 90 g).
Thus, we want to write 18 as the sum of two of 1, 2, 4, 8, 16 in order to determine the boxes from which the 90 g bars were taken.
We note that $18 = 2 + 16$ and so the 90 g bars must have been taken from box W and box Z.
Can you see why this is unique?

Answer: (B)

19. Since the average of a, b and c is 16, then $\dfrac{a+b+c}{3} = 16$ and so $a + b + c = 3 \times 16 = 48$.
Since the average of c, d and e is 26, then $\dfrac{c+d+e}{3} = 26$ and so $c + d + e = 3 \times 26 = 78$.
Since the average of a, b, c, d, and e is 20, then $\dfrac{a+b+c+d+e}{5} = 20$.
Thus, $a + b + c + d + e = 5 \times 20 = 100$.
We note that

$$(a + b + c) + (c + d + e) = (a + b + c + d + e) + c$$

and so $48 + 78 = 100 + c$ which gives $c = 126 - 100 = 26$.

Answer: (D)

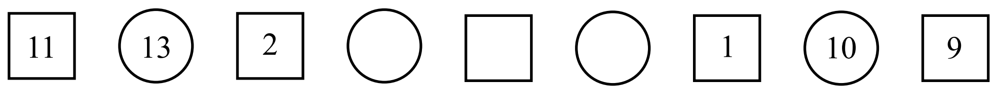
20. Each group of four jumps takes the grasshopper 1 cm to the east and 3 cm to the west, which is a net movement of 2 cm to the west, and 2 cm to the north and 4 cm to the south, which is a net movement of 2 cm to the south.
In other words, we can consider each group of four jumps, starting with the first, as resulting in a net movement of 2 cm to the west and 2 cm to the south.
We note that $158 = 2 \times 79$.
Thus, after 79 groups of four jumps, the grasshopper is $79 \times 2 = 158$ cm to the west and 158 cm to the south of its original position. (We need at least 79 groups of these because the grasshopper cannot be 158 cm to the south of its original position before the end of 79 such groups.)
The grasshopper has made $4 \times 79 = 316$ jumps so far.
After the 317th jump (1 cm to the east), the grasshopper is 157 cm west and 158 cm south of its original position.
After the 318th jump (2 cm to the north), the grasshopper is 157 cm west and 156 cm south of its original position.
After the 319th jump (3 cm to the west), the grasshopper is 160 cm west and 156 cm south of its original position.
After the 320th jump (4 cm to the south), the grasshopper is 160 cm west and 160 cm south of its original position.
After the 321st jump (1 cm to the east), the grasshopper is 159 cm west and 160 cm south of its original position.
After the 322nd jump (2 cm to the north), the grasshopper is 159 cm west and 158 cm south of its original position.
After the 323rd jump (3 cm to the west), the grasshopper is 162 cm west and 158 cm south of its original position, which is the desired position.
As the grasshopper continues jumping, each of its positions will always be at least 160 cm south of its original position, so this is the only time that it is at this position.
Therefore, $n = 323$. The sum of the squares of the digits of n is $3^2 + 2^2 + 3^2 = 9 + 4 + 9 = 22$.

ANSWER: (A)

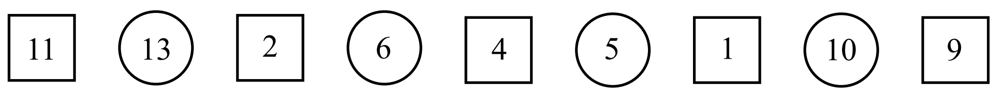
21. Since the line with equation $y = mx - 50$ passes through the point $(a, 0)$, then $0 = ma - 50$ or $ma = 50$.
Since m and a are positive integers whose product is 50, then m and a are a divisor pair of 50.
Therefore, the possible values of m are the positive divisors of 50, which are 1, 2, 5, 10, 25, 50.
The sum of the possible values of m is thus $1 + 2 + 5 + 10 + 25 + 50 = 93$.

ANSWER: 93

22. From the given information, if a and b are in two consecutive squares, then $a+b$ goes in the circle between them.
Since all of the numbers that we can use are positive, then $a+b$ is larger than both a and b.
This means that the largest integer in the list, which is 13, cannot be either x or y (and in fact cannot be placed in any square). This is because the number in the circle next to it must be smaller than 13 (because 13 is the largest number in the list) and so cannot be the sum of 13 and another positive number from the list.
Thus, for $x+y$ to be as large as possible, we would have x and y equal to 10 and 11 in some order. But here we have the same problem: there is only one larger number from the list (namely 13) that can go in the circles next to 10 and 11, and so we could not fill in the circle next to both 10 and 11.
Therefore, the next largest possible value for $x+y$ is when $x=9$ and $y=11$. (We could also swap x and y.)
Here, we could have $13=11+2$ and $10=9+1$, giving the following partial list:



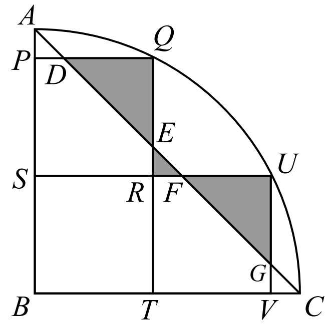
The remaining integers (4, 5 and 6) can be put in the shapes in the following way that satisfies the requirements.



This tells us that the largest possible value of $x+y$ is 20.

ANSWER: 20

23. Since AB and BC are both radii of the circle, then $AB = BC$.
Since ABC is a quarter-circle centred at B, then $\angle ABC = 90^\circ$.
Thus, $\triangle ABC$ is isosceles and right-angled, which means that $\angle BAC = \angle BCA = 45^\circ$.
We add some additional labels to the diagram:



We note that the angles between the straight lines at P, Q, R, and U are all right angles.
Since $\angle PAD = \angle BAC = 45^\circ$, this means that

$$\angle PAD = \angle ADP = \angle QDE = \angle DEQ = \angle REF = \angle EFR = \angle UFG = \angle UGF = 45^\circ$$

This in turn means that each of $\triangle APD$, $\triangle DQE$, $\triangle ERF$, and $\triangle FUG$ is right-angled and isosceles.
Since the side length of each square is 10, then $BT = 10$ and $TQ = TR + RQ = 20$.
Since $\angle BTQ = 90^\circ$, then by the Pythagorean Theorem,

$$BQ = \sqrt{BT^2 + TQ^2} = \sqrt{10^2 + 20^2} = \sqrt{500}$$

We note that $\sqrt{500} = \sqrt{100 \times 5} = \sqrt{100} \times \sqrt{5} = 10\sqrt{5}$.
Since BQ is a radius of the circle, then $BQ = BA = BC = 10\sqrt{5}$.
Since $BP = TQ = 20$, then $AP = BA - BP = 10\sqrt{5} - 20$.
Thus, $PD = AP = 10\sqrt{5} - 20$.
Since $PQ = 10$, then $DQ = PQ - PD = 10 - (10\sqrt{5} - 20) = 30 - 10\sqrt{5}$.
Thus, $QE = DQ = 30 - 10\sqrt{5}$.
Since $PQ = QR$ and $DQ = QE$, then $PD = ER = 10\sqrt{5} - 20$.
Using similar reasoning, $ER = RF = 10\sqrt{5} - 20$ and $UF = UG = 30 - 10\sqrt{5}$.
The total area, $\mathcal{A}$, of the shaded regions equals the sum of the areas of $\triangle DQE$, $\triangle ERF$ and $\triangle FUG$.
Therefore,

$$\begin{aligned}
\mathcal{A} &= \tfrac{1}{2} \times DQ \times QE + \tfrac{1}{2} \times ER \times RF + \tfrac{1}{2} UF \times UG \\
&= \tfrac{1}{2}(30 - 10\sqrt{5})^2 + \tfrac{1}{2}(10\sqrt{5} - 20)^2 + \tfrac{1}{2}(30 - 10\sqrt{5})^2 \\
&= (30 - 10\sqrt{5})^2 + \tfrac{1}{2}(10\sqrt{5} - 20)^2 \\
&= (30^2 - 2 \times 30 \times 10\sqrt{5} + (10\sqrt{5})^2) + \tfrac{1}{2}((10\sqrt{5})^2 - 2 \times 10\sqrt{5} \times 20 + 20^2) \\
&= (900 - 600\sqrt{5} + 500) + \tfrac{1}{2}(500 - 400\sqrt{5} + 400) \\
&= 1400 - 600\sqrt{5} + 450 - 200\sqrt{5} \\
&= 1850 - 800\sqrt{5} \\
&\approx 61.14
\end{aligned}$$

and so the integer closest to $\mathcal{A}$ is 61.

Answer: 61

24. We want to determine the probability that Carina wins 3 games before she loses 2 games.
This means that she either wins 3 and loses 0, or wins 3 and loses 1.
If Carina wins her first three games, we do not need to consider the case of Carina losing her fourth game, because we can stop after she wins 3 games.
Putting this another way, once Carina has won her third game, the outcomes of any later games do not affect the probability because wins or losses at that stage will not affect the question that is being asked.
Using W to represent a win and L to represent a loss, the possible sequence of wins and losses that we need to examine are WWW, LWWW, WLWW, and WWLW.

In the case of WWW, the probabilities of the specific outcome in each of the three games are $\frac{1}{2}$, $\frac{3}{4}$, $\frac{3}{4}$, because the probability of a win after a win is $\frac{3}{4}$.
Threfore, the probability of WWW is $\frac{1}{2} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{32}$.

In the case of LWWW, the probabilities of the specific outcome in each of the four games are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{4}$, $\frac{3}{4}$, because the probability of a loss in the first game is $\frac{1}{2}$, the probability of a win after a loss is $\frac{1}{3}$, and the probability of a win after a win is $\frac{3}{4}$.
Threfore, the probability of LWWW is $\frac{1}{2} \times \frac{1}{3} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{96} = \frac{3}{32}$.

Using similar arguments, the probability of WLWW is $\frac{1}{2} \times \frac{1}{4} \times \frac{1}{3} \times \frac{3}{4} = \frac{3}{96} = \frac{1}{32}$.
Here, we used the fact that the probability of a loss after a win is $1 - \frac{3}{4} = \frac{1}{4}$.

Finally, the probability of WWLW is $\frac{1}{2} \times \frac{3}{4} \times \frac{1}{4} \times \frac{1}{3} = \frac{3}{96} = \frac{1}{32}$.

Therefore, the probability that Carina wins 3 games before she loses 2 games is $\frac{9}{32}+\frac{3}{32}+\frac{1}{32}+\frac{1}{32} = \frac{14}{32} = \frac{7}{16}$, which is in lowest terms.
The sum of the numerator and denominator of this fraction is 23.

Answer: 23

25. *Solution 1*
Let $N = AB\,0AB$ and let t be the two-digit integer AB.
We note that $N = 1001t$, and that $1001 = 11 \cdot 91 = 11 \cdot 7 \cdot 13$.
Therefore, $N = t \cdot 7 \cdot 11 \cdot 13$.
We want to write N as the product of 5 distinct odd integers, each greater than 2, and to count the number of sets S of such odd integers whose product is N.
There are several situations to consider.
First, we look at possible sets S that include the integers 7, 11 and 13 (which we know are divisors).
Second, we look at possible sets S that include two of these integers and an odd multiple of the third.
Third, we rule out possible sets S that include one of these integers and odd multiples of the second and third.
Fourth, we rule out possible sets S that include the product of two or three of these integers and additional integers.

Case 1: $S = \{7, 11, 13, m, n\}$ where $m < n$ and $m, n \neq 7, 11, 13$

Here, $N = 7 \cdot 11 \cdot 13 \cdot m \cdot n = mn \cdot 1001$ and so $t = mn$. This tells us that mn is less than 100.
If $m = 3$, then the possible values for n are 5, 9, 15, 17, 19, 21, 23, 25, 27, 29, 31.
These give the following corresponding values of mn: 15, 27, 45, 51, 57, 63, 69, 75, 81, 87, 93.
Note that $n \neq 33$, since $m = 3$ and $n = 33$ gives $mn = 99$ which has two equal digits and so is not possible.
If $m = 5$, then the possible values for n are 9, 15, 17, 19.

If $m \geq 9$, then $n \geq 15$ since the integers in n are odd and distinct, and so $mn \geq 135$, which is not possible.
Therefore, in this case, there are 15 possible sets.

Case 2: $S = \{7q, 11, 13, m, n\}$ where $m < n$ and $q > 1$ is odd and $m, n \neq 7q, 11, 13$

Here, we have $N = 7q \cdot 11 \cdot 13 \cdot m \cdot n$ and so $N = 1001 \cdot mnq$ which gives $t = mnq$.
Note that $mnq \leq 99$.
Suppose that $q = 3$. This means that $mn \leq 33$.
If $m = 3$, then the possible values of n are 5 and 9 since m and n are odd, greater than 2, and distinct. ($n = 7$ is not possible, since this would give the set $\{21, 11, 13, 3, 7\}$ which is already counted in Case 1 above.)
If $m \geq 5$, then $n \geq 7$ which gives $mn \geq 35$, which is not posible.
Suppose that $q = 5$. This means that $mn \leq \frac{99}{5} = 19\frac{4}{5}$.
If $m = 3$, then $n = 5$. There are no further possibilities when $q = 5$.
Since $mn \geq 3 \cdot 5 = 15$ and $mnq \leq 99$, then we cannot have $q \geq 7$.
Therefore, in this case, there are 3 possible sets.

Case 3: $S = \{7, 11q, 13, m, n\}$ where $m < n$ and $q > 1$ is odd and $m, n \neq 7, 11q, 13$

Suppose that $q = 3$. This means that $mn \leq 33$.
If $m = 3$, then the possible values of n are 5 and 9. (Note that $n \neq 7$.) We cannot have $n = 11$ as this would give $mnq = 99$ and a product of 99 099 which has equal digits A and B.
We cannot have $m \geq 5$ since this gives $mn \geq 45$.
Suppose that $q = 5$. This means that $mn \leq \frac{99}{5}$.
If $m = 3$, then $n = 5$.
As in Case 2, we cannot have $q \geq 7$.
Therefore, in this case, there are 3 possible sets.

Case 4: $S = \{7, 11, 13q, m, n\}$ where $m < n$ and $q > 1$ is odd and $m, n \neq 7, 11, 13q$

Suppose that $q = 3$. This means that $mn \leq 33$.
If $m = 3$, the possible values of n are 5 and 9. (Again, $n \neq 11$ in this case.)
We cannot have $m \geq 5$ when $q = 3$ otherwise $mn \geq 45$.
If $q = 5$, we can have $m = 3$ and $n = 5$ but there are no other possibilities.
As in Cases 2 and 3, we cannot have $q \geq 7$.
Therefore, in this case, there are 3 possible sets.

Case 5: $S = \{7q, 11r, 13, m, n\}$ where $m < n$ and $q, r > 1$ are odd and $m, n \neq 7q, 11r, 13$

Here, $mnqr \leq 99$.
Since $q, r > 1$ are odd, then $qr \geq 9$ which means that $mn \leq 11$.
Since there do not exist two distinct odd integers greater than 1 with a product less than 15, there are no possible sets in this case.
A similarly argument rules out the products

$$N = 7q \cdot 11 \cdot 13r \cdot m \cdot n \qquad N = 7 \cdot 11q \cdot 13r \cdot m \cdot n \qquad N = 7q \cdot 11r \cdot 13s \cdot m \cdot n$$

where q, r, s are odd integers greater than 1.

Case 6: $S = \{77, 13, m, n, \ell\}$ where $m < n < \ell$ and $m, n, \ell \neq 77, 13$

Note that $77 = 7 \cdot 11$ since we know that N has divisors of 7 and 11.
Here, $mn\ell \leq 99$.
Since $mn\ell \geq 3 \cdot 5 \cdot 7 = 105$, there are no possible sets in this case, nor using $7 \cdot 143$ or $11 \cdot 91$ in the product or 1001 by itself or multiples of 77, 91 or 143.

Having considered all cases, there are $15 + 3 + 3 + 3 = 24$ possible sets.

Solution 2

We note first that $AB\,0AB = AB \cdot 1001$, and that $1001 = 11 \cdot 91 = 11 \cdot 7 \cdot 13$.
Therefore, $AB\,0AB = AB \cdot 7 \cdot 11 \cdot 13$.
Since $AB\,0AB$ is odd, then B is odd.
Since $A \neq 0$ and $A \neq B$ and B is odd, then we have the following possibilities for the two-digit integer AB:

$$13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 35, 37, 39, 41, 43, 45, 47, 49$$

$$51, 53, 57, 59, 61, 63, 65, 67, 69, 71, 73, 75, 79, 81, 83, 85, 87, 89, 91, 93, 95, 97$$

If the integer AB is a prime number, then $AB\,0AB$ cannot be written as the product of five different positive integers each greater than 2, since it would have at most four prime factors. Using this information, we can eliminate many possibilities for AB from our list to obtain the shorter list:

$$15, 21, 25, 27, 35, 39, 45, 49, 51, 57, 63, 65, 69, 75, 81, 85, 87, 91, 93, 95$$

Several of the integers in this shorter list are the product of two distinct prime numbers neither of which is equal to 7, 11 or 13. These integers are $15 = 3 \cdot 5$ and $51 = 3 \cdot 17$ and $57 = 3 \cdot 19$ and $69 = 3 \cdot 23$ and $85 = 5 \cdot 17$ and $87 = 3 \cdot 29$ and $93 = 3 \cdot 31$ and $95 = 5 \cdot 19$.
Thinking about each of these as $p \cdot q$ for some distinct prime numbers p and q, we have $AB\,0AB = p \cdot q \cdot 7 \cdot 11 \cdot 13$.
To write $AB\,0AB$ as the product of five different positive odd integers greater each greater than 2, these five integers must be the five prime factors. For each of these 8 integers (15, 51, 57, 69, 85, 87, 93, 95), there is 1 set of five distinct odd integers, since the order of the integers does not matter. This is 8 sets so far.
This leaves the integers 21, 25, 27, 35, 39, 45, 49, 63, 65, 75, 81, 91.

Seven of these remaining integers are equal to the product of two prime numbers, which are either equal primes or at least one of which is equal to 7, 11 or 13. These products are $21 = 3 \cdot 7$ and $25 = 5 \cdot 5$ and $35 = 5 \cdot 7$ and $39 = 3 \cdot 13$ and $49 = 7 \cdot 7$ and $65 = 5 \cdot 13$ and $91 = 7 \cdot 13$.
In each case, $AB\,0AB$ can then be written as a product of 5 prime numbers, at least 2 of which are the same. These 5 prime numbers cannot be grouped to obtain five different odd integers, each larger than 1, since the 5 prime numbers include duplicates and if two of the primes are combined, we must include 1 in the set. Consider, for example, $21 = 3 \cdot 7$. Here, $21\,021 = 3 \cdot 7 \cdot 7 \cdot 11 \cdot 13$. There is no way to group these prime factors to obtain five different odd integers, each larger than 1.
Similary, $25\,025 = 5 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ and $91\,091 = 7 \cdot 13 \cdot 7 \cdot 11 \cdot 13$. The remaining three possibilities (35, 49 and 65) give similar situations.
This leaves the integers 27, 45, 63, 75, 81 to consider.

Consider $27\,027 = 3^3 \cdot 7 \cdot 11 \cdot 13$. There are 6 prime factors to distribute among the five odd integers that form the product. Since there cannot be two 3's in the set, the only way to do this so that they are all different is $\{3, 9, 7, 11, 13\}$.
Consider $81\,081 = 3^4 \cdot 7 \cdot 11 \cdot 13$. There are 7 prime factors to distribute among the five odd integers that form the product.
Since there cannot be two 3s or two 9s in the set and there must be two powers of 3 in the set, there are four possibilities for the set S:

$$S = \{3, 27, 7, 11, 13\}, \{3, 9, 21, 11, 13\}, \{3, 9, 7, 33, 13\}, \{3, 9, 7, 11, 39\}$$

Consider $45\,045 = 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$. There are 6 prime factors to distribute among the five odd integers that form the product.

Since two of these prime factors are 3, they cannot each be an individual element of the set and so one of the 3s must always be combined with another prime giving the following possibilities:

$$S = \{9, 5, 7, 11, 13\}, \{3, 15, 7, 11, 13\}, \{3, 5, 21, 11, 13\}, \{3, 5, 7, 33, 13\}, \{3, 5, 7, 11, 39\}$$

Consider $75\,075 = 3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$.
Using a similar argument to that in the case of $45\,045$, we obtain

$$S = \{15, 5, 7, 11, 13\}, \{3, 25, 7, 11, 13\}, \{3, 5, 35, 11, 13\}, \{3, 5, 7, 55, 13\}, \{3, 5, 7, 11, 65\}$$

Finally, consider $63\,063 = 3^2 \cdot 7^2 \cdot 11 \cdot 13$. There are 6 prime factors to distribute among the five odd integers that form the product. Since we cannot have two 3s or two 7s in the product, the second 3 and the second 7 must be combined, and so there is only one set in this case, namely $S = \{3, 7, 21, 11, 13\}$.

We have determined that the total number of sets is thus $8 + 1 + 4 + 5 + 5 + 1 = 24$.

Answer: 24

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2022 Cayley Contest

(Grade 10)

Wednesday, February 23, 2022
(in North America and South America)

Thursday, February 24, 2022
(outside of North America and South America)

Solutions

1. Evaluating, $2 + (0 \times 2^2) = 2 + 0 = 2$.

   Answer: (B)

2. The ones digit of 119 is not even, so 119 is not a multiple of 2.
   The ones digit of 119 is not 0 or 5, so 119 is not a multiple of 5.
   Since $120 = 3 \times 40$, then 119 is 1 less than a multiple of 3 so is not itself a multiple of 3.
   Since $110 = 11 \times 10$ and $121 = 11 \times 11$, then 119 is between two consecutive multiples of 11, so is not itself a multiple of 11.
   Finally, $119 \div 7 = 17$, so 119 is a multiple of 7.

   Answer: (D)

3. The fractions $\frac{3}{10}$ and $\frac{5}{23}$ are each less than $\frac{1}{2}$ (which is choice (E)) so cannot be the greatest among the choices. (Note that $\frac{1}{2} = \frac{5}{10} = \frac{11.5}{23}$ which we can use to compare the given fractions to $\frac{1}{2}$.)
   The fractions $\frac{4}{7}$ and $\frac{2}{3}$ are each greater than $\frac{1}{2}$, so $\frac{1}{2}$ cannot be the greatest among the choices.
   This means that the answer must be either $\frac{4}{7}$ or $\frac{2}{3}$.
   Using a common denominator of $3 \times 7 = 21$, we re-write these fractions as $\frac{4}{7} = \frac{12}{21}$ and $\frac{2}{3} = \frac{14}{21}$ which shows that $\frac{2}{3}$ has the gretest value among the five choices.
   (Alternatively, we could have converted the fractions into decimals and used their decimal approximations to compare their sizes.)

   Answer: (D)

4. The sequence consists of a pattern of 5 shapes that are repeated.
   The first repetitions of this pattern end on the 5th, 10th, 15th, 20th, and 25th shapes.
   This means that the 22nd shape is the 2nd shape after the 20th shape, and so is the 2nd shape in the pattern.
   Thus, the 22nd shape is $\square$.

   Answer: (A)

5. The given sum includes 5 terms each equal to (5×5).
   Thus, the given sum is equal to $5 \times (5 \times 5)$ which equals 5×25 or 125.

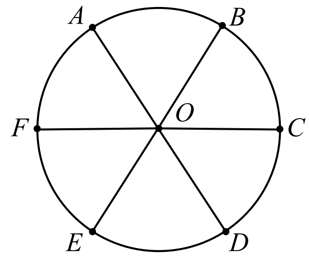
   Answer: (E)

6. Yihana is walking uphill exactly when the graph is increasing (that is, when the slope of the segment of the graph is positive).
   This is between 0 and 3 minutes and between 8 and 10 minutes, which correspond to lengths of time of 3 minutes and 2 minutes in these two cases, for a total of 5 minutes.

   Answer: (A)

7. *Solution 1*
   Since A, B, C, D, E, and F are equally spaced around the circle, moving from one point to the next corresponds to moving $\frac{1}{6}$ of the way around the circle.
   Therefore, moving from A to C corresponds to moving $\frac{2}{6}$ (or $\frac{1}{3}$) of the way around the circle.
   Since moving around the whole circle corresponds to moving through 360°, then moving $\frac{1}{3}$ of the way around the circle corresponds to moving through $\frac{1}{3} \times 360^\circ = 120^\circ$.
   Thus, $\angle AOC = 120^\circ$.

Solution 2
We join A, B, C, D, E, and F to O.



Since A, B, C, D, E, and F are equally spaced around the circle, the angles made at the centre by consecutive points are equal. That is,

$$\angle AOB = \angle BOC = \angle COD = \angle DOE = \angle EOF = \angle FOA$$

Since these 6 angles form a complete circle, the sum of their measures is 360°.
Therefore,

$$\angle AOB = \angle BOC = \angle COD = \angle DOE = \angle EOF = \angle FOA = \tfrac{1}{6} \times 360^\circ = 60^\circ$$

This means that $\angle AOC = \angle AOB + \angle BOC = 60^\circ + 60^\circ = 120^\circ$.

Answer: (D)

8. Since the rectangle has positive integer side lengths and an area of 24, its length and width must be a positive divisor pair of 24.
Therefore, the length and width must be 24 and 1, or 12 and 2, or 8 and 3, or 6 and 4.
Since the perimeter of a rectangle equals 2 times the sum of the length and width, the possible perimeters are

$$2(24+1) = 50 \qquad 2(12+2) = 28 \qquad 2(8+3) = 22 \qquad 2(6+4) = 20$$

These all appear as choices, which means that the perimeter of the rectangle cannot be 36, which is (E).

Answer: (E)

9. Using the definition, $3\nabla b = \dfrac{3+b}{3-b}$.
Assuming $b \neq 3$, the following equations are equivalent:

$$\begin{aligned} 3\nabla b &= -4 \\ \frac{3+b}{3-b} &= -4 \\ 3+b &= -4(3-b) \\ 3+b &= -12+4b \\ 15 &= 3b \end{aligned}$$

and so $b = 5$.

Answer: (A)

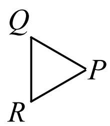
10. *Solution 1*
Since x is 20% of y, then $x = \frac{20}{100}y = \frac{1}{5}y$.
Since x is 50% of z, then $x = \frac{1}{2}z$.
Therefore, $\frac{1}{5}y = \frac{1}{2}z$ which gives $\frac{2}{5}y = z$.
Thus, $z = \frac{40}{100}y$ and so z is 40% of y.

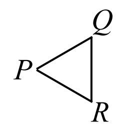
Solution 2
Since x is 20% of y, then $x = 0.2y$.
Since x is 50% of z, then $x = 0.5z$.
Therefore, $0.2y = 0.5z$ which gives $0.4y = z$.
Thus, $z = 0.4y$ and so z is 40% of y.

Answer: (D)

11. The store sells 250 g of jellybeans for \$7.50, which is 750 cents.
Therefore, 1 g of jellybeans costs $750 \div 250 = 3$ cents.
This means that \$1.80, which is 180 cents, will buy $180 \div 3 = 60$ g of jellybeans.

Answer: (C)

12. Starting with 

and flipping across QR results in 

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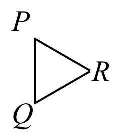
Now flipping across PR results in 

, which is the resulting position that Paola sees.

Answer: (E)

13. Since $2 \times 2 \times 2 = 8$, a cube with edge length 2 has volume 8. (We note also that $\sqrt[3]{8} = 2$.)
Therefore, each of the cubes with volume 8 have a height of 2.
This means that the larger cube has a height of $2 + 2 = 4$, which means that its volume is $4^3 = 4 \times 4 \times 4 = 64$.

Answer: (E)

14. Since 100 000 does not include the block of digits 178, each integer between 10 000 and 100 000 that includes the block of digits 178 has five digits.
Such an integer can be of the form $17\,8xy$ or of the form $x1\,78y$ or of the form $xy\,178$ for some digits x and y.
The leading digit of a five-digit integer has 9 possible values (any digit from 1 to 9, inclusive) while a later digit in a five-digit integer has 10 possible vlaues (0 or any digit from 1 to 9, inclusive).
This means that

- there are 100 integers of the form $17\,8xy$ (10 choices for each of x and y, and $10 \times 10 = 100$),
- there are 90 integers of the form $x1\,78y$ (9 choices for x and 10 choices for y, and $9 \times 10 = 90$), and
- there are 90 integers of the form $xy\,178$ (9 choices for x and 10 choices for y, and $9 \times 10 = 90$).

In total, there are thus $100 + 90 + 90 = 280$ integers between 10 000 and 100 000 that include the block of digits 178.

Answer: (A)

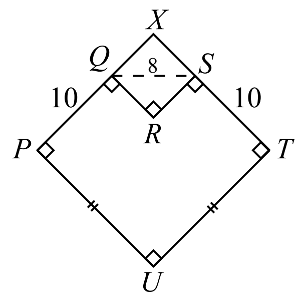
15. Since $a + 5 = b$, then $a = b - 5$.
Since $a = b - 5$ and $c = 5 + b$ and $b + c = a$, then

$$\begin{aligned} b + (5 + b) &= b - 5 \\ 2b + 5 &= b - 5 \\ b &= -10 \end{aligned}$$

(If $b = -10$, then $a = b - 5 = -15$ and $c = 5 + b = -5$ and $b + c = (-10) + (-5) = (-15) = a$, as required.)

Answer: (C)

16. Extend PQ and TS to meet at point X.



Since quadrilateral $QRSX$ has three right angles (at Q, R and S), it must have a fourth right angle at X.
Thus, $QRSX$ is a rectangle, which means that $XS = QR$ and $QX = RS$.
The perimeter of $PQRSTU$ is

$$\begin{aligned} PQ + QR + RS + ST + TU + UP &= PQ + XS + QX + ST + TU + UP \\ &= (PQ + QX) + (XS + ST) + TU + UP \\ &= PX + XT + TU + UP \end{aligned}$$

which is the perimeter of quadrilateral of $PXTU$.
But quadrilateral $PXTU$ has four right angles, and so is a rectangle.
Also, $PU = UT$, so $PXTU$ is a square, and so the perimeter of $PXTU$ equals

$$4 \times PX = 4 \times (PQ + QX) = 4 \times (10 + QX) = 40 + 4 \times QX$$

Finally, $QX = PX - PQ = PX - 10 = XT - 10 = XT - ST = XS$, which means that $\triangle QXS$ is isosceles as well as being right-angled at X.
By the Pythagorean Theorem, $QX^2 + XS^2 = QS^2$ and so $2 \times QX^2 = 8^2$ or $QX^2 = 32$.
Since $QX > 0$, then $QX = \sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$.
Thus, the perimeter of $PQRSTU$ is $40 + 4 \times 4\sqrt{2} = 40 + 16\sqrt{2} \approx 62.6$.
(We could have left this as $40 + 4\sqrt{32} \approx 62.6$.)
Of the given choices, this is closest to 63.

Answer: (C)

17. Zebadiah must remove at least 3 shirts.
If he removes 3 shirts, he might remove 2 red shirts and 1 blue shirt.
If he removes 4 shirts, he might remove 2 red shirts and 2 blue shirts.
Therefore, if he removes fewer than 5 shirts, it is not guaranteed that he removes either 3 of the same colour or 3 of different colours.
Suppose that he removes 5 shirts.
If 3 are of the same colour, the requirements are satisfied.
If no 3 of the 5 shirts are of the same colour, then at most 2 are of each colour. This means that he must remove shirts of 3 colours, since if he only removed shirts of 2 colours, he would remove at most $2 + 2 = 4$ shirts.
In other words, if he removes 5 shirts, it is guaranteed that there are either 3 of the same colours or shirts of all 3 colours.
Thus, the minimum number is 5.

Answer: (D)

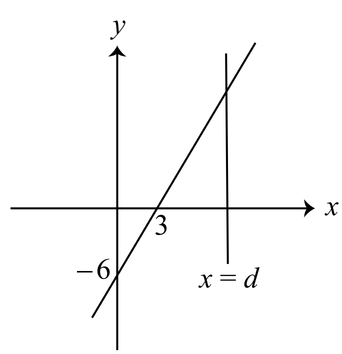
18. At the beginning of the first day, the box contains 1 black ball and 1 gold ball.
At the end of the first day, 2 black balls and 1 gold ball are added, so the box contains 3 black balls and 2 gold balls.
At the end of the second day, $2 \times 2 = 4$ black balls and $2 \times 1 = 2$ gold balls are added, so the box contains 7 black balls and 4 gold balls.
Continuing in this way, we find the following numbers of balls:

End of Day #	Black Balls	Gold Balls
2	7	4
3	$7 + 4 \times 2 = 15$	$4 + 4 = 8$
4	$15 + 8 \times 2 = 31$	$8 + 8 = 16$
5	$31 + 16 \times 2 = 63$	$16 + 16 = 32$
6	$63 + 32 \times 2 = 127$	$32 + 32 = 64$
7	$127 + 64 \times 2 = 255$	$64 + 64 = 128$

At the end of the 7th day, there are thus $255 + 128 = 383$ balls in the box.

Answer: (E)

19. The line with equation $y = 2x - 6$ has y-intercept -6.
Also, the x-intercept of $y = 2x - 6$ occurs when $y = 0$, which gives $0 = 2x - 6$ or $2x = 6$ which gives $x = 3$.
Therefore, the triangle bounded by the x-axis, the y-axis, and the line with equation $y = 2x-6$ has base of length 3 and height of length 6, and so has area $\frac{1}{2} \times 3 \times 6 = 9$.
We want the area of the triangle bounded by the x-axis, the vertical line with equation $x = d$, and the line with equation $y = 2x - 6$ to be 4 times this area, or 36.
This means that $x = d$ is to the right of the point $(3, 0)$, because the new area is larger. In other words, $d > 3$.



The base of this triangle has length $d-3$, and its height is $2d-6$, since the height is measured along the vertical line with equation $x = d$.
Thus, we want $\frac{1}{2}(d-3)(2d-6) = 36$ or $(d-3)(d-3) = 36$ which means $(d-3)^2 = 36$.
Since $d - 3 > 0$, then $d - 3 = 6$ which gives $d = 9$.
Alternatively, we could note that if similar triangles have areas in the ratio $4 : 1$ then their corresponding lengths are in the ratio $\sqrt{4} : 1$ or $2 : 1$.
Since the two triangles in question are similar (both are right-angled and they have equal angles at the point $(3, 0)$), the larger triangle has base of length $2 \times 3 = 6$ and so $d = 3 + 6 = 9$.

Answer: (A)

20. Since $3m^3$ is a multiple of 3, then $5n^5$ is a multiple of 3.
Since 5 is not a multiple of 3 and 3 is a prime number, then n^5 is a multiple of 3.
Since n^5 is a multiple of 3 and 3 is a prime number, then n is a multiple of 3, which means that $5n^5$ includes at least 5 factors of 3.
Since $5n^5$ includes at least 5 factors of 3, then $3m^3$ includes at least 5 factors of 3, which means that m^3 is a multiple of 3, which means that m is a multiple of 3.
Using a similar analysis, both m and n must be multiples of 5.
Therefore, we can write $m = 3^a5^bs$ for some positive integers a, b and s and we can write $n = 3^c5^dt$ for some positive integers c, d and t, where neither s nor t is a multiple of 3 or 5. (In other words, we have grouped all of the factors of 3 and 5 in each of m and n.)
From the given equation,

$$\begin{aligned} 3m^3 &= 5n^5 \\ 3(3^a5^bs)^3 &= 5(3^c5^dt)^5 \\ 3 \times 3^{3a}5^{3b}s^3 &= 5 \times 3^{5c}5^{5d}t^5 \\ 3^{3a+1}5^{3b}s^3 &= 3^{5c}5^{5d+1}t^5 \end{aligned}$$

Since s and t are not multiples of 3 or 5, we must have $3^{3a+1} = 3^{5c}$ and $5^{3b} = 5^{5d+1}$ and $s^3 = t^5$.
Since s and t are positive and m and n are to be as small as possible, we can set $s = t = 1$, which satisfy $s^3 = t^5$.
Since $3^{3a+1} = 3^{5c}$ and $5^{3b} = 5^{5d+1}$, then $3a + 1 = 5c$ and $3b = 5d + 1$.
Since m and n are to be as small as possible, we want to find the smallest positive integers a, b, c, d for which $3a + 1 = 5c$ and $3b = 5d + 1$.
Neither $a = 1$ nor $a = 2$ gives a value for $3a + 1$ that is a multiple of 5, but $a = 3$ gives $c = 2$.
Similarly, $b = 1$ does not give a value of $3b$ that equals $5d + 1$ for any positive integer d, but $b = 2$ gives $d = 1$.
Therefore, the smallest possible values of m and n are $m = 3^35^2 = 675$ and $n = 3^25^1 = 45$, which gives $m + n = 720$.
(We can verify by substitution that $m = 675$ and $n = 45$ satisfy the equation $3m^3 = 5n^5$.)

Answer: (C)

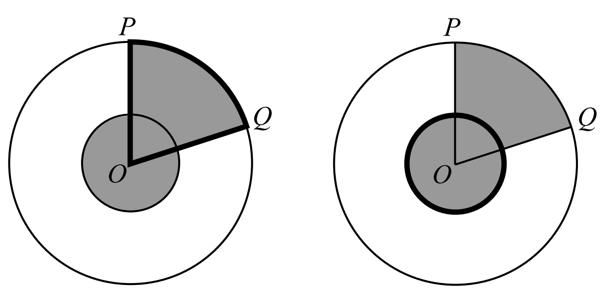
21. Since $20x + 11y = 881$, then $20x = 881 - 11y$ and $11y = 881 - 20x$.
Since x is an integer, then $20x$ is a multiple of 10 and so the units digit of $20x$ is 0 which means that the units digit of $881 - 20x$ is 1, and so the units digit of $11y$ is 1.
Since the units digit of $11y$ is 1, then the units digit of y is 1.
Since $20x$ is positive, then $11y = 881 - 20x$ is smaller than 881, which means that $11y < 881$ and so $y < \frac{881}{11} \approx 80.1$.
Thus, the possible values of y are 1, 11, 21, 31, 41, 51, 61, 71.
We check each of these:

y	$11y = 881 - 20x$	$20x$	x
1	11	870	Not an integer
11	121	760	38
21	231	650	Not an integer
31	341	540	27
41	451	430	Not an integer
51	561	320	16
61	671	210	Not an integer
71	781	100	5

Therefore, the sum of the smallest and largest of the permissible values of y is $11 + 71 = 82$.

ANSWER: 82

22. Since the shaded regions are equal, then when the unshaded sector in the small circle is shaded, the area of the now fully shaded sector of the larger circle must be equal to the area of the smaller circle.



The smaller circle has radius 1 and so has area $\pi \times 1^2 = \pi$.
The larger circle has radius 3 and so has area $\pi \times 3^2 = 9\pi$.
This means that the area of the shaded sector in the larger circle has area π, which means that it must be $\frac{1}{9}$ of the larger circle.
This means that $\angle POQ$ must be $\frac{1}{9}$ of a complete circle, and so $\angle POQ = \frac{1}{9} \times 360° = 40°$.
Thus, $x = 40$.

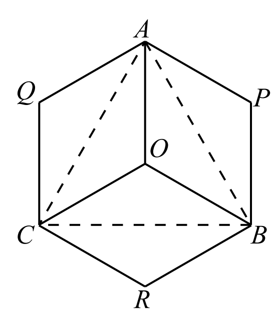
ANSWER: 40

23. Suppose that Andreas, Boyu, Callista, and Diane choose the numbers a, b, c, and d, respectively. There are 9 choices for each of a, b, c, and d, so the total number of quadruples (a, b, c, d) of choices is $9^4 = 6561$.
Among the 9 choices, 5 are odd $(1, 3, 5, 7, 9)$ and 4 are even $(2, 4, 6, 8)$.
If there are N quadruples (a, b, c, d) with $a + b + c + d$ even (that is, with the sum of their choices even), then the probability that the sum of their four integers is even is $\frac{N}{6561}$, which is in the desired form.
Therefore, we count the number of quadruples (a, b, c, d) with $a + b + c + d$ even.
Among the four integer a, b, c, d, either 0, 1, 2, 3, or 4 of these integers are even, with the remaining integers odd.
If 0 of a, b, c, d are even and 4 are odd, their sum is even.
If 1 of a, b, c, d is even and 3 are odd, their sum is odd.
If 2 of a, b, c, d are even and 2 are odd, their sum is even.
If 3 of a, b, c, d are even and 1 is odd, their sum is odd.
If 4 of a, b, c, d are even and 0 are odd, their sum is even.
Therefore, we need to count the number of quadruples (a, b, c, d) with 0, 2 or 4 even parts.

If 0 are even and 4 are odd, there are 5 choices for each of the parts, and so there are $5^4 = 625$ such quadruples.
If 4 are even and 0 are odd, there are 4 choices for each of the parts, and so there are $4^4 = 256$ such quadruples.
If 2 are even and 2 are odd, there are 4 choices for each of the even parts and 5 choices for each of the odd parts, and 6 pairs of locations for the even integers (ab, ac, ad, bc, bd, cd) with the odd integers put in the remaining two locations after the locations of the even integers are chosen. Thus, there are $4^2 \cdot 5^2 \cdot 6 = 2400$ such quadruples.
In total, this means that there are $625 + 256 + 2400 = 3281$ quadruples and so $N = 3281$.
The sum of the squares of the digits of N is equal to $3^2 + 2^2 + 8^2 + 1^2 = 9 + 4 + 64 + 1 = 78$.

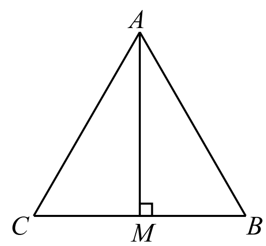
ANSWER: 78

24. Let O be the vertex of the cube farthest away from the table.
Let A, B and C be the vertices of the cube connected to O with edges.
Since the cube has edge length 8, then $OA = OB = OC = 8$.
Note that $\angle AOB = \angle AOC = \angle BOC = 90^\circ$, which means that each of $\triangle AOB$, $\triangle AOC$ and $\triangle BOC$ is a right-angled isosceles triangle, which means that $AB = \sqrt{2}AO = 8\sqrt{2}$, and so $AB = AC = BC = 8\sqrt{2}$.
Let P, Q and R be the vertices that complete the square faces $PAOB$, $QAOC$ and $RBOC$.
From directly above, the cube looks like this:



When the sun is directly overhead, the shadow of the cube will look exactly like the area of the "flat" hexagon $APBRCQ$. In mathematical terms, we are determining the area of what is called a *projection*.

To find the area of this figure, we need to know some lengths, but we have to be careful because not all of the edges in the edges in the diagram above are "flat".
We do know that points A, B and C are in the same horizontal plane, and we also know that $AB = AC = BC = 8\sqrt{2}$. This means that these are true lengths in that the points A, B and C are this distance apart.
Note that flat quadrilateral $PAOB$ is divided into two regions of equal area by AB. Similarly, $QAOC$ is divided into two regions of equal area by AC, and $RBOC$ is divided into two regions of equal area by BC.
In other words, the area of $\triangle ABC$ is half of the area of hexagon $APBRCQ$, so to find the area of $APBRCQ$, we double the area of $\triangle ABC$.
So we need to calculate the area of equilateral $\triangle ABC$ which has side length $8\sqrt{2}$.
Let M be the midpoint of BC. Thus, $BM = CM = 4\sqrt{2}$.



Since $\triangle ABC$ is equilateral, then AM is perpendicular to BC.
By the Pythagorean Theorem in $\triangle AMC$,

$$AM = \sqrt{AC^2 - MC^2} = \sqrt{(8\sqrt{2})^2 - (4\sqrt{2})^2} = \sqrt{128 - 32} = \sqrt{96}$$

Therefore, the area of $\triangle ABC$ is

$$\tfrac{1}{2} \cdot CB \cdot AM = \tfrac{1}{2} \cdot 8\sqrt{2} \cdot \sqrt{96} = 4\sqrt{192} = 4\sqrt{64 \cdot 3} = 4 \cdot 8\sqrt{3} = 32\sqrt{3}$$

This means that the area of the hexagonal shadow is $64\sqrt{3}$.
Since this is in the form $a\sqrt{b}$ where a and b are positive integers and b is not divisible by the square of an integer larger than 1, then $a = 64$ and $b = 3$ and so $a + b = 67$.

ANSWER: 67

25. We begin by tracing what happens when $T = 337$.
We start with tokens labelled $1, 2, 3, \ldots, 335, 336, 337$ arranged around a circle.
We remove the first token (1), move 2 tokens along and remove that token (3), move 2 tokens along and remove that token (5), and continue around the circle until we remove tokens 335 and 337.
This leaves tokens $2, 4, 6, \ldots, 332, 334, 336$. These tokens differ by 2.

On the second pass, we start with tokens labelled $2, 4, 6, \ldots, 332, 334, 336$, which differ by 2.
Because the last token was removed on the first pass (337), the first token is not removed on the second pass, which means that we remove every other token starting with 4.
This means that the remaining tokens differ by 4, and are $2, 6, 10, \ldots, 326, 330, 334$.

On the third pass, we start with $2, 6, 10, \ldots, 326, 330, 334$, which differ by 4.
Because the last token (336) was removed on the previous pass, we remove every other token starting with 6.
The remaining tokens differ by 8 and are $2, 10, 18, \ldots, 314, 322, 330$.

On the fourth pass, we start with $2, 10, 18, \ldots, 314, 322, 330$, which differ by 8.
Because the last token (334) was removed on the previous pass, we remove every other token starting with 10.
The remaining tokens differ by 16 and are $2, 18, 34, \ldots, 290, 306, 322$.

On the fifth pass, we start with $2, 18, 34, \ldots, 290, 306, 322$, which differ by 16.
Because the last token (330) was removed on the previous pass, we remove every other token starting with 18.
The remaining tokens differ by 32 and are $2, 34, 66, 98, 130, 162, 194, 226, 258, 290, 322$.

On the sixth pass, we start with $2, 34, 66, 98, 130, 162, 194, 226, 258, 290, 322$, which differ by 32.
Because the second last token (306) was removed on the previous pass, we remove ever other token starting with 2.
This leaves $34, 98, 162, 226, 290$, which differ by 64.

On the seventh pass, we remove starting with the second token, which leaves $34, 162, 290$, which differ by 128.

On the eighth pass, we remove starting with the first token, which leaves 162.

This tells us that the smallest possible value of T is at least 162 and at most 337.

Next, we will show that $T = 209$ also gives a final token of 162 by working through the various passes. (This is a place where how we *discover* the answer is more than likely different than how we *justify* the answer.)
Before first pass:

$$1, 2, 3, \ldots, 207, 208, 209$$

After first pass (removing every other token starting with 1):

$$2, 4, 6, \ldots, 204, 206, 208$$

After second pass (removing every other token starting with 4):

$$2, 6, 10, \ldots, 198, 202, 206$$

After third pass (removing every other token starting with 6):

$$2, 10, 18, \ldots, 186, 194, 202$$

After fourth pass (removing every other token starting with 10):

$$2, 18, 34, \ldots, 162, 178, 194$$

After fifth pass (removing every other token starting with 18):

$$2, 34, 66, 98, 130, 162, 194$$

After sixth pass (removing every other token starting with 2):

$$34, 98, 162$$

After seventh pass (removing 98):

$$34, 162$$

This leaves 162 after the eighth pass, since 34 will be removed.
Therefore, when $T = 209$ and when $T = 337$, the final token is 162.

Finally, we show that if the final token starting with T tokens is 162, then $T \geq 209$, which will tell us that the smallest value of T is 209.
Suppose that $T \leq 209$ and that the token remaining after the final pass is 162.
Before each pass, the remaining tokens differ by a power of 2, since we start by removing every other token from a list that differs by 1, then every other token from a list that differs by 2, and so on.
The smallest powers of 2 are 2, 4, 8, 16, 32, 64, 128, 256.

Since 162 is left after the last pass (this will turn out to be the eighth pass), the remaining tokens must have differed by 128 before the eighth pass, and thus were 34, 162. (Since $T \leq 209$, then there could not be a token numbered $162 + 128 = 290$.) Also, if the tokens differed by 64 before the eighth pass, there would have been tokens labelled 34 and 98 that were both removed.
Thus, before the eighth pass, the tokens were 34, 162 and 34 was removed.

Before the seventh pass, the tokens differed by 64.
Thus, these were 34, 98, 162. We note that the last token was not removed on this pass, and so the first token is removed on the eighth pass, as expected.
Also, there cannot be a token numbered $162 + 64 = 226$, since $T \leq 209$.

Before the sixth pass, the tokens differed by 32.
Thus, these were 2, 34, 66, 98, 130, 162, 194. The last token cannot have been 162 since the last token must be removed on this pass so that the second token (98) is removed on the seventh pass. Thus, $162 + 32 = 194$ must be the last token here. (Note that 226 was already rejected earlier.)

Before the fifth pass, the tokens differed by 16.
Since the first token (2) is removed on the sixth pass, the last token is not removed on the fifth pass.
This means that the tokens before this pass were

$$2, 18, 34, 50, 66, 82, 98, 114, 130, 146, 162, 178, 194$$

On this pass, 178 is the last token removed. (Note that $194 + 16 = 210$ is too large.)

Before the fourth pass, the tokens differed by 8.
Since the second token (18) is removed on the fifth pass, the last token is removed on the fourth pass.
This means that the tokens before this pass were

$$2, 10, 18, \ldots, 162, 170, 178, 186, 194, 202$$

The token 202 must be included since 194 remains for the fifth pass. (Note that $202 + 8 = 210$ is too large.)

Before the third pass, the tokens differed by 4.
Since the second token (10) is removed on the fourth pass, the last token is removed on the third pass.
This means that the tokens before this pass were

$$2, 6, 10, 14, 18, \ldots, 186, 190, 194, 198, 202, 206$$

The token 206 must be included since 202 remains for the fourth pass. (Note that $206 + 4 = 210$ is too large.)

Before the second pass, the tokens differed by 2.
Since the second token (6) is removed on the third pass, the last token is removed on the second pass.
This means that the tokens before this pass were

$$2, 4, 6, 8, \ldots, 198, 200, 202, 204, 206, 208$$

The token 208 must be included since 206 remains for the third pass. (Note that $208 + 2 = 210$ is too large.)

Before the first pass, the tokens differed by 1.
Since the second token (4) is removed on the second pass, the last token is removed on the first pass.
This means that the tokens before this pass were

$$1, 2, 3, 4, \ldots, 204, 205, 206, 207, 208, 209$$

The token 209 must be included since 208 remains for the second pass. (Note that $209+1 = 210$ is too large.)

Therefore, we must have at least 209 tokens for the final token to be 162, and so the smallest possible value of T is 209, whose rightmost two digits are 09.

Answer: 09

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2021 Cayley Contest

(Grade 10)

Tuesday, February 23, 2021
(in North America and South America)

Wednesday, February 24, 2021
(outside of North America and South America)

Solutions

1. Evaluating, $\dfrac{2+4}{1+2} = \dfrac{6}{3} = 2$.

   Answer: (C)

2. Since $542 \times 3 = 1626$, the ones digit of the result is 6.
   Note that the ones digit of a product only depends on the ones digits of the numbers being multiplied, so we could in fact multiply 2×3 and look at the ones digit of this product.

   Answer: (E)

3. The top, left and bottom unit squares each contribute 3 sides of length 1 to the perimeter.
   The remaining square contributes 1 side of length 1 to the perimeter.
   Therefore, the perimeter is $3 \times 3 + 1 \times 1 = 10$.
   Alternatively, the perimeter includes 3 vertical right sides, 3 vertical left sides, 2 horizontal top sides, and 2 horizontal bottom sides, which means that the perimeter is $3 + 3 + 2 + 2 = 10$.

   Answer: (A)

4. If $3x + 4 = x + 2$, then $3x - x = 2 - 4$ and so $2x = -2$, which gives $x = -1$.

   Answer: (D)

5. *Solution 1*
   10% of 500 is $\frac{1}{10}$ or 0.1 of 500, which equals 50.
   100% of 500 is 500.
   Thus, 110% of 500 equals $500 + 50$, which equals 550.

   Solution 2
   110% of 500 is equal to $\dfrac{110}{100} \times 500 = 110 \times 5 = 550$.

   Answer: (E)

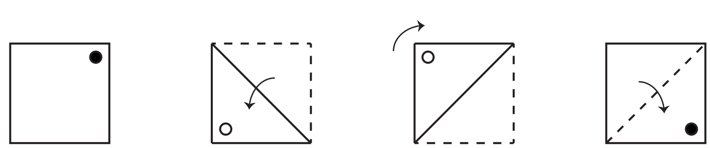
6. Since Eugene swam three times and had an average swim time of 34 minutes, he swam for $3 \times 34 = 102$ minutes in total.
   Since he swam for 30 minutes and 45 minutes on Monday and Tuesday, then on Sunday, he swam for $102 - 30 - 45 = 27$ minutes.

   Answer: (C)

7. If $x = 1$, then $x^2 = 1$ and $x^3 = 1$ and so $x^3 = x^2$.
   If $x > 1$, then x^3 equals x times x^2; since $x > 1$, then x times x^2 is greater than x^2 and so $x^3 > x^2$.
   Therefore, if x is positive with $x^3 < x^2$, we must have $0 < x < 1$. We note that if $0 < x < 1$, then both x, x^2 and x^3 are all positive, and $x^3 = x^2 \times x < x^2 \times 1 = x^2$.
   Of the given choices, only $x = \frac{3}{4}$ satisfies $0 < x < 1$, and so the answer is (B).
   We can verify by direct calculation that this is the only correct answer from the given choices.

   Answer: (B)

8. We draw an unshaded dot to represent the location of the dot when it is on the other side of the sheet of paper being shown. Therefore, the dot moves as follows:

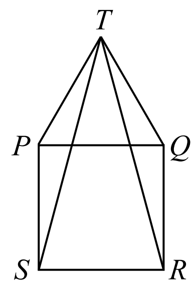


   Answer: (E)

9. Suppose that Janice is x years old now.
Two years ago, Janice was $x - 2$ years old.
In 12 years, Janice will be $x + 12$ years old.
From the given information $x + 12 = 8(x - 2)$ and so $x + 12 = 8x - 16$ which gives $7x = 28$ and so $x = 4$.
Checking, if Janice is 4 years old now, then 2 years ago, she was 2 years old and in 12 years she will be 16 years old; since $16 = 2 \times 8$, this is correct.

ANSWER: (A)

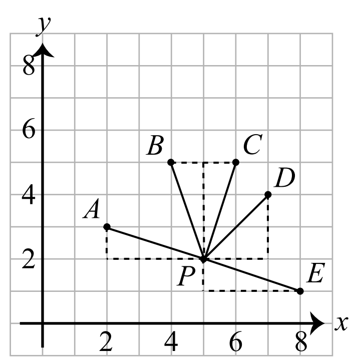
10. Join S to T and R to T.



Since $PQRS$ is a square, $\angle SPQ = 90^\circ$.
Since $\triangle PTQ$ is equilateral, $\angle TPQ = 60^\circ$.
Therefore, $\angle SPT = \angle SPQ + \angle TPQ = 90^\circ + 60^\circ$.
Since $PQRS$ is a square, $SP = PQ$.
Since $\triangle PTQ$ is equilateral, $TP = PQ$.
Since $SP = PQ$ and $TP = PQ$, then $SP = TP$ which means that $\triangle SPT$ is isosceles.
Thus, $\angle PTS = \frac{1}{2}(180^\circ - \angle SPT) = \frac{1}{2}(180^\circ - 150^\circ) = 15^\circ$.
Using a similar argument, we can show that $\angle QTR = 15^\circ$.
This means that $\angle STR = \angle PTQ - \angle PTS - \angle QTR = 60^\circ - 15^\circ - 15^\circ = 30^\circ$.

ANSWER: (D)

11. *Solution 1*
The points A, B, C, and E can each be reached from point P by moving 3 units in either the x- or y-direction and 1 unit in the other direction.
This means that the distance from P to each of these points is $\sqrt{3^2 + 1^2} = \sqrt{10}$, using the Pythagorean Theorem.



Therefore, the distance from P to D must be the distance that is different.

Solution 2
The coordinates of the 6 points are $A(2,3)$, $B(4,5)$, $C(6,5)$, $D(7,4)$, $E(8,1)$, $P(5,2)$.
Therefore, the distances from P to each of the five other points are

$$\begin{aligned} PA &= \sqrt{(5-2)^2+(2-3)^2} = \sqrt{3^2+(-1)^2} = \sqrt{10} \\ PB &= \sqrt{(5-4)^2+(2-5)^2} = \sqrt{1^2+(-3)^2} = \sqrt{10} \\ PC &= \sqrt{(5-6)^2+(2-5)^2} = \sqrt{(-1)^2+(-3)^2} = \sqrt{10} \\ PD &= \sqrt{(5-7)^2+(2-4)^2} = \sqrt{(-2)^2+(-2)^2} = \sqrt{8} \\ PE &= \sqrt{(5-8)^2+(2-1)^2} = \sqrt{(-3)^2+1^2} = \sqrt{10} \end{aligned}$$

This tells us that the distance from P to D is the one distance that is different.

ANSWER: (D)

12. Since $x = 2$ and $y = x^2 - 5$, then $y = 2^2 - 5 = 4 - 5 = -1$.
Since $y = -1$ and $z = y^2 - 5$, then $z = (-1)^2 - 5 = 1 - 5 = -4$.

ANSWER: (E)

13. Since PQR forms a straight angle, then

$$x^\circ + y^\circ + x^\circ + y^\circ + x^\circ = 180^\circ$$

which gives $3x + 2y = 180$.
Since $x + y = 76$ and $2x + 2y + x = 180$, then $2(76) + x = 180$ or $x = 180 - 152 = 28$.

ANSWER: (A)

14. *Solution 1*
To find the x-intercept of the original line, we set $y = 0$ to obtain $0 = 2x - 6$ or $2x = 6$ and so $x = 3$.
When the line is reflected in the y-axis, its x-intercept is reflected in the y-axis to become $x = -3$.

Solution 2
When a line is reflected in the y-axis, its slope changes signs (that is, is multiplied by -1).
When a line is reflected in the y-axis, its y-intercept (which is on the y-axis) does not change.
Thus, when the line with equation $y = 2x - 6$ is reflected in the y-axis, the equation of the resulting line is $y = -2x - 6$.
To find the x-intercept of this line, we set $y = 0$ to obtain $0 = -2x - 6$ or $2x = -6$ and so $x = -3$.

ANSWER: (D)

15. Amy bought and then sold $15n$ avocados.
Since she bought the avocados in groups of 3, she bought $\dfrac{15n}{3} = 5n$ groups of 3 avocados.
Since she paid \$2 for every 3 avocados, she paid $\$2 \times 5n = \$10n$.
Since she sold the avocados in groups of 5, she sold $\dfrac{15n}{5} = 3n$ groups of 5 avocados.
Since she sold every 5 avocados for \$4, she received $\$4 \times 3n = \$12n$.
In terms of n, Amy's profit is $\$12n - \$10n = \$2n$.
Since we know that Amy's profit was \$100, we get $\$2n = \$100$ and so $2n = 100$ or $n = 50$.

ANSWER: (C)

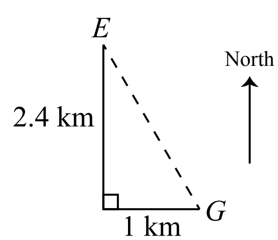
16. Using exponent laws, $3^{x+2} = 3^x \cdot 3^2 = 3^x \cdot 9$.
Since $3^x = 5$, then $3^{x+2} = 3^x \cdot 9 = 5 \cdot 9 = 45$.

ANSWER: (E)

17. We work backwards through the given information.
At the end, there is 1 candy remaining.
Since $\frac{5}{6}$ of the candies are removed on the fifth day, this 1 candy represents $\frac{1}{6}$ of the candies left at the end of the fourth day.
Thus, there were $6 \times 1 = 6$ candies left at the end of the fourth day.
Since $\frac{4}{5}$ of the candies are removed on the fourth day, these 6 candies represent $\frac{1}{5}$ of the candies left at the end of the third day.
Thus, there were $5 \times 6 = 30$ candies left at the end of the third day.
Since $\frac{3}{4}$ of the candies are removed on the third day, these 30 candies represent $\frac{1}{4}$ of the candies left at the end of the second day.
Thus, there were $4 \times 30 = 120$ candies left at the end of the second day.
Since $\frac{2}{3}$ of the candies are removed on the second day, these 120 candies represent $\frac{1}{3}$ of the candies left at the end of the first day.
Thus, there were $3 \times 120 = 360$ candies left at the end of the first day.
Since $\frac{1}{2}$ of the candies are removed on the first day, these 360 candies represent $\frac{1}{2}$ of the candies initially in the bag.
Thus, there were $2 \times 360 = 720$ in the bag at the beginning.

ANSWER: (B)

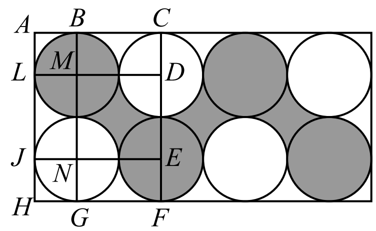
18. Elina and Gustavo start by running and walking for 12 minutes.
Since there are 60 minutes in 1 hour, 12 minutes equals $\frac{1}{5}$ of an hour.
When Elina runs at 12 km/h for $\frac{1}{5}$ of an hour, she runs 12 km/h $\times \frac{1}{5}$ h $= 2.4$ km to the north.
When Gustavo walks at 5 km/h for $\frac{1}{5}$ of an hour, he walks 5 km/h $\times \frac{1}{5}$ h $= 1$ km to the east.
At this point, Elina and Gustavo start to travel directly towards each other.



As they change direction, we use the Pythagorean Theorem to calculate their distance from each other. Using this, we obtain $\sqrt{(2.4 \text{ km})^2 + (1 \text{ km})^2} = 2.6$ km.
Since Elina continues to travel at 12 km/h, Gustavo continues to travel at 5 km/h, and they travel directly towards each other, they close the gap at a rate of 12 km/h+5 km/h = 17 km/h.
Thus, it takes $\dfrac{2.6 \text{ km}}{17 \text{ km/h}} \approx 0.153$ h for them to meet.
Since there are 60 minutes in an hour, 0.153 h is equivalent to roughly 9.18 minutes.
Since Elina and Gustavo leave at 3:00 p.m. and travel for 12 minutes and then for an additional 9 minutes, they meet again at approximately 3:21 p.m.

ANSWER: (E)

19. Each of the four shaded circles has radius 1, and so has area $\pi \cdot 1^2$ which equals π.
Next, we consider one of the three spaces. By symmetry, each of the three spaces has the same area.
Consider the leftmost of the three spaces.
Join the centres of the circles that bound this space to form a quadrilateral. Also, join the centres of these circles to the points where these circles touch the sides of the rectangle.



This quadrilateral is a square with side length 2.
The side length of this quadrilateral is 2, because each of MD, DE, EN, and NM is equal to the sum of lengths of two radii, or 2.
Next, we show that the angles of $DMNE$ are each 90°.
Consider quadrilateral $ABML$. The angle at A is 90°, since the larger shape is a rectangle. The angles at B and L are both 90° since radii are perpendicular to tangents at their points of tangency. Thus, $ABML$ has three 90° angles, which means that its fourth angle must also be 90°.
Now consider quadrilateral $BCDM$. The angles at B and C are both 90°, as above. Since $BM = CD = 1$, this means that $BCMD$ is actually a rectangle.
In a similar way, we can see that $JNGH$ is a square and $MLJN$ is a rectangle.
Finally, we can consider $DMNE$. At M, the three angles outside $DMNE$ are each 90°, which means that the angle at M inside $DMNE$ is 90°.
Similarly, the angle at N inside $DMNE$ is 90°.
Since these angles are both 90° and $MD = NE = 2$, then $DMNE$ is a rectangle. Since its four sides each have length 2, then this rectangle must be a square.

The area of the space between the four circles is thus equal to the area of the square minus the area of the four circular sectors inside the square. In fact, each of these four circular sectors is one-quarter of a circle of radius 1, since its angle at the centre of the circle is 90°.
Thus, the area of the space is equal to $2^2 - 4 \cdot \frac{1}{4} \cdot \pi \cdot 1^2$ which equals $4 - \pi$.
This means that the total area of the shaded region equals $4\pi + 3(4-\pi) = 4\pi + 12 - 3\pi = 12 + \pi$ which is approximately equal to 15.14.
Of the given choices, this is closest to 15.

Answer: (D)

20. An integer is divisible by both 12 and 20 exactly when it is divisible by the least common multiple of 12 and 20.
The first few positive multiples of 20 are 20, 40, 60. Since 60 is divisible by 12 and neither 20 nor 40 is divisible by 12, then 60 is the least common multiple of 12 and 20.
Since $60 \cdot 16 = 960$ and $60 \cdot 17 = 1020$, the smallest four-digit multiple of 60 is $60 \cdot 17$.
Since $60 \cdot 166 = 9960$ and $60 \cdot 167 = 10\,020$, the largest four-digit multiple of 60 is $60 \cdot 166$.
This means that there are $166 - 17 + 1 = 150$ four-digit multiples of 60.
Now, we need to remove the multiples of 60 that are also multiples of 16.
Since the least common multiple of 60 and 16 is 240, we need to remove the four-digit multiples of 240.
Since $240 \cdot 4 = 960$ and $240 \cdot 5 = 1200$, the smallest four-digit multiple of 240 is $240 \cdot 5$.
Since $240 \cdot 41 = 9840$ and $240 \cdot 42 = 10\,080$, the largest four-digit multiple of 240 is $240 \cdot 41$.
This means that there are $41 - 5 + 1 = 37$ four-digit multiples of 240.
Finally, this means that the number of four-digit integers that are multiples of 12 and 20 but are not multiples of 16 is $150 - 37 = 113$.

Answer: (B)

21. We systematically work through pairs of the given integers to see which pairs add up to a third given integer. Starting with the smallest possible pairs, we have:

$$4 + 27 = 31 \qquad 12 + 15 = 27 \qquad 12 + 27 = 39$$

There are no other pairs that add up to a third given integer.
This means that each of the three sums in the problem must be one of the three sums above.
In the sums above, the only integer that appears three times is 27.
In the sums in the problem, the only variable that appears three times is c.
Therefore, $c = 27$.
This also means that the sum $a + b = c$ must be the sum $12 + 15 = 27$.
Since 12 appears in two sums and 15 does not, then $a = 15$ and $b = 12$.
Matching the values that we know already with the equations that we have, we obtain

$$\begin{array}{lcr} a + b = c & \qquad\qquad & 15 + 12 = 27 \\ b + c = d & & 12 + 27 = 39 \\ c + e = f & & 27 + 4 = 31 \end{array}$$

Therefore, $a + c + f = 15 + 27 + 31 = 73$.

Answer: (C)

22. Suppose that the integer in the bottom left corner is n.
In this case, the sum of the integers in the first column is $64 + 70 + n$ or $n + 134$.
Thus, the sum of the integers in each row, in each column, and on each diagonal also equals $n + 134$. (This means that this square is in fact a magic square, since the sum of the numbers in each row, in each column, and on each diagonal is the same.)
Using the top row, the top right integer equals $(n + 134) - 64 - 10$ or $n + 60$.
Using the northeast diagonal, the centre integer equals $(n + 134) - n - (n + 60)$ or $74 - n$.
Using the second row, the middle integer in the right column equals $(n + 134) - 70 - (74 - n)$ or $2n - 10$.

Using the southeast diagonal, the bottom right integer equals $(n + 134) - 64 - (74 - n)$ or $2n - 4$.
Using the third row, the middle integer is $x = (n + 134) - n - (2n - 4) = 138 - 2n$.

64	10	$n + 60$
70	$74 - n$	$2n - 10$
n	$138 - 2n$	$2n - 4$

Using the third column,

$$\begin{aligned}(n + 60) + (2n - 10) + (2n - 4) &= n + 134\\ 5n + 46 &= n + 134\\ 4n &= 88\\ n &= 22\end{aligned}$$

Therefore, $x = 130 - 2n = 130 - 44 = 94$ and the complete grid is

64	10	82
70	52	34
22	94	40

.

ANSWER: (E)

23. Robbie has a score of 8 and Francine has a score of 10 after two rolls each. Thus, in order for Robbie to win (that is, to have a higher total score), his third roll must be at least 3 larger than that of Francine.
If Robbie rolls 1, 2 or 3, his roll cannot be 3 larger than that of Francine.
If Robbie rolls a 4 and wins, then Francine rolls a 1.
If Robbie rolls a 5 and wins, then Francine rolls a 1 or a 2.
If Robbie rolls a 6 and wins, then Francine rolls a 1 or a 2 or a 3.
We now know the possible combinations of rolls that lead to Robbie winning, and so need to calculate the probabilities.
We recall that Robbie and Francine are rolling a special six-sided dice.
Suppose that the probability of rolling a 1 is p.
From the given information, the probability of rolling a 2 is $2p$, of rolling a 3 is $3p$, and of rolling a 4, 5 and 6 is $4p$, $5p$ and $6p$, respectively.
Since the combined probability of rolling a 1, 2, 3, 4, 5, or 6 equals 1, we get the equation $p + 2p + 3p + 4p + 5p + 6p = 1$ which gives $21p = 1$ or $p = \frac{1}{21}$.
Thus, the probability that Robbie rolls a 4 and Francine rolls a 1 equals the product of the probabilities of each of these events, which equals $\frac{4}{21} \cdot \frac{1}{21}$.
Also, the probability that Robbie rolls a 5 and Francine rolls a 1 or 2 equals $\frac{5}{21} \cdot \frac{1}{21} + \frac{5}{21} \cdot \frac{2}{21}$.
Lastly, the probability that Robbie rolls a 6 and Francine rolls a 1, a 2, or a 3 equals

$$\tfrac{6}{21} \cdot \tfrac{1}{21} + \tfrac{6}{21} \cdot \tfrac{2}{21} + \tfrac{6}{21} \cdot \tfrac{3}{21}$$

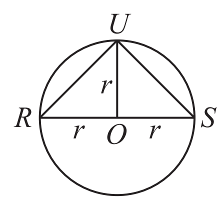
Therefore, the probability that Robbie wins is

$$\tfrac{4}{21} \cdot \tfrac{1}{21} + \tfrac{5}{21} \cdot \tfrac{1}{21} + \tfrac{5}{21} \cdot \tfrac{2}{21} + \tfrac{6}{21} \cdot \tfrac{1}{21} + \tfrac{6}{21} \cdot \tfrac{2}{21} + \tfrac{6}{21} \cdot \tfrac{3}{21} = \tfrac{4+5+10+6+12+18}{21 \cdot 21} = \tfrac{55}{441}$$

which is in lowest terms since $55 = 5 \cdot 11$ and $441 = 3^2 \cdot 7^2$.
Converting to the desired form, we see that $r = 55$ and $s = 41$ which gives $r + s = 96$.

ANSWER: (A)

24. Let O be the centre of the top face of the cylinder and let r be the radius of the cylinder.
We need to determine the value of QT^2.
Since RS is directly above PQ, then RP is perpendicular to PQ.
This means that $\triangle TPQ$ is right-angled at P.
Since PQ is a diameter, then $PQ = 2r$.
By the Pythagorean Theorem, $QT^2 = PT^2 + PQ^2 = n^2 + (2r)^2 = n^2 + 4r^2$.
So we need to determine the values of n and r. We will use the information about QU and UT to determine these values.
Join U to O.
Since U is halfway between R and S, then the arcs RU and US are each one-quarter of the circle that bounds the top face of the cylinder.
This means that $\angle UOR = \angle UOS = 90^\circ$.



We can use the Pythagorean Theorem in $\triangle UOR$ and $\triangle UOS$, which are both right-angled at O, to obtain

$$UR^2 = UO^2 + OR^2 = r^2 + r^2 = 2r^2 \qquad \text{and} \qquad US^2 = 2r^2$$

Since RP and QS are both perpendicular to the top face of the cylinder, we can use the Pythagorean Theorem in $\triangle TRU$ and in $\triangle QSU$ to obtain

$$QU^2 = QS^2 + US^2 = m^2 + 2r^2$$
$$UT^2 = TR^2 + UR^2 = (PR - PT)^2 + 2r^2 = (QS - n)^2 + 2r^2 = (m - n)^2 + 2r^2$$

Since $QU = 9\sqrt{33}$, then $QU^2 = 9^2 \cdot 33 = 2673$.
Since $UT = 40$, then $UT^2 = 1600$.
Therefore,

$$\begin{aligned} m^2 + 2r^2 &= 2673 \\ (m-n)^2 + 2r^2 &= 1600 \end{aligned}$$

Subtracting the second equation from the first, we obtain the equivalent equations

$$\begin{aligned} m^2 - (m-n)^2 &= 1073 \\ m^2 - (m^2 - 2mn + n^2) &= 1073 \\ 2mn - n^2 &= 29 \cdot 37 \\ n(2m - n) &= 29 \cdot 37 \end{aligned}$$

Since m and n are integers, then $2m - n$ is an integer. Thus, n and $2m - n$ are a factor pair of $29 \cdot 37 = 1073$.
Since 29 and 37 are prime numbers, the integer 1073 has only four positive divisors: 1, 29, 37, 1073.
This gives the following possibilities:

n	$2m-n$	$m = \frac{1}{2}((2m-n)+n)$
1	1073	537
29	37	33
37	29	33
1073	1	537

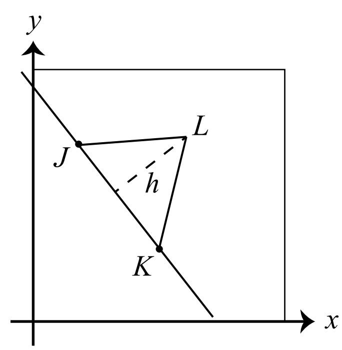
Since $m > n$, then n cannot be 37 or 1073.
Since $QU > QS$, then $m < 9\sqrt{33} \approx 51.7$.
This means that $n = 29$ and $m = 33$.
Since $(m-n)^2 + 2r^2 = 1600$, we obtain $2r^2 = 1600 - (m-n)^2 = 1600 - 4^2 = 1584$ and so

$$QT^2 = n^2 + 4r^2 = 29^2 + 2(2r^2) = 841 + 3168 = 4009$$

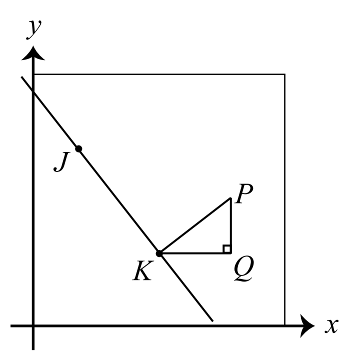
The remainder when QT^2 is divided by 100 is 9.

Answer: (C)

25. The distance between $J(2,7)$ and $K(5,3)$ is equal to $\sqrt{(2-5)^2 + (7-3)^2} = \sqrt{3^2+4^2} = 5$.
Therefore, if we consider $\triangle JKL$ as having base JK and height h, then we want $\frac{1}{2} \cdot JK \cdot h \leq 10$ which means that $h \leq 10 \cdot \frac{2}{5} = 4$.
In other words, $L(r,t)$ can be any point with $0 \leq r \leq 10$ and $0 \leq t \leq 10$ whose perpendicular distance to the line through J and K is at most 4.



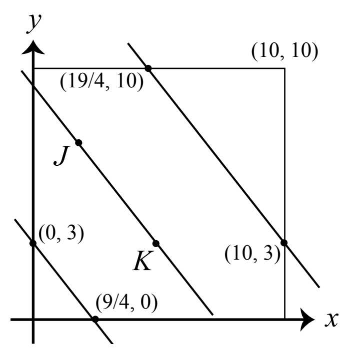
The slope of the line through $J(2,7)$ and $K(5,3)$ is equal to $\frac{7-3}{2-5} = -\frac{4}{3}$.
Therefore, this line has equation $y - 7 = -\frac{4}{3}(x-2)$.
Muliplying through by 3, we obtain $3y - 21 = -4x + 8$ or $4x + 3y = 29$.
We determine the equation of the line above this line that is parallel to it and a perpendicular distance of 4 from it.
The equation of this line will be of the form $4x + 3y = c$ for some real number c, since it is parallel to the line with equation $4x + 3y = 29$.
To determine the value of c, we determine the coordinates of one point on this line.
To determine such a point, we draw a perpendicular of length 4 from K to a point P above the line.
Since JK has slope $-\frac{4}{3}$ and KP is perpendicular to JK, then KP has slope $\frac{3}{4}$.
Draw a vertical line from P and a horizontal line from K, meeting at Q.



Since KP has slope $\frac{3}{4}$, then $PQ : QK = 3 : 4$, which means that $\triangle KQP$ is similar to a 3-4-5 triangle.
Since $KP = 4$, then $PQ = \frac{3}{5}KP = \frac{12}{5}$ and $QK = \frac{4}{3}KP = \frac{16}{5}$.
Thus, the coordinates of P are $\left(5 + \frac{16}{5}, 3 + \frac{12}{5}\right)$ or $\left(\frac{41}{5}, \frac{27}{5}\right)$.
Since P lies on the line with equation $4x + 3y = c$, then

$$c = 4 \cdot \frac{41}{5} + 3 \cdot \frac{27}{5} = \frac{164}{5} + \frac{81}{5} = \frac{245}{5} = 49$$

and so the equation of the line parallel to JK and 4 units above it is $4x + 3y = 49$.
In a similar way, we find that the line parallel to JK and 4 units below it has equation $4x + 3y = 9$. (Note that $49 - 29 = 29 - 9$.)
This gives us the following diagram:



The points L that satisfy the given conditions are exactly the points within the square, below the line $4x+3y = 49$ and above the line $4x+3y = 9$. In other words, the region $\mathcal{R}$ is the region inside the square and between these lines.
To find the area of $\mathcal{R}$, we take the area of the square bounded by the lines $x = 0$, $x = 10$, $y = 0$, and $y = 10$ (this area equals $10 \cdot 10$ or 100) and subtract the area of the two triangles inside the square and not between the lines.
The line with equation $4x + 3y = 9$ intersects the y-axis at $(0, 3)$ (we see this by setting $x = 0$) and the x-axis at $\left(\frac{9}{4}, 0\right)$ (we see this by setting $y = 0$).
The line with equation $4x + 3y = 49$ intersects the line $x = 10$ at $(10, 3)$ (we see this by setting $x = 10$) and the line $y = 10$ at $\left(\frac{19}{4}, 10\right)$ (we see this by setting $y = 10$).
The bottom triangle that is inside the square and outside $\mathcal{R}$ has area $\frac{1}{2} \cdot 3 \cdot \frac{9}{4} = \frac{27}{8}$.
The top triangle that is inside the square and outside $\mathcal{R}$ has horizontal base of length $10 - \frac{19}{4}$ or $\frac{21}{4}$ and vertical height of length $10 - 3$ or 7, and thus has area $\frac{1}{2} \cdot \frac{21}{4} \cdot 7 = \frac{147}{8}$.
Finally, this means that the area of $\mathcal{R}$ is

$$100 - \frac{27}{8} - \frac{147}{8} = 100 - \frac{174}{8} = 100 - \frac{87}{4} = \frac{313}{4}$$

which is in lowest terms since the only divisors of the denominator that are larger than 1 are 2 and 4, while the numerator is odd.

When we write this area in the form $\dfrac{300+a}{40-b}$ where a and b are positive integers, we obtain $a = 13$ and $b = 36$, giving $a + b = 49$.

Answer: (D)

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2020 Cayley Contest

(Grade 10)

Tuesday, February 25, 2020
(in North America and South America)

Wednesday, February 26, 2020
(outside of North America and South America)

Solutions

1. Simplifying, $\frac{20-20}{20+20} = \frac{0}{40} = 0$.

   Answer: (A)

2. When $x = 3$ and $y = 4$, we get $xy - x = 3 \times 4 - 3 = 12 - 3 = 9$.
   Alternatively, $xy - x = x(y - 1) = 3 \times 3 = 9$.

   Answer: (D)

3. Since $OPQR$ is a rectangle with two sides on the axes, then its sides are horizontal and vertical.
   Since PQ is horizontal, the y-coordinate of Q is the same as the y-coordinate of P, which is 3.
   Since QR is vertical, the x-coordinate of Q is the same as the x-coordinate of R, which is 5.
   Therefore, the coordinates of Q are $(5, 3)$.

   Answer: (B)

4. If $0 < a < 20$, then $\frac{1}{a} > \frac{1}{20}$. Therefore, $\frac{1}{15} > \frac{1}{20}$ and $\frac{1}{10} > \frac{1}{20}$.
   Also, $\frac{1}{20} = 0.05$ which is less than both 0.5 and 0.055.
   Lastly, $\frac{1}{20} > \frac{1}{25}$ since $0 < 20 < 25$.
   Therefore, $\frac{1}{25}$ is the only one of the choices that is less than $\frac{1}{20}$.

   Answer: (B)

5. Since QST is a straight angle, then $\angle QSP = 180^\circ - \angle TSP = 180^\circ - 50^\circ = 130^\circ$.
   Now $\angle RQS$ is an exterior angle for $\triangle QSP$.
   This means that $\angle RQS = \angle QSP + \angle SPQ$.
   Using the information that we know, $150^\circ = 130^\circ + x^\circ$ and so $x = 150 - 130 = 20$.
   (Alternatively, we could have noted that $\angle RQS$ and $\angle SQP$ are supplementary and then used the sum of the angles in $\triangle QSP$.)

   Answer: (E)

6. From the bar graph, Matilda saw 6 goldfinches, 9 sparrows, and 5 grackles.
   In total, she saw $6 + 9 + 5 = 20$ birds.
   This means that the percentage of birds that were goldfinches is $\frac{6}{20} \times 100\% = \frac{3}{10} \times 100\% = 30\%$.

   Answer: (C)

7. Since the average of m and n is 5, then $\frac{m+n}{2} = 5$ which means that $m + n = 10$.
   In order for n to be as large as possible, we need to make m as small as possible.
   Since m and n are positive integers, then the smallest possible value of m is 1, which means that the largest possible value of n is $n = 10 - m = 10 - 1 = 9$.

   Answer: (C)

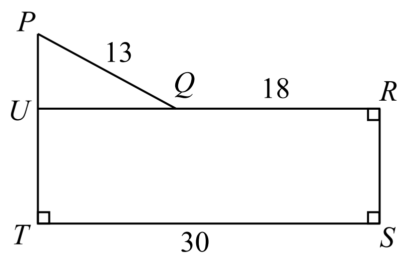
8. To determine 30% of Roman's \$200 prize, we calculate $\$200 \times 30\% = \$200 \times \frac{30}{100} = \$2 \times 30 = \$60$.
   After Roman gives \$60 to Jackie, he has $\$200 - \$60 = \$140$ remaining.
   He splits 15% of this between Dale and Natalia.
   The total that he splits is $\$140 \times 15\% = \$140 \times 0.15 = \$21$.
   Since Roman splits \$21 equally between Dale and Natalia, then Roman gives Dale a total of $\$21 \div 2 = \$10.50$.

   Answer: (A)

9. The 1st row has 0 shaded squares and 1 unshaded square.
The 2nd row has 1 shaded square and 2 unshaded squares.
The 3rd row has 2 shaded squares and 3 unshaded squares.
The 4th row has 3 shaded squares and 4 unshaded squares.
Because each row has 2 more squares than the previous row and the squares in each row alternate betweeen unshaded and shaded, then each row has exactly 1 more shaded square than the previous row.
This means that, moving from the 4th row to the 2020th row, a total of $2020 - 4 = 2016$ additional shaded squares are added.
Thus, the 2020th row has $3 + 2016 = 2019$ shaded squares.

ANSWER: (D)

10. We extend RQ to the left until it meets PT at point U, as shown.



Because quadrilateral $URST$ has three right angles, then it must have four right angles and so is a rectangle.
Thus, $UT = RS$ and $UR = TS = 30$.
Since $UR = 30$, then $UQ = UR - QR = 30 - 18 = 12$.
Now $\triangle PQU$ is right-angled at U.
By the Pythagorean Theorem, since $PU > 0$, we have

$$PU = \sqrt{PQ^2 - UQ^2} = \sqrt{13^2 - 12^2} = \sqrt{169 - 144} = \sqrt{25} = 5$$

Since the perimeter of $PQRST$ is 82, then $13 + 18 + RS + 30 + (UT + 5) = 82$.
Since $RS = UT$, then $2 \times RS = 82 - 13 - 18 - 30 - 5 = 16$ and so $RS = 8$.
Finally, we can calculate the area of $PQRST$ by splitting it into $\triangle PQU$ and rectangle $URST$.
The area of $\triangle PQU$ is $\frac{1}{2} \times UQ \times PU = \frac{1}{2} \times 12 \times 5 = 30$.
The area of rectangle $URST$ is $RS \times TS = 8 \times 30 = 240$.
Therefore, the area of pentagon $PQRST$ is $30 + 240 = 270$.

ANSWER: (E)

11. Since

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

then

$$5 + 10 + 15 + \cdots + 40 + 45 = 5(1 + 2 + 3 + \cdots + 8 + 9) = 5(45) = 225$$

ANSWER: (A)

12. Suppose that the length, width and height of the prism are the positive integers a, b and c.
Since the volume of the prism is 21, then $abc = 21$.
We note that each of a, b and c is a positive divisor of 21.
The positive divisors of 21 are 1, 3, 7, and 21, and the only way to write 21 as a product of three different integers is $1 \times 3 \times 7 = 21$.
Therefore, the length, width and height of the prism must be 1, 3, and 7, in some order.
The sum of these is $1 + 3 + 7 = 11$.

Answer: (A)

13. Since $8 = 2 \times 2 \times 2 = 2^3$, then $8^{20} = (2^3)^{20} = 2^{3\times 20} = 2^{60}$.
Thus, if $2^n = 8^{20}$, then $n = 60$.

Answer: (B)

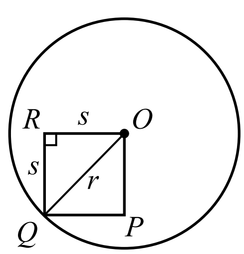
14. Since $3 \times 5 \times 7 = 105$, then the greatest possible value of n is *at least* 105.
In particular, the greatest possible value of n must be positive.
For the product of three numbers to be positive, either all three numbers are positive (that is, none of the numbers is negative) or one number is positive and two numbers are negative. (If there were an odd number of negative factors, the product would be negative.)
If all three numbers are positive, the product is as large as possible when the three numbers are each as large as possible. In this case, the greatest possible value of n is $3 \times 5 \times 7 = 105$.
If one number is positive and two numbers are negative, their product is as large as possible if the positive number is as large as possible (7) and the product of the two negative numbers is as large as possible.
The product of the two negative numbers will be as large as possible when the negative numbers are each "as negative as possible" (that is, as far from 0 as possible). In this case, these numbers are thus -4 and -6 with product $(-4)\times(-6) = 24$. (We can check the other possible products of two negative numbers and see that none is as large.)
So the greatest possible value of n in this case is $7 \times (-4) \times (-6) = 7 \times 24 = 168$.
Combining the two cases, we see that the greatest possible value of n is 168.

Answer: (A)

15. Since the ratio of green marbles to yellow marbles to red marbles is $3 : 4 : 2$, then we can let the numbers of green, yellow and red marbles be $3n$, $4n$ and $2n$ for some positive integer n.
Since 63 of the marbles in the bag are not red, then the sum of the number of green marbles and the number of yellow marbles in the bag is 63.
Thus, $3n + 4n = 63$ and so $7n = 63$ or $n = 9$, which means that the number of red marbles in the bag is $2n = 2 \times 9 = 18$.

Answer: (B)

16. Let s be the side length of the square. Therefore, $OR = RQ = s$.
Let r be the radius of the circle. Therefore, $OQ = r$ since O is the centre of the circle and Q is on the circumference of the circle.
Since the square has a right-angle at each of its vertices, then $\triangle ORQ$ is right-angled at R.



By the Pythagorean Theorem, $OR^2 + RQ^2 = OQ^2$ and so $s^2 + s^2 = r^2$ or $2s^2 = r^2$.
In terms of r, the area of the circle is πr^2.
Since we are given that the area of the circle is 72π, then $\pi r^2 = 72\pi$ or $r^2 = 72$.
Since $2s^2 = r^2 = 72$, then $s^2 = 36$.
In terms of s, the area of the square is s^2, so the area of the square is 36.

ANSWER: (E)

17. Suppose that Carley buys x boxes of chocolates, y boxes of mints, and z boxes of caramels.
In total, Carley will then have $50x$ chocolates (since there are 50 chocolates in a box of chocolates), $40y$ mints (since there are 40 mints in a box of mints), and $25z$ caramels (since there are 25 caramels in a box of caramels).
Since the contents of each bag was the same and Carley made no incomplete treat bags and there were no left-over candies, then it must be the case that $50x = 40y = 25z$.
We want to find the minimum possible positive value of $x + y + z$ given this condition.
Dividing by the common factor of 5, the equation $50x = 40y = 25z$ becomes $10x = 8y = 5z$.
Since $10x$ is a multiple of 10 and $8y$ is a multiple 8 and $10x = 8y$, we look for the smallest multiple of 10 which is also a multiple of 8.
Since 10, 20 and 30 are not multiples of 8, and 40 is a multiple of 8, then the smallest possible value of $10x$ appears to be 40.
In this case, $x = 4$, $y = 5$ and $z = 8$ gives $10x = 8y = 5z = 40$, and these are the smallest positive integers that create this equality.
Since x, y and z are each the smallest possible, then their sum $x + y + z$ is also the smallest possible.
Thus, the minimum number of boxes that Carley could have bought is $4 + 5 + 8 = 17$.

ANSWER: (B)

18. *Solution 1*
Suppose that when Nate arrives on time, his drive takes t hours.
When Nate arrives 1 hour early, he arrives in $t - 1$ hours.
When Nate arrives 1 hour late, he arrives in $t + 1$ hours.
Since the distance that he drives is the same in either case and distance equals speed multiplied by time, then $(60 \text{ km/h}) \times ((t - 1) \text{ h}) = (40 \text{ km/h}) \times ((t + 1) \text{ h})$.
Expanding, we obtain $60t - 60 = 40t + 40$ and so $20t = 100$ or $t = 5$.
The total distance that Nate drives is thus $(60 \text{ km/h}) \times (4 \text{ h}) = 240$ km.
When Nate drives this distance in 5 hours at a constant speed, he should drive at $\dfrac{240 \text{ km}}{5 \text{ h}}$ which equals 48 km/h.

Solution 2
Suppose that the distance that Nate drives is d km.
Since driving at 40 km/h causes Nate to arrive 1 hour late and driving at 60 km/h causes Nate to arrive 1 hour early, then the difference between the lengths of time at these two speeds is 2 hours.
Since time equals distance divided by speed, then

$$\frac{d \text{ km}}{40 \text{ km/h}} - \frac{d \text{ km}}{60 \text{ km/h}} = 2 \text{ h}$$

Multiplying both sides of the equation by 120 km/h, we obtain

$$3d \text{ km} - 2d \text{ km} = 240 \text{ km}$$

which gives us $d = 240$.
Thus, the distance that Nate drives is 240 km.
At 40 km/h, the trip takes 6 hours and Nate arrives 1 hour late.
To arrive just in time, it should take 5 hours.
To drive 240 km in 5 hours, Nate should drive at a constant speed of $\dfrac{240 \text{ km}}{5 \text{ h}} = 48$ km/h.

Answer: (D)

19. For each of the 10 questions, each correct answer is worth 5 points, each unanswered question is worth 1 point, and each incorrect answer is worth 0 points.
If 10 of 10 questions are answered correctly, the total score is $10 \times 5 = 50$ points.
If 9 of 10 questions are answered correctly, either 0 or 1 questions can be unanswered. This means that the total score is either $9 \times 5 = 45$ points or $9 \times 5 + 1 = 46$ points.
If 8 of 10 questions are answered correctly, either 0 or 1 or 2 questions can be unanswered. This means that the total score is either $8 \times 5 = 40$ points or $8 \times 5 + 1 = 41$ points or $8 \times 5 + 2 = 42$ points.
If 7 of 10 questions are answered correctly, either 0 or 1 or 2 or 3 questions can be unanswered. This means that the total score is one of 35, 36, 37, or 38 points.
If 6 of 10 questions are answered correctly, either 0 or 1 or 2 or 3 or 4 questions can be unanswered. This means that the total score is one of 30, 31, 32, 33, or 34 points.
So far, we have seen that the following point totals between 30 and 50, inclusive, are possible:

$$30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 45, 46, 50$$

which means that

$$39, 43, 44, 47, 48, 49$$

are not possible.
If 5 or fewer questions are answered correctly, is it possible to obtain a total of at least 39 points?
The answer is no, because in this case, the number of correct answers is at most 5 and the number of unanswered questions is at most 10 (these both can't happen at the same time) which together would give at most $5 \times 5 + 10 = 35$ points.
Therefore, there are exactly 6 integers between 30 and 50, inclusive, that are not possible total scores.

Answer: (D)

20. We determine when $3^m + 7^n$ is divisible by 10 by looking at the units (ones) digits of $3^m + 7^n$. To do this, we first look individually at the units digits of 3^m and 7^n.

The units digits of powers of 3 cycle $3, 9, 7, 1, 3, 9, 7, 1, \ldots$.
To see this, we note that the first few powers of 3 are

$$3^1 = 3 \quad 3^2 = 9 \quad 3^3 = 27 \quad 3^4 = 81 \quad 3^5 = 243 \quad 3^6 = 729$$

Since the units digit of a product of integers depends only on the units digits of the integers being multiplied and we multiply by 3 to get from one power to the next, then once a units digit recurs in the sequence of units digits, the following units digits will follow the same pattern.
This means that the units digits of powers of 3 cycle every four powers of 3.
Therefore, of the 100 powers of 3 of the form 3^m with $1 \leq m \leq 100$, exactly 25 will have a units digit of 3, exactly 25 will have a units digit of 9, exactly 25 will have a units digit of 7, and exactly 25 will have a units digit of 1.

The units digits of powers of 7 cycle $7, 9, 3, 1, 7, 9, 3, 1, \ldots$.
To see this, we note that the first few powers of 7 are

$$7^1 = 7 \quad 7^2 = 49 \quad 7^3 = 343 \quad 7^4 = 2401 \quad 7^5 = 16\,807 \quad 7^6 = 117\,649$$

Using the same argument as above, the units digits of powers of 7 cycle every four powers of 7.
Since 101 is 1 more than a multiple of 4, then the power 7^{101} is at the beginning of one of these cycles, and so the units digit of 7^{101} is a 7.
Therefore, of the 105 powers of 7 of the form 7^n with $101 \leq n \leq 205$, exactly 27 will have a units digit of 7, exactly 26 will have a units digit of 9, exactly 26 will have a units digit of 3, and exactly 26 will have a units digit of 1. (Here, 105 powers include 26 complete cycles of 4 plus one additional term.)

For $3^m + 7^n$ to have a units digit of 0 (and thus be divisible by 10), one of the following must be true:

- the units digit of 3^m is 3 (25 possible values of m) and the units digit of 7^n is 7 (27 possible values of n), or
- the units digit of 3^m is 9 (25 possible values of m) and the units digit of 7^n is 1 (26 possible values of n), or
- the units digit of 3^m is 7 (25 possible values of m) and the units digit of 7^n is 3 (26 possible values of n), or
- the units digit of 3^m is 1 (25 possible values of m) and the units digit of 7^n is 9 (26 possible values of n).

The number of possible pairs (m, n) is therefore

$$27 \times 25 + 26 \times 25 + 26 \times 25 + 26 \times 25 = 25 \times (27 + 26 + 26 + 25) = 25 \times 105 = 2625$$

ANSWER: (E)

21. To determine the number of points (x, y) on the line with equation $y = 4x + 3$ that lie inside this region, we determine the number of integers x with $25 \leq x \leq 75$ that have the property that $y = 4x + 3$ is an integer between 120 and 250.
In other words, we determine the number of integers x with $25 \leq x \leq 75$ for which $120 \leq 4x + 3 \leq 250$ and is an integer.
We note that, as x increases, the value of the expression $4x + 3$ increases.
Also, when $x = 29$, we get $4x + 3 = 119$, and when $x = 30$, we get $4x + 3 = 123$.
Further, when $x = 61$, we get $4x + 3 = 247$, and when $x = 62$, we get $4x + 3 = 251$.
Therefore, $4x + 3$ is between 120 and 250 exactly when $30 \leq x \leq 61$.
There are $61 - 30 + 1 = 32$ such values of x, and so there are 32 points that satisfy the given conditions.

Answer: (D)

22. Since $PT = 1$ and $TQ = 4$, then $PQ = PT + TQ = 1 + 4 = 5$.
$\triangle PSQ$ is right-angled at S and has hypotenuse PQ.
We can thus apply the Pythagorean Theorem to obtain $PS^2 = PQ^2 - QS^2 = 5^2 - 3^2 = 16$.
Since $PS > 0$, then $PS = 4$.
Consider $\triangle PSQ$ and $\triangle RTQ$.
Each is right-angled and they share a common angle at Q. Thus, these two triangles are similar.
This tells us that $\dfrac{PQ}{QS} = \dfrac{QR}{TQ}$.
Using the lengths that we know, $\dfrac{5}{3} = \dfrac{QR}{4}$ and so $QR = \dfrac{4 \cdot 5}{3} = \dfrac{20}{3}$.
Finally, $SR = QR - QS = \dfrac{20}{3} - 3 = \dfrac{11}{3}$.

Answer: (B)

23. Suppose that N is an integer that satisfies the given properties.
The first digit of N must be a 1, since there must be at least one 1 before the first 2, at least one 2 before the first 3, and at least one 3 before the 4, which means that we cannot have a 2, a 3, or a 4 before the first 1.
Since there are three 1s in N, then N can begin with 1, 11 or 111.
The first digit of N that is not a 1 must be a 2.
Thus, N begins 12, 112, or 1112.

Case 1: N begins 12
Since no two 2s can be next to each other, we next place the remaining two 2s.
Reading from the left, these 2s can go in positions 4 and 6, 4 and 7, 4 and 8, 4 and 9, 5 and 7, 5 and 8, 5 and 9, 6 and 8, 6 and 9, or 7 and 9.
In other words, there are 10 possible pairs of positions in which the 2s can be placed.
Once the 2s are placed, there are 5 positions left open.
Next, we place the remaining two 1s.
If we call these 5 positions A, B, C, D, E, we see that there are 10 pairs of positions in which the 1s can be placed: A and B, A and C, A and D, A and E, B and C, B and D, B and E, C and D, C and E, D and E.
This leaves 3 positions in which the two 3s and one 4 must be placed. Reading from the left, a 3 must be placed in the first empty position, since there must be a 3 before the 4.
This leaves 2 positions, in which the remaining digits (3 and 4) can be placed in any order; there are 2 such orders (3 and 4, or 4 and 3).
In this case, there are $10 \times 10 \times 2 = 200$ possible integers N.

Case 2: N begins 112
Since no two 2s can be next to each other, we next place the remaining 2s.
Reading from the left, these 2s can go in positions 5 and 7, 5 and 8, 5 and 9, 6 and 8, 6 and 9, or 7 and 9.
In other words, there are 6 possible pairs of positions in which the 2s can be placed.
Once the 2s are placed, there are 4 positions left open.
Next, we place the remaining 1. There are 4 possible positions in which the 1 can be placed.
This leaves 3 positions in which the two 3s and one 4 must be placed. Reading from the left, a 3 must be placed in the first empty position, since there must be a 3 before the 4.
This leaves 2 positions, in which the remaining digits (3 and 4) can be placed in any order; there are 2 such orders.
In this case, there are $6 \times 4 \times 2 = 48$ possible integers N.

Case 3: N begins 1112
Since no two 2s can be next to each other, we next place the remaining 2s.
Reading from the left, these 2s can go in positions 6 and 8, 6 and 9, or 7 and 9.
In other words, there are 3 possible pairs of positions in which the 2s can be placed.
Once the 2s are placed, there are 3 positions left open with only the two 3s and the 4 remaining to be placed.
Reading from the left, a 3 must be placed in the first empty position, since there must be a 3 before the 4.
This leaves 2 positions, in which the remaining digits (3 and 4) can be placed in any order; there are 2 such orders.
In this case, there are $3 \times 2 = 6$ possible integers N.

Combining the three cases, we see that there are $200 + 48 + 6 = 254$ possible integers N.

ANSWER: (C)

24. Suppose that $GP = x$.
Since the edge length of the cube is 200, then $HP = 200 - x$.
We consider tetrahedron (that is, triangle-based pyramid) $FGMP$ and calculate its volume in two different ways.
The volume of a tetrahedron is equal to one-third times the area of its triangular base times the length of its perpendicular height.
First, we consider tetrahedron $FGMP$ as having base $\triangle FGM$ and height GP, which is perpendicuar to the base.
$\triangle FGM$ is right-angled at G and has $FG = GM = 200$, so its area is $\frac{1}{2} \times FG \times GM$ which equals $\frac{1}{2} \times 200 \times 200$ which equals $20\,000$.
Thus, the volume of $FGMP$ is $\frac{1}{3} \times 20\,000 \times x$.

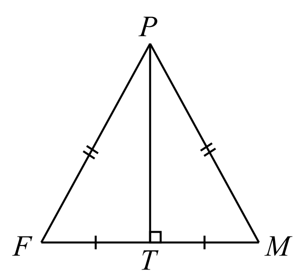
Next, we consider tetrahedron $FGMP$ as having base $\triangle PFM$.
From the given information, the shortest distance from G to a point inside this triangle is 100.
This means that the height of tetrahedron $FGMP$ when considered to have base $\triangle PFM$ is 100.
We need to calcualte the area of $\triangle PFM$.
Since $\triangle FGM$ is right-angled at G and $FM > 0$, then by the Pythagorean Theorem,

$$FM = \sqrt{FG^2 + GM^2} = \sqrt{200^2 + 200^2} = \sqrt{200^2 \times 2} = 200\sqrt{2}$$

Since $\triangle FGP$ is right-angled at G and $FP > 0$, then by the Pythagorean Theorem,

$$FP = \sqrt{FG^2 + GP^2} = \sqrt{200^2 + x^2} = \sqrt{x^2 + 40\,000}$$

Similarly, $MP = \sqrt{x^2 + 40\,000}$.
This means that $\triangle PFM$ is isosceles with $FP = MP$.
Let T be the midpoint of FM.
Then $FT = TM = 100\sqrt{2}$.
Since $\triangle PFM$ is isosceles, then PT is perpendicular to FM.



By the Pythagorean Theorem,

$$PT = \sqrt{FP^2 - FT^2} = \sqrt{\left(\sqrt{x^2 + 40\,000}\right)^2 - \left(100\sqrt{2}\right)^2} = \sqrt{x^2 + 40\,000 - 20\,000} = \sqrt{x^2 + 20\,000}$$

Therefore, the area of $\triangle PFM$ is $\frac{1}{2} \times FM \times PT$ which equals $\frac{1}{2} \times 200\sqrt{2} \times \sqrt{x^2 + 20\,000}$.
This means that the volume of tetrahedron $FGMP$ is equal to

$$\frac{1}{3} \times \left(\frac{1}{2} \times 200\sqrt{2} \times \sqrt{x^2 + 20\,000}\right) \times 100$$

We can now equate the two expressions for the volume of $FGMP$ to solve for x:

$$\begin{aligned}
\tfrac{1}{3} \times 20\,000 \times x &= \tfrac{1}{3} \times \left(\tfrac{1}{2} \times 200\sqrt{2} \times \sqrt{x^2 + 20\,000}\right) \times 100 \\
20\,000 \times x &= \left(\tfrac{1}{2} \times 200\sqrt{2} \times \sqrt{x^2 + 20\,000}\right) \times 100 \\
x &= \tfrac{1}{2} \times \sqrt{2} \times \sqrt{x^2 + 20\,000} \\
2x &= \sqrt{2} \times \sqrt{x^2 + 20\,000} \\
4x^2 &= 2(x^2 + 20\,000) \\
2x^2 &= 40\,000 \\
x^2 &= 20\,000
\end{aligned}$$

Since $x > 0$, then $x = \sqrt{20\,000} = \sqrt{10\,000 \times 2} = \sqrt{100^2 \times 2} = 100\sqrt{2}$.
This means that $HP = 200 - x = 200 - 100\sqrt{2} \approx 58.58$.
Of the given answers, this is closest to 59, which is (D).

Answer: (D)

25. We will use the result that if a positive integer N has prime factorization $N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ for some distinct prime numbers $p_1, p_2, \ldots, p_k$ and positive integers $a_1, a_2, \ldots, a_k$, then N has exactly $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$ positive divisors.

This result is based on the following facts:

F1. Every positive integer greater than 1 can be written as a product of prime numbers in a unique way. (If the positive integer is itself prime, this product consists of only the prime number.) This fact is called the "Fundamental Theorem of Arithmetic". This fact is often seen implicitly in finding a "factor tree" for a given integer. For example, 1500 is equal to $2^2 \times 3^1 \times 5^3$ and there is no other way of writing 1500 as a product of prime numbers. Note that rearranging the same prime factors in a different order does not count as a different factorization.

F2. If n is a positive integer and d is a positive integer that is a divisor of n, then the only possible prime factors of d are the prime factors of n. For example, if d is a positive divisor of $n = 1500$, then the only possible prime factors of d are 2, 3 and 5. This means, for example, that d cannot be divisible by 7 or by 11 or by any other prime number not equal to 2, 3 or 5. d might or might not be divisible by each of 2, 3 or 5.

F3. If n is a positive integer, d is a positive integer that is a divisor of n, and p is a prime factor of both n and d, then p cannot divide d "more times" than it divides n. For example, if d is a positive divisor of $n = 1500 = 2^2 \times 3^1 \times 5^3$ that is divisible by 5, then d can be divisible by 5 or by 5^2 or by 5^3 but cannot be divisible by 5^4 or by 5^5 or by any larger power of 5.

From these facts, the positive divisors of $N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ are the integers of the form

$$d = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$$

where $b_1, b_2, \ldots, b_k$ are non-negative integers with $0 \le b_1 \le a_1$, $0 \le b_2 \le a_2$, and so on.
This means that there are $a_1 + 1$ possible values for b_1, namely $0, 1, 2, \ldots, a_1$.
Similarly, there are $a_2 + 1$ possible values for b_2, $a_3 + 1$ possible values for b_3, and so on.
Since every combination of these possible values gives a different divisor d, then there are $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$ positive divisors.

Suppose that n has prime factorization $2^r 5^s p_3^{a_3} p_4^{a_4} \cdots p_k^{a_k}$ for some distinct prime numbers $p_3, p_4, \ldots, p_k$ none of which equal 2 or 5, for some positive integers $a_3, a_4, \ldots, a_k$, and some non-negative integers r and s.
We have written n this way to allow us to pay special attention to the possible prime factors of 2 and 5.
This means that

$$2n = 2^{r+1} 5^s p_3^{a_3} p_4^{a_4} \cdots p_k^{a_k}$$
$$5n = 2^r 5^{s+1} p_3^{a_3} p_4^{a_4} \cdots p_k^{a_k}$$

Since $2n$ has 64 positive divisors and $5n$ has 60 positive divisors, then

$$(r+2)(s+1)(a_3+1)(a_4+1)\cdots(a_k+1) = 64$$
$$(r+1)(s+2)(a_3+1)(a_4+1)\cdots(a_k+1) = 60$$

Since every factor in the two expressions on the left is a positive integer, then

$$(a_3+1)(a_4+1)\cdots(a_k+1)$$

is a positive common divisor of 64 and of 60.
The positive divisors of 64 are 1, 2, 4, 8, 16, 32, 64.
Of these, only 1, 2, 4 are divisors of 60.
Therefore, $(a_3+1)(a_4+1)\cdots(a_k+1)$ equals 1, 2 or 4.
Since each of $a_3, a_4, \ldots, a_k$ is a positive integer, then each of $a_3+1, a_4+1, \ldots, a_k+1$ is at least 2.

Case 1: $(a_3+1)(a_4+1)\cdots(a_k+1) = 4$

The only ways in which 4 can be written as a product of positive integers each at least 2 are 2×2 and 4 (a product of one integer is itself).
Thus, either $k=4$ with $a_3+1 = a_4+1 = 2$ (giving $a_3 = a_4 = 1$), or $k=3$ with $a_3+1=4$ (giving $a_3=3$).
Since

$$(r+2)(s+1)(a_3+1)(a_4+1)\cdots(a_k+1) = 64$$
$$(r+1)(s+2)(a_3+1)(a_4+1)\cdots(a_k+1) = 60$$

then, after simplification, we have

$$(r+2)(s+1) = 16$$
$$(r+1)(s+2) = 15$$

Expanding the left sides of the two equations, we obtain

$$rs + r + 2s + 2 = 16$$
$$rs + 2r + s + 2 = 15$$

Subtracting the second of these equations from the first, we obtain $-r+s=1$ and so $s=r+1$.
Substituting into the equation $(r+2)(s+1)=16$, we obtain $(r+2)(r+2)=16$.
Since $r>0$, then $(r+2)^2 = 16$ gives $r+2=4$ and so $r=2$, which gives $s=3$.
Therefore, we could have $r=2$, $s=3$.

Combining with the possible values of a_3 and a_4, this means that we could have $n = 2^2 5^3 p_3 p_4$ for some primes p_3 and p_4 not equal to 2 or 5, or $n = 2^2 5^3 p_3^3$ for some prime p_3 not equal to 2 or 5.
We can verify that $2n$ and $5n$ have the correct number of positive divisors in each case.

Case 2: $(a_3+1)(a_4+1)\cdots(a_k+1) = 2$
The only way in which 2 can be written as a product of positive integers each at least 2 is 2.
Thus, $k = 3$ with $a_3 + 1 = 2$ (giving $a_3 = 1$).
Since

$$(r+2)(s+1)(a_3+1)(a_4+1)\cdots(a_k+1) = 64$$
$$(r+1)(s+2)(a_3+1)(a_4+1)\cdots(a_k+1) = 60$$

then

$$(r+2)(s+1) = 32$$
$$(r+1)(s+2) = 30$$

We could proceed as in Case 1.
Alternatively, knowing that r and s are non-negative integers, then the possibilities from the first equation are

$r+2$	$s+1$	r	s	$r+1$	$s+2$	$(r+2)(s+1)$
32	1	30	0	31	2	62
16	2	14	1	15	3	45
8	4	6	3	7	5	35
4	8	2	7	3	9	27
2	16	0	15	1	17	17
1	32	-1	31	0	33	0

There are no values of r and s that work in this case.

Case 3: $(a_3+1)(a_4+1)\cdots(a_k+1) = 1$
Since each factor on the left side is supposedly at least 2, what can this mean? This actually means that there are no factors on the left side. In other words, $k = 2$ and $n = 2^r 5^s$.
(See if you can follow the argument before Case 1 through to verify that there are no contradictions.)
Here,

$$(r+2)(s+1) = 64$$
$$(r+1)(s+2) = 60$$

Knowing that r and s are non-negative integers, then the possibilities from the first equation are

$r+2$	$s+1$	r	s	$r+1$	$s+2$	$(r+2)(s+1)$
64	1	62	0	63	2	126
32	2	30	1	31	3	93
16	4	14	3	15	5	45
8	8	6	7	7	9	63
4	16	2	13	3	15	51
2	32	0	31	1	33	33
1	64	-1	63	0	65	0

There are no values of r and s that work in this case.

Therefore, combining the results of the three cases, the positive integer n satisfies the given conditions exactly when

- $n = 2^2 5^3 p_3 p_4 = 500 p_3 p_4$ for some primes p_3 and p_4 not equal to 2 or 5, or
- $n = 2^2 5^3 p_3^3 = 500 p_3^3$ for some prime p_3 not equal to 2 or 5.

Since $n \leq 20\,000$, then either

- $500 p_3 p_4 \leq 20\,000$ which gives $p_3 p_4 \leq 40$, or
- $500 p_3^3 \leq 20\,000$ which gives $p_3^3 \leq 40$.

It remains to determine the number of pairs of primes p_3 and p_4 that are not equal to 2 or 5 with product less than 40, and the number of primes p_3 that are not equal to 2 or 5 whose cube is less than 40.
In the first case, the possibilities are:

$$3 \times 7 = 21 \qquad 3 \times 11 = 33 \qquad 3 \times 13 = 39$$

The order of p_3 and p_4 does not matter as switching the order gives the same value for n. We note as well that we cannot have both p_3 and p_4 at least 7 and have their product at most 40.
In the second case, the only possibility is $p_3^3 = 3^3$.

This means that there are 4 possible values of n that satisfy the given conditions.

Answer: (A)

The CENTRE for EDUCATION
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2019 Cayley Contest

(Grade 10)

Tuesday, February 26, 2019
(in North America and South America)

Wednesday, February 27, 2019
(outside of North America and South America)

Solutions

1. Evaluating, $2 \times 0 + 1 - 9 = 0 + 1 - 9 = -8$.

   Answer: (A)

2. Kai was born 25 years before 2020 and so was born in the year $2020 - 25 = 1995$.

   Answer: (C)

3. Since 38% of students received a muffin, then $100\% - 38\% = 62\%$ of students did not receive a muffin.
   Alternatively, using the percentages of students who received yogurt, fruit or a granola bar, we see that $10\% + 27\% + 25\% = 62\%$ did not receive a muffin.

   Answer: (D)

4. Re-arranging the order of the numbers being multiplied,

$$(2 \times \tfrac{1}{3}) \times (3 \times \tfrac{1}{2}) = 2 \times \tfrac{1}{2} \times 3 \times \tfrac{1}{3} = (2 \times \tfrac{1}{2}) \times (3 \times \tfrac{1}{3}) = 1 \times 1 = 1$$

   Answer: (C)

5. Since $10d + 8 = 528$, then $10d = 520$ and so $\dfrac{10d}{5} = \dfrac{520}{5}$ which gives $2d = 104$.

   Answer: (A)

6. The line with equation $y = x + 4$ has a y-intercept of 4.
   When the line is translated 6 units downwards, all points on the line are translated 6 units down.
   This moves the y-intercept from 4 to $4 - 6 = -2$.

   Answer: (E)

7. Since the average of 2, x and 10 is x, then $\dfrac{2 + x + 10}{3} = x$.
   Multiplying by 3, we obtain $2 + x + 10 = 3x$.
   Re-arranging, we obtain $x + 12 = 3x$ and then $2x = 12$ which gives $x = 6$.

   Answer: (E)

8. To get from P to A, Alain travels 5 units right and 4 units up, for a total distance of $5 + 4 = 9$ units. (Any path from P to A that only moves right and up will have this same length.)
   To get from P to B, Alain travels 6 units right and 2 units up, for a total distance of 8 units.
   To get from P to C, Alain travels 3 units right and 3 units up, for a total distance of 6 units.
   To get from P to D, Alain travels 5 units right and 1 unit up, for a total distance of 6 units.
   To get from P to E, Alain travels 1 unit right and 4 units up, for a total distance of 5 units.
   Therefore, the shortest path is from P to E.

   Answer: (E)

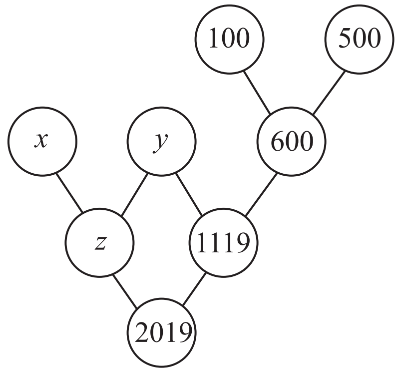
9. Since $(pq)(qr)(rp) = 16$, then $pqqrrp = 16$ or $ppqqrr = 16$ which gives $p^2q^2r^2 = 16$.
   Thus, $(pqr)^2 = 16$ and so $pqr = \pm 4$.
   Using the given answers, pqr is positive and so $pqr = 4$.

   Answer: (C)

10. Matilda and Ellie each take $\frac{1}{2}$ of the wall.
    Matilda paints $\frac{1}{2}$ of her half, or $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ of the entire wall.
    Ellie paints $\frac{1}{3}$ of her half, or $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ of the entire wall.
    Therefore, $\frac{1}{4} + \frac{1}{6} = \frac{3}{12} + \frac{2}{12} = \frac{5}{12}$ of the wall is painted red.

    Answer: (A)

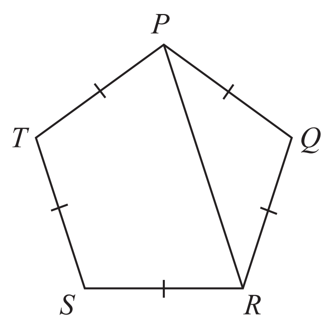
11. We let the values in the two unlabelled circles be y and z, as shown.



From the given rules, $y + 600 = 1119$ and so $y = 519$.
Also, $z + 1119 = 2019$ and so $z = 900$.
Finally, $x + y = z$ and so $x = z - y = 900 - 519 = 381$.

ANSWER: (B)

12. Join P to R.



Since $PQRST$ is a regular pentagon, then $\angle PQR = \angle QRS = 108^\circ$.
Since $PQ = QR$, then $\triangle PQR$ is isosceles with $\angle QPR = \angle QRP$.
Since $\angle PQR = 108^\circ$, then

$$\begin{aligned} \angle PQR + \angle QPR + \angle QRP &= 180^\circ \\ 108^\circ + 2\angle QRP &= 180^\circ \\ 2\angle QRP &= 72^\circ \\ \angle QRP &= 36^\circ \end{aligned}$$

Since $\angle QRS = 108^\circ$, then $\angle PRS = \angle QRS - \angle QRP = 108^\circ - 36^\circ = 72^\circ$.

ANSWER: (A)

13. From the ones column, we see that $3 + 2 + q$ must have a ones digit of 2.
Since q is between 1 and 9, inclusive, then $3 + 2 + q$ is beween 6 and 14.
Since its ones digit is 2, then $3 + 2 + q = 12$ and so $q = 7$.
This also means that there is a carry of 1 into the tens column.
From the tens column, we see that $1 + 6 + p + 8$ must have a ones digit of 4.
Since p is between 1 and 9, inclusive, then $1 + 6 + p + 8$ is beween 16 and 24.
Since its ones digit is 4, then $1 + 6 + p + 8 = 24$ and so $p = 9$.

This also means that there is a carry of 2 into the hundreds column.
From the hundreds column, we see that $2 + n + 7 + 5$ must have a ones digit of 0.
Since n is between 1 and 9, inclusive, then $2 + n + 7 + 5$ is beween 15 and 23.
Since its ones digit is 0, then $2 + n + 7 + 5 = 20$ and so $n = 6$.
This also means that there is a carry of 2 into the thousands column.
This means that $m = 2$.
This gives

$$\begin{array}{rr} & 6\,6\,3 \\ & 7\,9\,2 \\ + & 5\,8\,7 \\ \hline & 2\,0\,4\,2 \end{array}$$

Thus, we have $m + n + p + q = 2 + 6 + 9 + 7 = 24$.

ANSWER: (B)

14. Each letter A, B, C, D, E appears exactly once in each column and each row.
The entry in the first column, second row cannot be A or E or B (the entries already present in that column) and cannot be C or A (the entries already present in that row).
Therefore, the entry in the first column, second row must be D.
This means that the entry in the first column, fourth row must be C.
The entry in the fifth column, second row cannot be D or C or A or E and so must be B.
This means that the entry in the second column, second row must be E.
Using similar arguments, the entries in the first row, third and fourth columns must be D and B, respectively.
This means that the entry in the second column, first row must be C.
Using similar arguments, the entries in the fifth row, second column must be A.
Also, the entry in the third row, second column must be D.
This means that the letter that goes in the square marked with $*$ must be B.
We can complete the grid as follows:

A	C	D	B	E
D	E	C	A	B
E	D	B	C	A
C	B	A	E	D
B	A	E	D	C

ANSWER: (B)

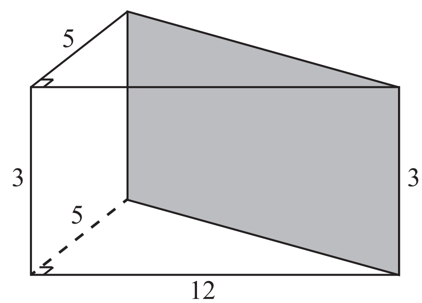
15. The slope of line segment PR is $\dfrac{2-1}{0-4}$ which equals $-\dfrac{1}{4}$.
Since $\angle QPR = 90^\circ$, then PQ and PR are perpendicular.
This means that the slopes of PQ and PR have a product of -1.
Since the slope of PR is $-\dfrac{1}{4}$, then the slope of PQ is 4.
Since the "run" of PQ is $2 - 0 = 2$, then the "rise" of PQ must be $4 \times 2 = 8$.
Thus, $s - 2 = 8$ and so $s = 10$.

ANSWER: (C)

16. Suppose that there are p people behind Kaukab.
This means that there are $2p$ people ahead of her.
Including Kaukab, the total number of people in line is $n = p + 2p + 1 = 3p + 1$, which is one more than a multiple of 3.
Of the given choices (23, 20, 24, 21, 25), the only one that is one more than a multiple of 3 is 25, which equals $3 \times 8 + 1$.
Therefore, a possible value for n is 25.

Answer: (E)

17. Consider the triangular-based prism on the front of the rectangular prism.
This prism has five faces: a rectangle on the front, a rectangle on the left, a triangle on the bottom, a triangle on the top, and a rectangle on the back.



The rectangle on the front measures 3×12 and so has area 36.
The rectangle on the left measures 3×5 and so has area 15.
The triangles on the top and bottom each are right-angled and have legs of length 5 and 12. This means that each has area $\frac{1}{2} \times 12 \times 5 = 30$.
The rectangle on the back has height 3. The length of this rectangle is the length of the diagonal of the bottom face of the rectangular prism. By the Pythagorean Theorem, this length is $\sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$. Thus, this rectangle is 3×13 and so has area 39.
In total, the surface area of the triangular prism is thus $36 + 15 + 2 \times 30 + 39 = 150$.

Answer: (D)

18. André runs for 10 seconds at a speed of y m/s.
Therefore, André runs $10y$ m.
Carl runs for 20 seconds before André starts to run and then 10 seconds while André is running.
Thus, Carl runs for 30 seconds.
Since Carl runs at a speed of x m/s, then Carl runs $30x$ m.
Since André and Carl run the same distance, then $30x$ m $= 10y$ m, which means that $\dfrac{y}{x} = 3$.
Thus, $y : x = 3 : 1$.

Answer: (D)

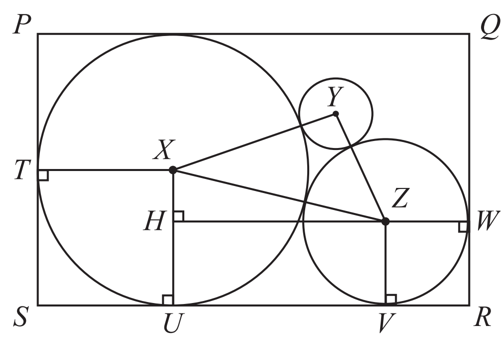
19. Using exponent laws, the expression $\dfrac{2^{x+y}}{2^{x-y}} = 2^{(x+y)-(x-y)} = 2^{2y}$.
Since x and y are positive integers with $xy = 6$, then the possible values of y are the positive divisors of 6, namely 1, 2, 3, or 6. (These correspond to $x = 6, 3, 2, 1$.)
The corresponding values of 2^{2y} are $2^2 = 4$, $2^4 = 16$, $2^6 = 64$, and $2^{12} = 4096$.
Therefore, the sum of the possible values of $\dfrac{2^{x+y}}{2^{x-y}}$ is $4 + 16 + 64 + 4096 = 4180$.

Answer: (A)

20. Let the radii of the circles with centres X, Y and Z be x, y and z, respectively.
The distance between the centres of two touching circles equals the sum of the radii of these circles.
Therefore, $XY = x + y$ which means that $x + y = 30$.
Also, $XZ = x + z$ which gives $x + z = 40$ and $YZ = y + z$ which gives $y + z = 20$.
Adding these three equations, we obtain $(x + y) + (x + z) + (y + z) = 30 + 40 + 20$ and so $2x + 2y + 2z = 90$ or $x + y + z = 45$.
Since $x + y = 30$ and $x + y + z = 45$, then $30 + z = 45$ and so $z = 15$.
Since $y + z = 20$, then $y = 20 - z = 5$.
Since $x + z = 40$, then $x = 40 - z = 25$.
Knowing the radii of the circles will allow us to calculate the dimensions of the rectangle.

The height of rectangle $PQRS$ equals the "height" of the circle with centre X, which is length of the diameter of the circle, or $2x$.
Thus, the height of rectangle $PQRS$ equals 50.

To calculate the width of rectangle $PQRS$, we join X to the points of tangency (that is, the points where the circle touches the rectangle) T and U on PS and SR, respectively, Z to the points of tangency V and W on SR and QR, respectively, and draw a perpendicular from Z to H on XU.
Since radii are perpendicular to tangents at points of tangency, then XT, XU, ZV, and ZW are perpendicular to the sides of the rectangle.



Each of $XTSU$ and $ZWRV$ has three right angles, and so must have four right angles and so are rectangles.
Thus, $SU = TX = 25$ (the radius of the circle with centre X) and $VR = ZW = 15$ (the radius of the circle with centre Z).
By a similar argument, $HUVZ$ is also a rectangle.
Thus, $UV = HZ$ and $HU = ZV = 15$.
Since $XH = XU - HU$, then $XH = 10$.
By the Pythagorean Theorem, $HZ = \sqrt{XZ^2 - XH^2} = \sqrt{40^2 - 10^2} = \sqrt{1500} = 10\sqrt{15}$ and so $UV = 10\sqrt{15}$.
This means that $SR = SU + UV + VR = 25 + 10\sqrt{15} + 15 = 40 + 10\sqrt{15}$.
Therefore, the area of rectangle $PQRS$ is $50 \times (40 + 10\sqrt{15}) = 2000 + 500\sqrt{15} \approx 3936.5$.
Of the given choices, this answer is closest to (E) 3950.

ANSWER: (E)

21. *Solution 1*

We start with the ones digits.
Since $4 \times 4 = 16$, then $T = 6$ and we carry 1 to the tens column.
Looking at the tens column, since $4 \times 6 + 1 = 25$, then $S = 5$ and we carry 2 to the hundreds column.
Looking at the hundreds column, since $4 \times 5 + 2 = 22$, then $R = 2$ and we carry 2 to the thousands column.
Looking at the thousands column, since $4 \times 2 + 2 = 10$, then $Q = 0$ and we carry 1 to the ten thousands column.
Looking at the ten thousands column, since $4 \times 0 + 1 = 1$, then $P = 1$ and we carry 0 to the hundred thousands column.
Looking at the hundred thousands column, $4 \times 1 + 0 = 4$, as expected.
This gives the following completed multiplication:

$$\begin{array}{r} 1\,0\,2\,5\,6\,4 \\ \times \qquad\quad 4 \\ \hline 4\,1\,0\,2\,5\,6 \end{array}$$

Finally, $P + Q + R + S + T = 1 + 0 + 2 + 5 + 6 = 14$.

Solution 2

Let x be the five-digit integer with digits $PQRST$.
This means that $PQRST0 = 10x$ and so $PQRST4 = 10x + 4$.
Also, $4PQRST = 400\,000 + PQRST = 400\,0000 + x$.
From the given multiplication, $4(10x + 4) = 400\,000 + x$ which gives $40x + 16 = 400\,000 + x$ or $39x = 399\,984$.
Thus, $x = \dfrac{399\,984}{39} = 10\,256$.
Since $PQRST = 10\,256$, then $P + Q + R + S + T = 1 + 0 + 2 + 5 + 6 = 14$.

ANSWER: (A)

22. Here is one way in which the seven friends can ride on four buses so that the seven restrictions are satisfied:

Bus 1	Bus 2	Bus 3	Bus 4
Abu	Bai	Don	Gia
Cha	Fan	Eva	

At least 3 buses are needed because of the groups of 3 friends who must all be on different buses.
We will now show that it is impossible for the 7 friends to travel on only 3 buses.
Suppose that the seven friends could be put on 3 buses.
Since Abu, Bai and Don are on 3 different buses, then we assign them to three buses that we can call Bus 1, Bus 2 and Bus 3, respectively. (See the table below.)
Since Abu, Bai and Eva are on 3 different buses, then Eva must be on Bus 3.
Since Cha and Bai are on 2 different buses and Cha and Eva are on 2 different buses, then Cha cannot be on Bus 2 or Bus 3, so Cha is on Bus 1.
So far, this gives

Bus 1	Bus 2	Bus 3
Abu	Bai	Don
Cha		Eva

The remaining two friends are Fan and Gia.
Since Fan, Cha and Gia are on 3 different buses, then neither Fan nor Gia is on Bus 1.
Since Don, Gia and Fan are on 3 different buses, then neither Fan nor Gia is on Bus 3.
Since Gia and Fan are on separate buses, they cannot both be on Bus 2, which means that the seven friends cannot be on 3 buses only.
Therefore, the minimum number of buses needed is 4.

ANSWER: (B)

23. Since the wheel turns at a constant speed, then the percentage of time when a shaded part of the wheel touches a shaded part of the path will equal the percentange of the total length of the path where there is "shaded on shaded" contact.
Since the wheel has radius 2 m, then its circumference is $2\pi \times 2$ m which equals 4π m.
Since the wheel is divided into four quarters, then the portion of the circumference taken by each quarter is π m.
We call the left-hand end of the path 0 m.
As the wheel rotates once, the first shaded section of the wheel touches the path between 0 m and $\pi \approx 3.14$ m.
As the wheel continues to rotate, the second shaded section of the wheel touches the path between $2\pi \approx 6.28$ m and $3\pi \approx 9.42$ m.
While the wheel makes 3 complete rotations, a shaded quarter will be in contact with the path over 6 intervals (2 intervals per rotation).
The path is shaded for 1 m starting at each odd multiple of 1 m, and unshaded for 1 m starting at each even multiple of 1 m.
We make a chart of the sections where shaded quarters touch the path and the parts of these intervals that are shaded:

Beginning of quarter (m)	End of quarter (m)	Shaded parts of path (m)
0	$\pi \approx 3.14$	1 to 2; 3 to π
$2\pi \approx 6.28$	$3\pi \approx 9.42$	7 to 8; 9 to 3π
$4\pi \approx 12.57$	$5\pi \approx 15.71$	13 to 14; 15 to 5π
$6\pi \approx 18.85$	$7\pi \approx 21.99$	19 to 20; 21 to 7π
$8\pi \approx 25.13$	$9\pi \approx 28.27$	8π to 26; 27 to 28
$10\pi \approx 31.42$	$11\pi \approx 34.56$	10π to 32; 33 to 34

Therefore, the total length of "shaded on shaded", in metres, is

$$1 + (\pi - 3) + 1 + (3\pi - 9) + 1 + (5\pi - 15) + 1 + (7\pi - 21) + (26 - 8\pi) + 1 + (32 - 10\pi) + 1$$

which equals $(16 - 2\pi)$ m.
The total length of the path along which the wheel rolls is $3 \times 4\pi$ m or 12π m.
This means that the required percentage of time equals $\dfrac{(16 - 2\pi)\text{ m}}{12\pi\text{ m}} \times 100\% \approx 25.8\%$.
Of the given choices, this is closest to 26%, or choice (E).

ANSWER: (E)

24. We let A be the set $\{2, 3, 4, 5, 6, 7, 8, 9\}$.
First, we note that the integer s that Roberta chooses is of the form $s = 11m$ for some integer m from the set A, and the integer t that Roberta chooses is of the form $t = 101n$ for some integer n from the set A.
This means that the product rst is equal to $r(11m)(101n) = 11 \times 101 \times rmn$ where each of r, m, n comes from the set A.
This means that the number of possible values for rst is equal to the number of possible values of rmn, and so we count the number of possible values of rst by counting the number of possible values of rmn.

We note that A contains only one multiple of 5 and one multiple of 7. Furthermore, these multiples include only one factor of 5 and 7 each, respectively.
We count the number of possible values for rmn by considering the different possibilities for the number of factors of 5 and 7 in rmn.
Let y be the number of factors of 5 in rmn and let z be the number of factors of 7 in rmn.
Note that since each of r, m and n includes at most one factor of 5 and at most one factor of 7 and each of r, m and n cannot contain both a factor of 5 and a factor of 7, then $y + z$ is at most 3.

<u>Case 1: $y = 3$</u>
If rmn includes 3 factors of 5, then $r = m = n = 5$ and so $rmn = 5^3$.
This means that there is only one possible value for rmn.

<u>Case 2: $z = 3$</u>
Here, it must be the case that $rmn = 7^3$ and so there is only one possible value for rmn.

<u>Case 3: $y = 2$ and $z = 1$</u>
Here, it must be the case that two of r, m, n are 5 and the other is 7.
In other words, rmn must equal $5^2 \times 7$.
This means that there is only one possible value for rmn.

<u>Case 4: $y = 1$ and $z = 2$</u>
Here, rmn must equal 5×7^2.
This means that there is only one possible value for rmn.

<u>Case 5: $y = 2$ and $z = 0$</u>
Here, two of r, m, n must equal 5 and the third cannot be 5 or 7.
This means that the possible values for the third of these are $2, 3, 4, 6, 8, 9$.
This means that there are 6 possible values for rmn in this case.

<u>Case 6: $y = 0$ and $z = 2$</u>
As in Case 5, there are 6 possible values for rmn.

<u>Case 7: $y = 1$ and $z = 1$</u>
Here, one of r, m, n equals 5, one equals 7, and the other must be one of $2, 3, 4, 6, 8, 9$.
This means that there are 6 possible values for rmn in this case.

<u>Case 8: $y = 1$ and $z = 0$</u>
Here, one of r, m, n equals 5 and none of these equal 7. Suppose that $r = 5$.
Each of m and n equals one of $2, 3, 4, 6, 8, 9$.

We make a multiplication table to determine the possible values of mn:

$\times$	2	3	4	6	8	9
2	4	6	8	12	16	18
3	6	9	12	18	24	27
4	8	12	16	24	32	36
6	12	18	24	36	48	54
8	16	24	32	48	64	72
9	18	27	36	54	72	81

The different values in this table are

$$4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48, 54, 64, 72, 81$$

of which there are 16.
Therefore, there are 16 possible values of rmn in this case.

Case 9: $y = 0$ and $z = 1$
As in Case 8, there are 16 possible values of rmn.

Case 10: $y = 0$ and $z = 0$
Here, none of r, m, n is 5 or 7.
This means that each of r, m, n equals one of $2, 3, 4, 6, 8, 9$.
This means that the only possible prime factors of rmn are 2 and 3.
Each of $2, 3, 4, 6, 8, 9$ includes at most 2 factors of 3 and only 9 includes 2 factors of 3.
This means that rmn contains at most 6 factors of 3.
Let w be the number of factors of 3 in rmn.

- If $w = 6$, then $r = m = n = 9$ and so $rmn = 9^3$. There is one value of rmn here.
- If $w = 5$, then two of r, m, n equal 9 and the third is 3 or 6. This means that $rmn = 9^2 \times 3$ or $rmn = 9^2 \times 6$. There are two values of rmn here.
- If $w = 4$, then either two of r, m, n equal 9 and the third does not include a factor of 3, or one of r, m, n equals 9 and the second and third are each 3 or 6.
  Thus, rmn can equal $9^2 \times 2$, $9^2 \times 4$, $9^2 \times 8$, $9 \times 3 \times 3$, $9 \times 3 \times 6$, or $9 \times 6 \times 6$.
  There is duplication in this list and so the values of rmn are 9^2, $9^2 \times 2$, $9^2 \times 4$, $9^2 \times 8$.
  There are four values of rmn here.
- If $w = 3$, we can have one of r, m, n equal to 9, one equal to 3 or 6, and the last equal to 2, 4 or 8, or we can have each of r, m, n equal to 3 or 6.
  In the first situation, rmn can be $9 \times 3 \times 2$, $9 \times 3 \times 4$, $9 \times 3 \times 8$, $9 \times 6 \times 2$, $9 \times 6 \times 4$, $9 \times 6 \times 8$.
  These can be written as 27×2^1, 27×2^2, 27×2^3, 27×2^4.
  In the second situation, rmn can be $3 \times 3 \times 3$, $3 \times 3 \times 6$, $3 \times 6 \times 6$, $6 \times 6 \times 6$.
  These equal 27, 27×2^1, 27×2^2, 27×2^3.
  Combining lists, we get 27, 27×2^1, 27×2^2, 27×2^3, 27×2^4.
  Therefore, in this case there are 5 possible values for rmn.
- If $w = 2$, then either one of r, m, n equals 9 or two of r, m, n equal 3 or 6.
  In the first situation, the other two of r, m, n equal 2, 4 or 8.
  Note that $2 = 2^1$ and $4 = 2^2$ and $8 = 2^3$.
  Thus rmn contains at least 2 factors of 2 (for example, $rmn = 9 \times 2 \times 2$) and contains at most 6 factors of 2 ($rmn = 9 \times 8 \times 8$).

If two of r, m, n equal 3 or 6, then the third equals 2, 4 or 8.
Thus, rmn contains at least 1 factor of 2 ($rmn = 3 \times 3 \times 2$) and at most 5 factors of 2 ($rmn = 6 \times 6 \times 8$).
Combining these possibilities, rmn can equal $9 \times 2, 9 \times 2^2, \ldots, 9 \times 2^6$, and so there are 6 possible values of rmn in this case.

- If $w = 1$, then one of r, m, n equals 3 or 6, and the other two equal 2, 4 or 8.
  Thus, rmn can contain at most 7 factors of 2 ($rmn = 6 \times 8 \times 8$) and must contain at least 2 factors of 2 ($rmn = 3 \times 2 \times 2$).
  Each of the number of factors of 2 between 2 and 7, inclusive, is possible, so there are 6 possible values of rmn.
- If $w = 0$, then none of r, m, n can equal 3, 6 or 9.
  Thus, each of r, m, n equals 2^1, 2^2 or 2^3.
  Thus, rmn must be a power of 2 and includes at least 3 and no more than 9 factors of 2. Each of these is possible, so there are 7 possible values of rmn.

In total, the number of possible values of rmn (and hence of rst) is

$$2 \times 1 + 2 \times 1 + 2 \times 6 + 6 + 2 \times 16 + (1 + 2 + 4 + 5 + 6 + 6 + 7)$$

which equals 85.

Answer: (A)

25. Suppose that $PQ = a$, $PS = b$ and $PU = c$.
Since $PQRSTUVW$ is a rectangular prism, then $QR = PS = b$ and $ST = QV = PU = c$.
By the Pythagorean Theorem, $PR^2 = PQ^2 + QR^2$ and so $1867^2 = a^2 + b^2$.
By the Pythagorean Theorem, $PV^2 = PQ^2 + QV^2$ and so $2019^2 = a^2 + c^2$.
By the Pythagorean Theorem, $PT^2 = PS^2 + ST^2$ and so $x^2 = b^2 + c^2$.
Adding these three equations, we obtain

$$\begin{aligned} 1867^2 + 2019^2 + x^2 &= (a^2 + b^2) + (a^2 + c^2) + (b^2 + c^2) \\ 1867^2 + 2019^2 + x^2 &= 2a^2 + 2b^2 + 2c^2 \\ a^2 + b^2 + c^2 &= \frac{1867^2 + 2019^2 + x^2}{2} \end{aligned}$$

Since $a^2 + b^2 = 1867^2$, then

$$c^2 = (a^2 + b^2 + c^2) - (a^2 + b^2) = \frac{1867^2 + 2019^2 + x^2}{2} - 1867^2 = \frac{-1867^2 + 2019^2 + x^2}{2}$$

Since $a^2 + c^2 = 2019^2$, then

$$b^2 = (a^2 + b^2 + c^2) - (a^2 + c^2) = \frac{1867^2 + 2019^2 + x^2}{2} - 2019^2 = \frac{1867^2 - 2019^2 + x^2}{2}$$

Since $b^2 + c^2 = x^2$, then

$$a^2 = (a^2 + b^2 + c^2) - (b^2 + c^2) = \frac{1867^2 + 2019^2 + x^2}{2} - x^2 = \frac{1867^2 + 2019^2 - x^2}{2}$$

Since a, b and c are edge lengths of the prism, then $a, b, c > 0$.
Since $a^2 > 0$, then $\frac{1867^2 + 2019^2 - x^2}{2} > 0$ and so $1867^2 + 2019^2 - x^2 > 0$ or $x^2 < 2019^2 + 1867^2$.
Since $b^2 > 0$, then $\frac{1867^2 - 2019^2 + x^2}{2} > 0$ and so $1867^2 - 2019^2 + x^2 > 0$ or $x^2 > 2019^2 - 1867^2$.
Since $c^2 > 0$, then $\frac{-1867^2 + 2019^2 + x^2}{2} > 0$ and so $-1867^2 + 2019^2 + x^2 > 0$ which means that $x^2 > 1867^2 - 2019^2$.
Since the right side of this last inequality is negative and the left side is non-negative, then this inequality is always true.
Therefore, it must be true that $2019^2 - 1867^2 < x^2 < 2019^2 + 1867^2$.
Since all three parts of this inequality are positive, then $\sqrt{2019^2 - 1867^2} < x < \sqrt{2019^2 + 1867^2}$.
Since $\sqrt{2019^2 - 1867^2} \approx 768.55$ and $\sqrt{2019^2 + 1867^2} \approx 2749.92$ and x is an integer, then $769 \leq x \leq 2749$.
The number of integers x in this range is $2749 - 769 + 1 = 1981$.
Every such value of x gives positive values for a^2, b^2 and c^2 and so positive values for a, b and c, and so a rectangular prism $PQRSTUVW$ with the correct lengths of face diagonals.
Therefore, there are 1981 such integers x.

Answer: (E)

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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2018 Cayley Contest

(Grade 10)

Tuesday, February 27, 2018
(in North America and South America)

Wednesday, February 28, 2018
(outside of North America and South America)

Solutions

1. Since $3 \times n = 6 \times 2$, then $3n = 12$ or $n = \frac{12}{3} = 4$.

   Answer: (E)

2. The 4×5 grid contains 20 squares that are 1×1.
   For half of these to be shaded, 10 must be shaded.
   Since 3 are already shaded, then $10 - 3 = 7$ more need to be shaded.

   Answer: (C)

3. Since the number line between 0 and 2 is divided into 8 equal parts, then the portions between 0 and 1 and between 1 and 2 are each divided into 4 equal parts.
   In other words, the divisions on the number line mark off quarters, and so $S = 1 + 0.25 = 1.25$.

   Answer: (D)

4. Since $9 = 3 \times 3$, then $9^4 = (3 \times 3)^4 = 3^4 \times 3^4 = 3^8$.
   Alternatively, we can note that

   $$9^4 = 9 \times 9 \times 9 \times 9 = (3 \times 3) \times (3 \times 3) \times (3 \times 3) \times (3 \times 3) = 3^8$$

   Answer: (D)

5. The entire central angle in a circle measures 360°.
   In the diagram, the central angle which measures 120° represents $\frac{120^\circ}{360^\circ} = \frac{1}{3}$ of the entire central angle.
   Therefore, the area of the sector is $\frac{1}{3}$ of the area of the entire circle, or $\frac{1}{3} \times 9\pi = 3\pi$.

   Answer: (B)

6. For any value of x, we have $x^2 + 2x - x(x+1) = x^2 + 2x - x^2 - x = x$.
   When $x = 2018$, the value of this expression is thus 2018.

   Answer: (B)

7. We want to calculate the percentage increase from 24 to 48.
   The percentage increase from 24 to 48 is equal to $\dfrac{48 - 24}{24} \times 100\% = 1 \times 100\% = 100\%$.
   Alternatively, since 48 is twice 24, then 48 represents an increase of 100% over 24.

   Answer: (D)

8. A line segment joining two points is parallel to the x-axis exactly when the y-coordinates of the two points are equal.
   Here, this means that $2k + 1 = 4k - 5$ and so $6 = 2k$ or $k = 3$.
   (We can check that when $k = 3$, the coordinates of the points are $(3, 7)$ and $(8, 7)$.)

   Answer: (B)

9. Since $5, a, b$ have an average of 33, then $\dfrac{5 + a + b}{3} = 33$.
   Multiplying by 3, we obtain $5 + a + b = 3 \times 33 = 99$, which means that $a + b = 94$.
   The average of a and b is thus equal to $\dfrac{a+b}{2} = \dfrac{94}{2} = 47$.

   Answer: (E)

10. Of the given uniform numbers,

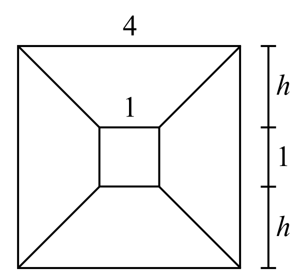
- 11 and 13 are prime numbers
- 16 is a perfect square
- 12, 14 and 16 are even

Since Karl's and Liu's numbers were prime numbers, then their numbers were 11 and 13 in some order.
Since Glenda's number was a perfect square, then her number was 16.
Since Helga's and Julia's numbers were even, then their numbers were 12 and 14 in some order. (The number 16 is already taken.)
Thus, Ioana's number is the remaining number, which is 15.

ANSWER: (D)

11. *Solution 1*
The large square has side length 4, and so has area $4^2 = 16$.
The small square has side length 1, and so has area $1^2 = 1$.
The combined area of the four identical trapezoids is the difference in these areas, or $16-1 = 15$.
Since the 4 trapezoids are identical, then they have equal areas, each equal to $\frac{15}{4}$.

Solution 2
Suppose that the height of each of the four trapezoids is h.
Since the side length of the outer square is 4, then $h + 1 + h = 4$ and so $h = \frac{3}{2}$.
Each of the four trapezoids has parallel bases of lengths 1 and 4, and height $\frac{3}{2}$.
Therefore, the area of each is $\frac{1}{2}(1+4)(\frac{3}{2}) = \frac{5}{2}\left(\frac{3}{2}\right) = \frac{15}{4}$.



ANSWER: (D)

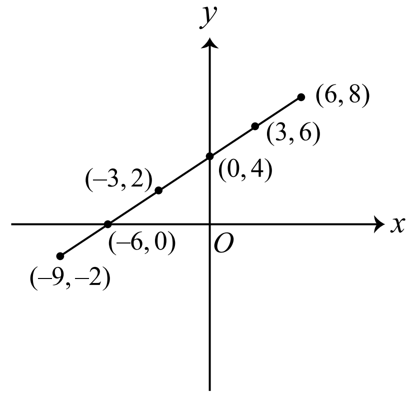
12. We are told that 1 Zed is equal in value to 16 Exes.
We are also told that 2 Exes are equal in value to 29 Wyes.
Since 16 Exes is 8 groups of 2 Exes, then 16 Exes are equal in value to $8 \times 29 = 232$ Wyes.
Thus, 1 Zed is equal in value to 232 Wyes.

ANSWER: (C)

13. The problem is equivalent to determining all values of x for which $x + 1$ is a divisor of 3.
The divisors of 3 are $3, -3, 1, -1$.
If $x + 1 = 3, -3, 1, -1$, then $x = 2, -4, 0, -2$, respectively. There are 4 such values.

ANSWER: (A)

14. *Solution 1*



The line segment with endpoints $(-9, -2)$ and $(6, 8)$ has slope $\dfrac{8-(-2)}{6-(-9)} = \dfrac{10}{15} = \dfrac{2}{3}$.
This means that starting at $(-9, -2)$ and moving "up 2 and right 3" (corresponding to the rise and run of 2 and 3) repeatedly will give other points on the line that have coordinates which are both integers.
These points are $(-9, -2), (-6, 0), (-3, 2), (0, 4), (3, 6), (6, 8)$.
So far, this gives 6 points on the line with integer coordinates.
Are there any other such points?
If there were such a point between $(-9, -2)$ and $(6, 8)$, its y-coordinate would have to be equal to one of $-1, 1, 3, 5, 7$, the other integer possibilities between -2 and 8.
Consider the point on this line segment with y-coordinate 7.
Since this point has y-coordinate halfway between 6 and 8, then this point must be the midpoint of $(3, 6)$ and $(6, 8)$, which means that its x-coordinate is $\frac{1}{2}(3+6) = 4.5$, which is not an integer.
In a similar way, the points on the line segment with y-coordinates $-1, 1, 3, 5$ do not have integer x-coordinates.
Therefore, the 6 points listed before are the only points on this line segment with integer coordinates.

Solution 2
The line segment with endpoints $(-9, -2)$ and $(6, 8)$ has slope $\dfrac{8-(-2)}{6-(-9)} = \dfrac{10}{15} = \dfrac{2}{3}$.
Since the line passes through $(6, 8)$, its equation can be written as $y - 8 = \frac{2}{3}(x-6)$ or $y = \frac{2}{3}x + 4$.
Suppose that a point (x, y) lies along with line between $(-9, -2)$ and $(6, 8)$ and has both x and y integers.
Since y is an integer and $\frac{2}{3}x = y - 4$, then $\frac{2}{3}x$ is an integer.
This means that x must be a multiple of 3.
Since x is between -9 and 6, inclusive, then the possible values for x are $-9, -6, -3, 0, 3, 6$.
This leads to the points listed in Solution 1, and justifies why there are no additional points.
Therefore, there are 6 such points.

Answer: (E)

15. Since $\triangle PQS$ is equilateral, then $\angle QPS = 60^\circ$.
Since $\angle RPQ$, $\angle RPS$ and $\angle QPS$ completely surround point P, then the sum of their measures is 360°.
Since $\angle RPQ = \angle RPS$, this means that $2\angle RPQ + \angle QPS = 360^\circ$ or $2\angle RPQ = 360^\circ - 60^\circ$, which means that $\angle RPQ = \angle RPS = 150^\circ$.
Since $PR = PQ$, then in isosceles $\triangle PQR$, we have

$$\angle PRQ = \angle PQR = \tfrac{1}{2}(180^\circ - \angle RPQ) = \tfrac{1}{2}(180^\circ - 150^\circ) = 15^\circ$$

Similarly, $\angle PRS = \angle PSR = 15^\circ$.
This means that $\angle QRS = \angle PRQ + \angle PRS = 15^\circ + 15^\circ = 30^\circ$.

Answer: (A)

16. Elisabeth climbs a total of 5 rungs by climbing either 1 or 2 rungs at a time.
Since there are only 5 rungs, then she cannot climb 2 rungs at a time more than 2 times.
Therefore, she must climb 2 rungs either 0, 1 or 2 times.

If she climbs 2 rungs 0 times, then each step consists of 1 rung and so she climbs $1, 1, 1, 1, 1$ to get to the top.

If she climbs 2 rungs 1 time, then she climbs $2, 1, 1, 1$, since the remaining 3 rungs must be made up of steps of 1 rung.
But she can climb these numbers of rungs in several different orders.
Since she takes four steps, she can climb 2 rungs as any of her 1st, 2nd, 3rd, or 4th step.
Putting this another way, she can climb $2, 1, 1, 1$ or $1, 2, 1, 1$ or $1, 1, 2, 1$ or $1, 1, 1, 2$.
There are 4 possibilities in this case.

If she climbs 2 rungs 2 times, then she climbs $2, 2, 1$.
Again, she can climb these numbers of rungs in several different orders.
Since she takes three steps, she can climb 1 rung as any of her 1st, 2nd or 3rd step.
Putting this another way, she can climb $1, 2, 2$ or $2, 1, 2$ or $2, 2, 1$.
There are 3 possibilities in this case.

In total, there are $1 + 4 + 3 = 8$ ways in which she can climb.

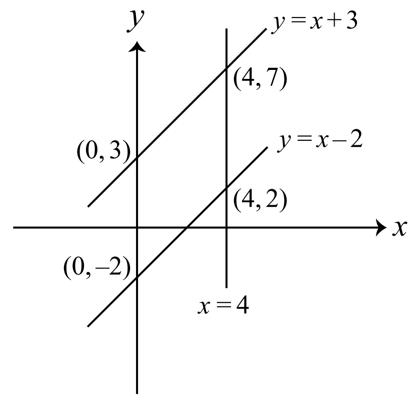
Answer: (E)

17. Since $\dfrac{x-y}{x+y} = 5$, then $x - y = 5(x + y)$.
This means that $x - y = 5x + 5y$ and so $0 = 4x + 6y$ or $2x + 3y = 0$.
Therefore, $\dfrac{2x+3y}{3x-2y} = \dfrac{0}{3x-2y} = 0$.
(A specific example of x and y that works is $x = 3$ and $y = -2$.
This gives $\dfrac{x-y}{x+y} = \dfrac{3-(-2)}{3+(-2)} = \dfrac{5}{1} = 5$ and $\dfrac{2x+3y}{3x-2y} = \dfrac{2(3)+3(-2)}{3(3)-2(-2)} = \dfrac{0}{13} = 0$.)
We should also note that if $\dfrac{x-y}{x+y} = 5$, then $2x+3y = 0$ (from above) and x and y cannot both be 0 (for $\dfrac{x-y}{x+y}$ to be well-defined).
Since $2x + 3y = 0$, then $x = -\frac{3}{2}y$ which means that it is not possible for only one of x and y to be 0, which means that neither is 0.
Also, the denominator of our desired expression $\dfrac{2x+3y}{3x-2y}$ is $3x - 2y = 3(-\frac{3}{2}y)) - 2y = -\frac{13}{2}y$ which is not 0, since $y \neq 0$.
Therefore, if $\dfrac{x-y}{x+y} = 5$, then $\dfrac{2x+3y}{3x-2y} = 0$.

Answer: (B)

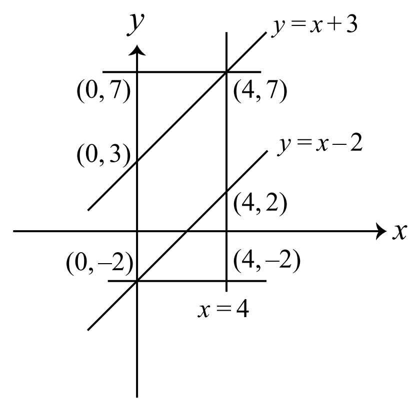
18. *Solution 1*

The lines with equations $x = 0$ and $x = 4$ are parallel vertical lines.
The lines with equations $y = x - 2$ and $y = x + 3$ are parallel lines with slope 1.
Since the quadrilateral has two sets of parallel sides, it is a parallelogram.
Thus, its area equals the length of its base times its height.
We consider the vertical side along the y-axis as its base.
Since the two sides of slope 1 have y-intercepts -2 and 3, then the length of the vertical base is $3 - (-2) = 5$.
Since the parallel vertical sides lie along the lines with equations $x = 0$ and $x = 4$, then these sides are a distance of 4 apart, which means that the height of the parallelogram is 4.
Therefore, the area of the quadrilateral is $5 \times 4 = 20$.



Solution 2

The lines with equations $x = 0$ and $x = 4$ are parallel vertical lines.
The lines with equations $y = x - 2$ and $y = x + 3$ are parallel lines with slope 1.
The lines with equations $y = x - 2$ and $y = x + 3$ intersect the line with equation $x = 0$ at their y-intercepts, namely -2 and 3, respectivley.
The lines with equations $y = x - 2$ and $y = x + 3$ intersect the line with equation $x = 4$ at $(4, 2)$ and $(4, 7)$, respectively (the point on each line with x-coordinate 4).
We draw horizontal lines through $(0, -2)$, intersecting $x = 4$ at $(4, -2)$, and through $(4, 7)$, intersecting $x = 0$ at $(0, 7)$.



The area of the quadrilateral equals the area of the large rectangle with vertices $(0, 7)$, $(4, 7)$, $(4, -2)$, $(0, -2)$ minus the combined area of the two triangles.
This rectangle has side lengths $4 - 0 = 4$ and $7 - (-2) = 9$ and so has area $4 \times 9 = 36$.
The two triangles are right-angled and have bases of length $4 - 0 = 4$ and heights of length $7 - 3 = 4$ and $2 - (-2) = 4$. This means that they can be combined to form a square with side length 4, and so have combined area $4^2 = 16$.
Therefore, the area of the quadrilateral is $36 - 16 = 20$.

Answer: (E)

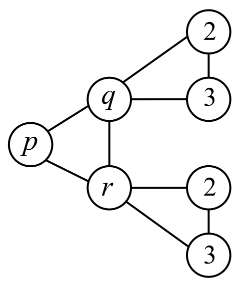
19. Suppose that L is the area of the large circle, S is the area of the small circle, and A is the area of the overlapped region.
Since the area of the overlapped region is $\frac{3}{5}$ of the area of the small circle, then $A = \frac{3}{5}S$.
Since the area of the overlapped region is $\frac{6}{25}$ of the area of the large circle, then $A = \frac{6}{25}L$.
Therefore, $\frac{3}{5}S = \frac{6}{25}L$.
Multiplying both sides by 25 to clear denominators, we obtain $15S = 6L$.
Dividing both sides by 3, we obtain $5S = 2L$. Therefore, $\dfrac{5S}{L} = 2$ or $\dfrac{S}{L} = \dfrac{2}{5}$.
This means that the ratio of the area of the small circle to the area of the large circle is $2 : 5$.

Answer: (D)

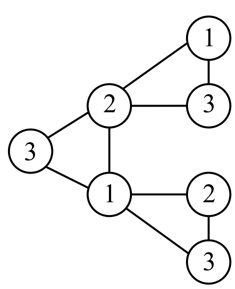
20. When the product of the three integers is calculated, either the product is a power of 2 or it is not a power of 2.
If p is the probability that the product is a power of 2 and q is the probability that the product is not a power of 2, then $p + q = 1$.
Therefore, we can calculate q by calculating p and noting that $q = 1 - p$.
For the product of the three integers to be a power of 2, it can have no prime factors other than 2. In particular, this means that each of the three integers must be a power of 2.
In each of the three sets, there are 3 powers of 2 (namely, 2, 4 and 8) and 2 integers that are not a power of 2 (namely, 6 and 10).
This means that the probability of choosing a power of 2 at random from each of the sets is $\frac{3}{5}$.
Since Abigail, Bill and Charlie choose their numbers independently, then the probability that each chooses a power of 2 is $\left(\frac{3}{5}\right)^3 = \frac{27}{125}$.
In other words, $p = \frac{27}{125}$ and so $q = 1 - p = 1 - \frac{27}{125} = \frac{98}{125}$.

ANSWER: (C)

21. Since each of s, t, u, v is equal to one of 1, 2 or 3, and s and t are different and u and v are different, then their sum cannot be any larger than $2 + 3 + 2 + 3 = 10$.
This can only happen if s and t are 2 and 3 in some order, and if u and v are 2 and 3 in some order.



But q, s, t are $1, 2, 3$ in some order and r, u, v are $1, 2, 3$ in some order.
So if s and t are 2 and 3, then $q = 1$. Similarly, if u and v are 2 and 3, then $r = 1$.
But p, q, r are $1, 2, 3$ in some order, so we cannot have $q = r = 1$.
Therefore, we cannot have $s + t + u + v = 10$.
The next largest possible value of $s + t + u + v$ would be 9.
We can construct the diagram with this value of $s + t + u + v$ by letting $s = 1$, $t = 3$, $u = 2$, $v = 3$, and proceeding as follows:



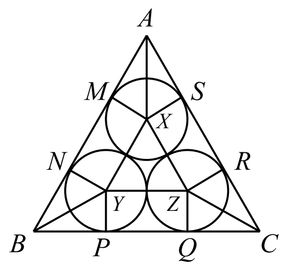
Therefore, the maximum possible value of $s + t + u + v$ is 9.

ANSWER: (B)

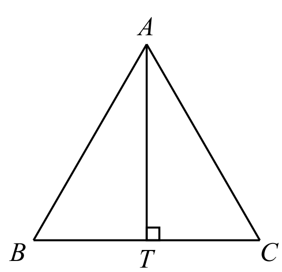
22. For the given expression to be equal to an integer, each prime factor of the denominator must divide out of the product in the numerator.
In other words, each prime number that is a factor of the denominator must occur as a factor at least as many times in the numerator as in the denominator.
We note that $25 = 5^2$ and so $25^y = 5^{2y}$.
Also, $36 = 6^2 = 2^2 3^2$ and so $36^x = (2^2 3^2)^x = 2^{2x} 3^{2x}$.
Therefore, the given expression is equal to $\dfrac{30!}{2^{2x}3^{2x}5^{2y}}$.
We count the number of times that each of 5, 3 and 2 is a factor of the numerator.
The product equal to 30! includes six factors which are multiples of 5, namely $5, 10, 15, 20, 25, 30$.
Each of these factors includes 1 factor of 5, except for 25 which includes 2 factors.
Therefore, the numerator includes 7 factors of 5.
For the numerator to include at least as many factors of 5 as the denominator, we must have $7 \geq 2y$.
Since y is an integer, then $y \leq 3$.
The product equal to 30! includes 10 factors which are multiples of 3, namely $3, 6, 9, 12, 15, 18, 21, 24, 27, 30$.
Seven of these include exactly one factor of 3, namely $3, 6, 12, 15, 21, 24, 30$.
Two of these include exactly two factors of 3, namely $9, 18$.
One of these includes exactly three factors of 3, namely 27.
Therefore, the numerator includes $7(1) + 2(2) + 1(3) = 14$ factors of 3.
For the numerator to include at least as many factors of 3 as the denominator, we must have $14 \geq 2x$.
Since x is an integer, then $x \leq 7$.
If $x \leq 7$, then the denominator includes at most 14 factors of 2. Since the product equal to 30! includes 15 even numbers, then the numerator includes at least 15 factors of 2, and so includes more factors of 2 than the denominator. This means that the number of 2s does not limit the value of x.
Since $x \leq 7$ and $y \leq 3$, then $x + y \leq 7 + 3 = 10$.
We note that if $x = 7$ and $y = 3$, then the given expression is an integer so this maximum can be achieved.

Answer: (A)

23. To determine the volume of the prism, we calculate the area of its base and the height of the prism.
First, we calculate the area of its base.
Any cross-section of the prism parallel to its base has the same shape, so we take a cross-section 1 unit above the base.
Since each of the spheres has radius 1, this triangular cross-section will pass through the centre of each of the spheres and the points of tangency between the spheres and the rectangular faces.
Let the vertices of the triangular cross-section be A, B and C, the centres of the spheres be X, Y and Z, and the points of tangency of the spheres (circles) to the faces of the prism (sides of the triangle) be M, N, P, Q, R, and S, as shown:

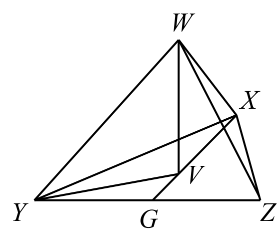


Join X to M, S, Y, Z, A, join Y to N, P, Z, B, and Z to Q, R, C.
We determine the length of BC. A similar calculation will determine the lengths of AB and AC. By symmetry, $AB = AC = BC$.
Note that YP and ZQ are perpendicular to BC since radii are perpendicular to tangents.
Also, $YP = ZQ = 1$ since the radius of each sphere (and hence of each circle) is 1.
Since $ZYPQ$ has right angles at P and Q and has $YP = ZQ = 1$, then $ZYPQ$ is a rectangle.
Therefore, $YZ = PQ$.
Since YZ passes through the point at which the circles touch, then YZ equals the sum of the radii of the circles, or 2.
Thus, $PQ = YZ = 2$.
Since $AB = BC = CA$, then $\triangle ABC$ is equilateral and so $\angle ABC = 60^\circ$.
Now YB bisects $\angle ABC$, by symmetry.
This means that $\angle YBP = 30^\circ$, which means that $\triangle BYP$ is a 30°-60°-90° triangle.
Since $YP = 1$, then $BP = \sqrt{3}$, using the ratios of side lengths in a 30°-60°-90° triangle.
Similarly, $QC = \sqrt{3}$.
Therefore, $BC = BP + PQ + QC = \sqrt{3} + 2 + \sqrt{3} = 2 + 2\sqrt{3}$.
This means that $AB = BC = CA = 2 + 2\sqrt{3}$.
To calculate the area of $\triangle ABC$, we drop a perpendicular from A to T on BC.



Since $\angle ABC = 60^\circ$, then $\triangle ABT$ is a 30°-60°-90° triangle.
Since $AB = 2 + 2\sqrt{3}$, then $AT = \frac{\sqrt{3}}{2}AB = \frac{\sqrt{3}}{2}(2 + 2\sqrt{3})$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(BC)(AT) = \frac{1}{2}(2 + 2\sqrt{3})\left(\frac{\sqrt{3}}{2}(2 + 2\sqrt{3})\right) = \frac{\sqrt{3}}{4}(2 + 2\sqrt{3})^2$.

This means that the area of the base of the prism (which is $\triangle ABC$) is $\frac{\sqrt{3}}{4}(2+2\sqrt{3})^2$.
Now we calculate the height of the prism.
Let the centre of the top sphere be W.
The vertical distance from W to the top face of the prism equals the radius of the sphere, which is 1.
Similarly, the vertical distance from the bottom face to the plane through X, Y and Z is 1.
To finish calculating the height of the prism, we need to determine the vertical distance between the cross-section through X, Y, Z and the point W.
The total height of the prism equals 2 plus this height.
Since the four spheres touch, then the distance between any pair of centres is the sum of the radii, which is 2.
Therefore, $WX = XY = YZ = WZ = WY = XZ = 2$. (This means that $WXYZ$ is a tetrahedron with equal edge lengths.)
We need to calculate the height of this tetrahedron.
Join W to the centre, V, of $\triangle XYZ$.
By symmetry, W is directly above V.



Let G be the midpoint of YZ. This means that $YG = GZ = 1$.
Join V to Y and G.
Since V is the centre of $\triangle XYZ$, then VG is perpendicular to YZ at G.
Since $\triangle XYZ$ is equilateral, then $\angle VYG = \frac{1}{2}\angle XYZ = 30^\circ$. This is because V, the centre of equilateral $\triangle XYZ$, lies on the angle bisectors of each of the angles.
$\triangle YVG$ is another 30°-60°-90° triangle, and so $YV = \frac{2}{\sqrt{3}}YG = \frac{2}{\sqrt{3}}$.
Finally, $\triangle WYV$ is right-angled at V.
Thus, $WV = \sqrt{WY^2 - YV^2} = \sqrt{2^2 - \frac{4}{3}} = \sqrt{\frac{8}{3}}$.
This means that the height of the prism is $1 + 1 + \sqrt{\frac{8}{3}}$.
The volume of the prism is equal to the area of its base times its height, which is equal to $\frac{\sqrt{3}}{4}(2+2\sqrt{3})^2 \cdot \left(2 + \sqrt{\frac{8}{3}}\right)$ which is approximately 46.97.
Of the given answers, this is closest to 47.00.

Answer: (E)

24. Since there must be at least 2 gold socks (G) between any 2 black socks (B), then we start by placing the n black socks with exactly 2 gold socks in between each pair:

$$BGGBGGBGGB \cdots BGGB$$

Since there are n black socks, then there are $n-1$ "gaps" between them and so $2(n-1)$ or $2n-2$ gold socks have been used.
This means that there are 2 gold socks left to place. There are $n+1$ locations in which these socks can be placed: either before the first black sock, after the last black sock, or in one of the $n-1$ gaps.
These 2 socks can either be placed together in one location or separately in two locations.
If the 2 socks are placed together, they can be placed in any one of the $n+1$ locations, and so there are $n+1$ ways to do this. (The placement of the gold socks within these locations does not matter as the gold socks are all identical.)
The other possibility is that the 2 gold socks are placed separately in two of these $n+1$ spots. There are $n+1$ possible locations for the first of these socks. For each of these, there are then n possible locations for the second of these socks (any other than the location of the first sock). Since these two socks are identical, we have double-counted the total number of possibilities, and so there are $\frac{1}{2}(n+1)n$ ways for these two gold socks to be placed.
In total, there are $(n+1)+\frac{1}{2}(n+1)n = (n+1)(1+\frac{1}{2}n) = \frac{1}{2}(n+1)(n+2)$ ways in which the socks can be arranged.
We want to determine the smallest positive integer n for which this total is greater than 1 000 000.
This is equivalent to determining the smallest positive integer n for which $(n+1)(n+2)$ is greater than 2 000 000.
We note that $(n+1)(n+2)$ increases as n increases, since each of $n+1$ and $n+2$ is positive and increasing, and so their product is increasing.
When $n = 1412$, we have $(n+1)(n+2) = 1\,997\,982$.
When $n = 1413$, we have $(n+1)(n+2) = 2\,000\,810$.
Since $(n+1)(n+2)$ is increasing, then $n = 1413$ is the smallest positive integer for which $(n+1)(n+2)$ is greater than 2 000 000, and so it is the smallest positive integer for which there are more than 1 000 000 arrangements of the socks.
The sum of the digits of $n = 1413$ is $1+4+1+3 = 9$.

ANSWER: (A)

25. Since the terms in each such sequence can be grouped to get both positive and negative sums, then there must be terms that are positive and there must be terms that are negative.
Since the 15 terms have at most two different values and there are terms that are positive and terms that are negative, then the terms have exactly two different values, one positive and one negative.
We will call these values x and y. We know that both are integers and we will add the condition that $x > 0$ and $y < 0$.
Consider one of these sequences and label the terms

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}$$

Since the sum of six consecutive terms is always positive, then $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 > 0$.
Since the sum of eleven consecutive terms is always negative, then $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} < 0$.
Since $a_1+a_2+a_3+a_4+a_5+a_6 > 0$ and $(a_1+a_2+a_3+a_4+a_5+a_6)+(a_7+a_8+a_9+a_{10}+a_{11}) < 0$, then $a_7 + a_8 + a_9 + a_{10} + a_{11} < 0$.
The six term condition also tells us that $a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} > 0$.
Since $a_7 + a_8 + a_9 + a_{10} + a_{11} < 0$ and $a_6 + (a_7 + a_8 + a_9 + a_{10} + a_{11}) > 0$, then $a_6 > 0$ and so $a_6 = x$.
The six term condition also tells us that $a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} > 0$.
Since $a_7 + a_8 + a_9 + a_{10} + a_{11} < 0$, then $a_{12} > 0$ which gives $a_{12} = x$.
We can repeat this argument by shifting all of the terms one further along in the sequence.
Starting with $a_2+a_3+a_4+a_5+a_6+a_7 > 0$ and $a_2+a_3+a_4+a_5+a_6+a_7+a_8+a_9+a_{10}+a_{11}+a_{12} < 0$ and using the same arguments as above will give $a_7 = a_{13} = x$.
By continuing to shift one more term further along twice more, we obtain $a_8 = a_{14} = x$ and $a_9 = a_{15} = x$.
So far this gives the sequence

$$a_1, a_2, a_3, a_4, a_5, x, x, x, x, a_{10}, a_{11}, x, x, x, x$$

We can continue to repeat this argument by starting at the right-hand end of the sequence.
Starting with $a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} > 0$ and $a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} < 0$, will allow us to conclude that $a_{10} = x$ and $a_4 = x$.
Shifting back to the left and repeating this argument gives $a_1 = a_2 = a_3 = a_4 = a_{10} = x$.
This means that the sequence has the form

$$x, x, x, x, a_5, x, x, x, x, x, a_{11}, x, x, x, x$$

At least one of a_5 and a_{11} must equal y, otherwise all of the terms in the sequence would be positive.
In fact, it must be the case that both of these terms equal y.
To see this, suppose that $a_5 = y$ but $a_{11} = x$.
In this case, the sum of the first 6 terms in the sequence is $5x + y$ which is positive (that is, $5x + y > 0$).
Also, the sum of the first 11 terms in the sequence is $10x + y$ which is negative (that is, $10x + y < 0$).
But $x > 0$ and so $10x + y = 5x + (5x + y) > 0$, which contradicts $10x + y < 0$.
We would obtain the same result if we considered the possibility that $a_5 = x$ and $a_{11} = y$.
Therefore, $a_5 = a_{11} = y$ and so the sequence must be

$$x, x, x, x, y, x, x, x, x, x, y, x, x, x, x$$

In this case, each group of 6 consecutive terms includes exactly 5 x's and so the sum of each group of 6 consecutive terms is $5x+y$.
We are told that $5x+y>0$.
Also, the sum of each group of 11 consecutive terms is $9x+2y$.
We are told that $9x+2y<0$.
We have now changed the original problem into an equivalent problem: count the number of pairs (x,y) of integers with $x>0$ and $y<0$ and $5x+y>0$ and $9x+2y<0$ and either x is between 1 and 16, inclusive, or y is between -16 and -1, inclusive.
Suppose that $1\leq x\leq 16$.
From $5x+y>0$ and $9x+2y<0$, we obtain $-5x<y<-4.5x$.
We can now make a chart that enumerates the values of x from 1 to 16, the corresponding bounds on y, and the resulting possible values of y:

x	$-5x$	$-4.5x$	Possible y
1	-5	-4.5	None
2	-10	-9	None
3	-15	-13.5	-14
4	-20	-18	-19
5	-25	-22.5	$-24,-23$
6	-30	-27	$-29,-28$
7	-35	-31.5	$-34,-33,-32$
8	-40	-36	$-39,-38,-37$
9	-45	-40.5	$-44,-43,-42,-41$
10	-50	-45	$-49,-48,-47,-46$
11	-55	-49.5	$-54,-53,-52,-51,-50$
12	-60	-54	$-59,-58,-57,-56,-55$
13	-65	-58.5	$-64,-63,-62,-61,-60,-59$
14	-70	-63	$-69,-68,-67,-66,-65,-64$
15	-75	-67.5	$-74,-73,-72,-71,-70,-69,-68$
16	-80	-72	$-79,-78,-77,-76,-75,-74,-73$

Therefore, when $1\leq x\leq 16$, there are $2(1+2+3+4+5+6+7)=56$ pairs (x,y) and so there are 56 sequences.
For example, if $x=5$ and $y=-24$, we get the sequence

$$5,5,5,5,-24,5,5,5,5,5,-24,5,5,5,5$$

which satisfies the five given conditions.

Are there any additional sequences with $-16\leq y\leq -1$?
Since $5x+y>0$, then $x>-0.2y$. Since $y\geq -16$, then $x<0.2(16)=3.2$.
Since $9x+2y<0$, then $x<-\frac{2}{9}y$. Since $y\leq -1$, then $x>\frac{2}{9}(1)=\frac{2}{9}$.
Since $\frac{2}{9}<x<3.2$, then any sequence with $-16\leq y\leq -1$ also has $1\leq x\leq 16$, and so we have counted this already.
This means that N, the total number of such sequences, equals 56.

Answer: (E)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2017 Cayley Contest

(Grade 10)

Tuesday, February 28, 2017
(in North America and South America)

Wednesday, March 1, 2017
(outside of North America and South America)

Solutions

1. Evaluating, $6 \times 111 - 2 \times 111 = 666 - 222 = 444$.
   Alternatively, we note that $6 \times 111 - 2 \times 111 = (6-2) \times 111 = 111(4) = 444$.

   ANSWER: (C)

2. Evaluating, $\dfrac{5^2 - 9}{5 - 3} = \dfrac{25 - 9}{2} = \dfrac{16}{2} = 8$.
   Alternatively, we note that $5^2 - 9 = 5^2 - 3^2 = (5-3)(5+3)$ and so
   $$\frac{5^2 - 9}{5 - 3} = \frac{(5-3)(5+3)}{5-3} = 5 + 3 = 8$$

   ANSWER: (D)

3. The height of the snowman equals the sum of the lengths of the diameters of the three spheres.
   Since the radii of the spheres are 10 cm, 20 cm and 30 cm, then the lengths of their diameters are 20 cm, 40 cm and 60 cm.
   Thus, the height of the snowman is 20 cm + 40 cm + 60 cm = 120 cm.

   ANSWER: (D)

4. We write each of the choices in lowest terms:
   $$\tfrac{44444}{55555} = \tfrac{4(11111)}{5(11111)} = \tfrac{4}{5} \qquad \tfrac{5555}{6666} = \tfrac{5(1111)}{6(1111)} = \tfrac{5}{6} \qquad \tfrac{666}{777} = \tfrac{6(111)}{7(111)} = \tfrac{6}{7} \qquad \tfrac{77}{88} = \tfrac{7(11)}{8(11)} = \tfrac{7}{8} \qquad \tfrac{8}{9}$$
   (The last choice was already in lowest terms.)
   Next, we note that $\frac{4}{5} = 1 - \frac{1}{5}$ and $\frac{5}{6} = 1 - \frac{1}{6}$ and $\frac{6}{7} = 1 - \frac{1}{7}$ and $\frac{7}{8} = 1 - \frac{1}{8}$ and $\frac{8}{9} = 1 - \frac{1}{9}$.
   The fraction with the greatest value will be the one that is equal to 1 minus the smallest fraction.
   Since $\frac{1}{9} < \frac{1}{8} < \frac{1}{7} < \frac{1}{6} < \frac{1}{5}$, then the fraction with the greatest value is $\frac{8}{9}$.

   ANSWER: (E)

5. Since 300 litres drains in 25 hours, then the rate at which water is leaving the tank equals $\dfrac{300\text{ L}}{25\text{ h}}$ or 12 L/h.

   ANSWER: (A)

6. When Penelope folds the paper in half, the number of layers doubles.
   Starting with 4 layers of paper, then after the next three folds, there are 8 and then 16 and then 32 layers of paper. Additional folds create more layers.
   Of the given choices (which are all less than 32), only 16 is a possible number of layers.

   ANSWER: (D)

7. By definition, $2 \diamondsuit 7 = 2^2(7) - 2(7^2) = 4(7) - 2(49) = 28 - 98 = -70$.

   ANSWER: (B)

8. In option (A), the first and third cards have 0 numbers in common, so (A) is not correct.
   In option (B), the second and third cards have 2 numbers in common, so (B) is not correct.
   In option (C), the first and third cards have 2 numbers in common, so (C) is not correct.
   In option (E), the first and third cards have 2 numbers in common, so (E) is not correct.
   In option (D), the first and second cards share exactly 1 number (namely, 4), the first and third cards share exactly one number (namely, 7), and the second and third cards share exactly one number (namely, 2). Thus, (D) is correct.

   ANSWER: (D)

9. Since the bill including a 13% tip was \$226, then \$226 is 113% of the bill before tax.
Thus, the bill before tax was $\frac{\$226}{1.13} = \$200$ before tax.
The amount of the tip is 15% of the bill before tax, or $\$200 \times 0.15 = \$30$.

ANSWER: (C)

10. Since PQR is a straight angle, then $\angle PQT + \angle RQT = 180^\circ$.
Therefore, $x^\circ + (x-50)^\circ = 180^\circ$ and so $2x - 50 = 180$ or $2x = 230$, which gives $x = 115$.
Therefore, $\angle TUR = (x+25)^\circ = 140^\circ$.
Since TU and PS are parallel, then $\angle URS$ and $\angle TUR$ are alternating angles, which means that $\angle URS = \angle TUR = 140^\circ$.

ANSWER: (B)

11. Since 10 identical squares have a total area of 160 cm^2, then the area of each square is $\frac{160 \text{ cm}^2}{10}$ or 16 cm^2.
Since the area of each of the 10 identical squares is 16 cm^2, then the side length of each of these squares is $\sqrt{16 \text{ cm}^2} = 4$ cm.
The perimeter of the given figure equals 22 square side lengths (4 on the left side, 4 on the bottom, 4 on the right side, 2 separate ones on the top, and 8 in the "U" shape in the middle).
Therefore, the perimeter of the figure is $4 \text{ cm} \times 22 = 88$ cm.

ANSWER: (C)

12. Since the mean of p, q and r is 9, then $\frac{p+q+r}{3} = 9$ and so $p+q+r = 27$.
Since the mean of s and t is 14, then $\frac{s+t}{2} = 14$ or $s+t = 28$.
Therefore, the mean of the five integers is $\frac{p+q+r+s+t}{5} = \frac{(p+q+r)+(s+t)}{5} = \frac{27+28}{5}$, which equals 11.

ANSWER: (A)

13. First, we consider the column of units (ones) digits.
From this column, we see that the units digit of $Z+Z+Z$ (or $3Z$) must be 5.
By trying the possible digits from 0 to 9, we find that Z must equal 5.
When $Z = 5$, we get $3Z = 15$, and so there is a "carry" of 1 to the column of tens digits.
From this column, we see that the units digit of $1+Y+Y+Y$ (or $3Y+1$) must be 7.
By trying the possible digits from 0 to 9, we find that Y must equal 2.
There is no carry created when $Y = 2$.
Looking at the remaining digits, we see that $2X = 16$ and so $X = 8$.
Checking, if $X = 8$ and $Y = 2$ and $Z = 5$, we obtain $825 + 825 + 25$ which equals 1675, as required.
Therefore, $X + Y + Z = 8 + 2 + 5 = 15$.

ANSWER: (B)

14. Since Igor is shorter than Jie, then Igor cannot be the tallest.
Since Faye is taller than Goa, then Goa cannot be the tallest.
Since Jie is taller than Faye, then Faye cannot be the tallest.
Since Han is shorter than Goa, then Han cannot be the tallest.
The only person of the five who has not been eliminated is Jie, who must thus be the tallest.

ANSWER: (E)

15. Since there are 32 red marbles in the bag and the ratio of red marbles to blue marbles is $4:7$, then there are $\frac{7}{4}(32) = 56$ blue marbles in the bag.
Since there are 56 blue marbles in the bag and the ratio of blue marbles to purple marbles is $2:3$, then there are $\frac{3}{2}(56) = 84$ purple marbles in the bag.
Since the bag contains only red, blue and purple marbles, then there are $32 + 56 + 84 = 172$ marbles in the bag.

ANSWER: (E)

16. Since $x + 2y = 30$, then

$$\begin{aligned}
\frac{x}{5} + \frac{2y}{3} + \frac{2y}{5} + \frac{x}{3} &= \frac{x}{5} + \frac{2y}{5} + \frac{x}{3} + \frac{2y}{3} \\
&= \frac{1}{5}x + \frac{1}{5}(2y) + \frac{1}{3}x + \frac{1}{3}(2y) \\
&= \frac{1}{5}(x+2y) + \frac{1}{3}(x+2y) \\
&= \frac{1}{5}(30) + \frac{1}{3}(30) \\
&= 6 + 10 \\
&= 16
\end{aligned}$$

ANSWER: (B)

17. First, we factor 1230 as a product of prime numbers:

$$1230 = 2 \times 615 = 2 \times 5 \times 123 = 2 \times 5 \times 3 \times 41$$

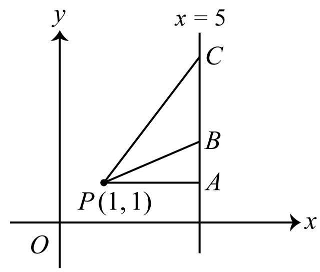
We are looking for three positive integers r, s, t with $r \times s \times t = 1230$ and whose sum is as small as possible. We note that all of the possibilities given for the smallest possible value of $r+s+t$ are less than 60.
Since 41 is a prime factor of 1230, then one of r, s, t must be a multiple of 41.
Since all of the possibilities given for the minimum sum $r+s+t$ are less than 60 and the second smallest multiple of 41 is 82, then the multiple of 41 in the list r, s, t must be 41 itself.
Thus, we can let $t = 41$.
Now we are looking for positive integers r and s with $r \times s = 2 \times 3 \times 5 = 30$ and whose sum $r + s$ is as small as possible. (Since t is now fixed, then minimizing $r + s + t$ is equivalent to minimizing $r + s$.)
The possible pairs of values for r and s in some order are $1, 30$ and $2, 15$ and $3, 10$ and $5, 6$.
The pair with the smallest sum is $5, 6$.
Therefore, we set $r = 5$ and $s = 6$. This gives $r + s + t = 5 + 6 + 41 = 52$.

ANSWER: (B)

18. Since $\frac{1}{7}$ is positive and $\frac{1}{7} \leq \frac{6}{n}$, then $\frac{6}{n}$ is positive and so n is positive.
Since $\frac{1}{7} = \frac{6}{42}$ and $\frac{1}{4} = \frac{6}{24}$, then the given inequality is equivalent to $\frac{6}{42} \leq \frac{6}{n} \leq \frac{6}{24}$.
Since the fractions are all positive and $n > 0$, then this is true when $24 \leq n \leq 42$. (If two fractions have the same numerator, then the larger fraction has a smaller denominator.)
Since n is an integer, then there are $42 - 24 + 1 = 19$ possible values for n. (We could count the integers from 24 to 42, inclusive, to confirm this.)

ANSWER: (C)

19. We start by drawing a graph that includes the point $(1, 1)$, the lines with slopes $\frac{1}{4}$ and $\frac{5}{4}$ that pass through this point, the vertical line with equation $x = 5$, and the horizontal line $y = 1$ (which passes through $(1, 1)$ and is perpendicular to the vertical line with equation $x = 5$).
We label the various points of intersection (A, B, C) as shown.



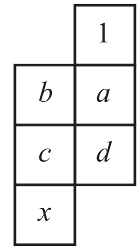
We want to determine the area of $\triangle PBC$.
Since P has coordinates $(1, 1)$ and A has coordinates $(5, 1)$, then $PA = 4$.
Since the slope of PB is $\frac{1}{4}$ and $PA = 4$, then thinking about slope as "rise over run", we see that $AB = 1$.
Since the slope of PC is $\frac{5}{4}$ and $PA = 4$, then $AC = 5$.
Since $AC = 5$ and $AB = 1$, then $BC = AC - AB = 5 - 1 = 4$.
We can view $\triangle PBC$ as having base BC and perpendicular height PA. (This is because the length of PA is the perpendicular distance from the line through B and C to the point P.)
Therefore, the area of this triangle is $\frac{1}{2}(4)(4)$ which equals 8.
Alternatively, we could note that the area of $\triangle PBC$ equals the area of $\triangle PAC$ minus the area of $\triangle PAB$.
$\triangle PAC$ has base $PA = 4$ and height $AC = 5$ and so has area $\frac{1}{2}(4)(5) = 10$.
$\triangle PAB$ has base $PA = 4$ and height $AB = 1$ and so has area $\frac{1}{2}(4)(1) = 2$.
Thus, the area of $\triangle PBC$ is $10 - 2 = 8$.

Answer: (C)

20. Since there are 60 seconds in 1 minute, then t seconds is equivalent to $\dfrac{t}{60}$ minutes.
Since there are 60 minutes in 1 hour, then $\dfrac{t}{60}$ minutes is equivalent to $\dfrac{t}{60 \times 60}$ hours or $\dfrac{t}{3600}$ hours.
Consider the distances that Car X and Car Y travel between the instant when the front of Car Y is lined up with the back of Car X and the instant when the back of Car Y is lined up with the front of Car X.
Since the length of Car X is 5 m and the length of Car Y is 6 m, then during this interval of time, Car Y travels $5 + 6 = 11$ m farther than Car X. (The front of Car Y must, in some sense, travel all of the way along the length of the Car X and be 6 m ahead of the front of Car X so that the back of Car Y is lined up with the front of Car X.)
Since there are 1000 metres in 1 km, then 11 m is equivalent to 0.011 km.
Since Car X travels at 90 km/h, then in $\dfrac{t}{3600}$ hours, Car X travels $\dfrac{90t}{3600}$ km.
Since Car Y travels at 91 km/h, then in $\dfrac{t}{3600}$ hours, Car Y travels $\dfrac{91t}{3600}$ km.
Therefore, $\dfrac{91t}{3600} - \dfrac{90t}{3600} = 0.011$, or $\dfrac{t}{3600} = 0.011$ and so $t = 3600 \times 0.011 = 36 \times 1.1 = 39.6$.

Answer: (A)

21. We label the remaining unknown entries as a, b, c, d, as shown.
Now a, b, c, d, x must equal $2, 3, 4, 5, 6$ in some order, with the restriction that no two integers that differ by 1 may be in squares that share an edge.
In particular, we see first that $a \neq 2$. Therefore, a can equal 3, 4, 5, or 6.



If $a = 3$, then neither b nor d can equal 2 or 4.
Thus, b and d equal 5 and 6 in some order.
In this case, c cannot be 4 (since one of b and d is 5) so $c = 2$ and so $x = 4$.

If $a = 4$, then neither b nor d can equal 3 or 5.
Thus, b and d equal 2 and 6 in some order.
In this case, c cannot equal 3 or 5, since it is adjacent to 2 and 6, so this case is not possible.

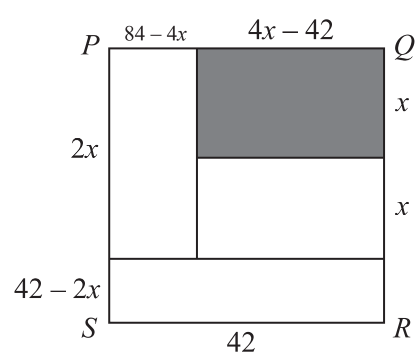
If $a = 5$, then neither b nor d can equal 4 or 6.
Thus, b and d equal 2 and 3 in some order.
In this case, c cannot be 4 (since one of b and d is 3) so $c = 6$ and so $x = 4$.

If $a = 6$, then neither b nor d can equal 5.
Thus, b and d equal two of 2, 3 or 4 in some order.
If b and d are 2 and 4, then c cannot be 3 or 5 (the remaining numbers), so there is no solution.
If b and d are 3 and 4, then c cannot be 2 or 5 (the remaining numbers), so there is no solution.
If b and d are 2 and 3, then we can have $c = 5$, in which case we must have $x = 4$, which is not possible.

Having considered all possible cases, we see that there is only one possible value for x, namely $x = 4$.

Answer: (A)

22. Suppose that the height of the top left rectangle is $2x$.
Since square $PQRS$ has side length 42, then the bottom rectangle has height $42 - 2x$.
Since the width of the bottom rectangle is 42 (the side length of the square), then the perimeter of the bottom rectangle is $2(42) + 2(42 - 2x) = 168 - 4x$.
Suppose that the width of the top left rectangle is y.
Since each of the small rectangles has the same perimeter, then $2(2x) + 2y = 168 - 4x$. (That is, the perimeter of the top left rectangle equals the perimeter of the bottom rectangle.)
Thus, $2y = 168 - 8x$ or $y = 84 - 4x$.
Since the side length of the square is 42, then the width of the top right rectangle is $42 - (84 - 4x) = 4x - 42$.
Since the two rectangles on the top right have the same perimeter and the same width, then they must have the same height, which is equal to half of the height of the top left rectangle.
Thus, the height of each of the two rectangles on the right is x.
Finally, the perimeter of the top right rectangle must equal also equal $168 - 4x$.
Thus, $2(4x - 42) + 2x = 168 - 4x$ or $8x - 84 + 2x = 168 - 4x$ and so $14x = 252$ or $x = 18$.
This tells us that the shaded rectangle has dimensions $x = 18$ by $4x - 42 = 4(18) - 42 = 30$ and so its area is $18 \times 30 = 540$.



(We can check by substitution that the resulting rectangles in the original diagram are 6×42, 36×12 and 30×18, which all have the same perimeter.)

Answer: (E)

23. To solve this problem, we need to use two facts about a triangle with positive area and side lengths $a \leq b \leq c$:

- $a + b > c$. This is called the Triangle Inequality and is based on the fact that the shortest distance between two points is a straight line. In this case, the shortest distance between the two vertices of the triangle at opposite ends of the side with length c is this length c. Any other path between these paths is longer. In particular, travelling between these two points along the other two sides is longer. This path has length $a + b$ and so $a + b > c$. (This is where the condition of "positive area" is used.)
- If the triangle is obtuse, then $a^2 + b^2 < c^2$; if the triangle is acute, $a^2 + b^2 > c^2$. This makes sense from the given examples and the fact that if $a^2 + b^2 = c^2$, the triangle is right-angled. We justify these facts further at the end.

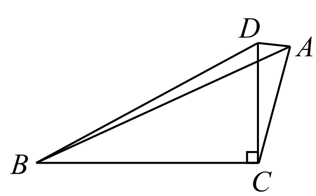
Suppose that the unknown side length x of the obtuse triangle is the longest side length; that is, suppose that $10 \leq 17 \leq x$.
Here, we must have $10 + 17 > x$ and $10^2 + 17^2 < x^2$.
From the first inequality, $x < 27$. Since x is an integer, then $x \leq 26$.
From the second inequality, $x^2 > 389$ and so $x > \sqrt{389}$. Since $\sqrt{389} \approx 19.72$ and x is an integer, then $x \geq 20$. This gives the possible values $x = 20, 21, 22, 23, 24, 25, 26$

Suppose that the unknown side length x of the obtuse triangle is not the longest side length. That is, $x \leq 17$. (We know that $10 \leq 17$ as well. Note also that the relative size of 10 and x is not important.)
Here, we must have $x + 10 > 17$ and $10^2 + x^2 < 17^2$.
From the first inequality, $x > 7$. Since x is an integer, then $x \geq 8$.
From the second inequality, $x^2 < 189$ and so $x < \sqrt{189}$. Since $\sqrt{189} \approx 13.75$ and x is an integer, then $x \leq 13$. This gives the possible values $x = 8, 9, 10, 11, 12, 13$.

Therefore, the possible values of x are $8, 9, 10, 11, 12, 13, 20, 21, 22, 23, 24, 25, 26$.
The sum of these possible values is 224.

We still need to justify the second fact. We show that if the triangle is obtuse, then $a^2 + b^2 < c^2$. The second half of the fact can be shown in a similar way. While we could use the cosine law to justify this statement, we show this using angles and side lengths:

Consider $\triangle ABC$ with $\angle ACB$ obtuse and with $AB = c$, $AC = b$ and $BC = a$.
At C, draw a perpendicular line segment CD with length b. Draw DB and DA.



Since $\triangle BCD$ is right-angled at C, then $BD = \sqrt{BC^2 + CD^2} = \sqrt{a^2 + b^2}$, by the Pythagorean Theorem.
Now $\triangle ACD$ is isosceles with $CA = CD$.
Thus, $\angle CDA = \angle CAD$.
But $\angle BDA > \angle CDA = \angle CAD > \angle BAD$.
Since $\angle BDA > \angle BAD$, then in $\triangle BDA$, we have $BA > BD$.
This means that $c = BA > BD = \sqrt{a^2 + b^2}$ and so $c^2 > a^2 + b^2$, as required.

Answer: (E)

24. We think of each allowable sequence of moves as a string of X's, Y's and Z's. For example, the string ZZYXZ would represent moving Z one space to the right, then Z, then Y, then X, then Z.
For each triple (x, y, z) of integers with $0 \leq x \leq 3$ and $0 \leq y \leq 3$ and $0 \leq z \leq 3$, we define $S(x, y, z)$ to be the number of sequences of moves which result in X moving x spaces to the right and Y moving y spaces to the right and Z moving z spaces to the right.
For example, $S(1, 0, 0) = 0$ and $S(0, 1, 0) = 0$ since X and Y are not allowable sequences, and $S(0, 0, 1) = 1$ since Z is allowable and is only sequence of 1 move.
We want to find $S(3, 3, 3)$.
Next, we note that if $x > y$ or $y > z$ or $x > z$, then $S(x, y, z) = 0$, since any allowable sequence has to have at least as many Z's as Y's and at least as many Y's as X's, because no coin can jump another coin. In other words, we need to have $0 \leq x \leq y \leq z \leq 3$.
We now make the key observation that if $0 \leq x \leq y \leq z \leq 3$, then

$$S(x, y, z) = S(x - 1, y, z) + S(x, y - 1, z) + S(x, y, z - 1)$$

where we use the convention that if $x = 0$, then $S(x - 1, y, z) = 0$, and if $y = 0$, then $S(x, y - 1, z) = 0$, and if $z = 0$, then $S(x, y, z - 1) = 0$.
This rule is true because:

- Every allowable sequence counted by $S(x, y, z)$ ends with X, Y or Z.
- If an allowable sequence counted by $S(x, y, z)$ ends in X, then the sequence formed by removing this X was allowable and is counted by $S(x - 1, y, z)$.
- Furthermore, if $x - 1 \leq y \leq z$ (that is, there are sequences consisting of $x - 1$ X's, y Y's and z Z's, making $S(x - 1, y, z) > 0$) and $x \leq y \leq z$, then every sequence counted by $S(x - 1, y, z)$ can have an X put on the end to make a sequence counted by $S(x, y, z)$ ending in an X.
- The last two bullets together tell us that the number of sequences counted by $S(x, y, z)$ and ending in X is exactly equal to $S(x - 1, y, z)$.
- Similarly, the sequences counted by $S(x, y, z)$ and ending in Y is $S(x, y - 1, z)$ and the number of sequences counted by $S(x, y, z)$ and ending in Z is $S(x, y, z - 1)$.
- Therefore, $S(x, y, z) = S(x - 1, y, z) + S(x, y - 1, z) + S(x, y, z - 1)$.

Using this rule, we can create tables of values of $S(x, y, z)$. We create one table for each of $z = 1$, $z = 2$ and $z = 3$. In each table, values of y are marked along the left side and values of x along the top. In each table, there are several 0s in spots where $x > y$ or $x > z$ or $y > z$.

y \ x	0	1	2	3
0	1	0	0	0
1	1	1	0	0
2	0	0	0	0
3	0	0	0	0

$z = 1$

y \ x	0	1	2	3
0	1	0	0	0
1	2	3	0	0
2	2	5	5	0
3	0	0	0	0

$z = 2$

y \ x	0	1	2	3
0	1	0	0	0
1	3	6	0	0
2	5	16	21	0
3	5	21	42	42

$z = 3$

The non-zero entries are filled from top to bottom and left to right, noting that each entry is the sum of the entries (if applicable) to the left in the same table (this is $S(x - 1, y, z)$), above

in the same table (this is $S(x, y-1, z)$), and in the same position in the previous table (this is $S(x, y, z-1)$).
Also, $S(0,0,1) = 1$ (the sequence Z), $S(0,1,1) = 1$ (the sequence ZY), and $S(1,1,1) = 1$ (the sequence ZYX.
From these tables, we see that $S(3,3,3) = 42$ and so the number of different sequences is 42.

ANSWER: (C)

25. We proceed using several steps.

Step 1: Least common multiples
For each positive integer $n \geq 3$ and positive integer $x < n$, we define $L(n, x)$ to be the least common multiple of the $n-2$ integers $1, 2, 3, \ldots, x-2, x-1, x+2, x+3, \ldots, n-1, n$.
One way to calculate the least common multiple of a list of integers is to determine the prime factorization of each of the integers in the list and then to create the product of the largest of each of the prime powers that occurs among the integers in the list.
For example, if $n = 9$ and $x = 6$, then $L(9, 6)$ is the least common multiple of the five integers $1, 2, 3, 4, 5, 8, 9$.
The prime factorizations of the integers larger than 1 in this list are $2 = 2^1$, $3 = 3^1$, $4 = 2^2$, $5 = 5^1$, $8 = 2^3$, $9 = 3^2$ and so $L(9, 6) = 2^3 3^2 5^1 = 360$. In this case, $L(9, 6)$ is not divisible by $x + 1 = 7$ but is divisible by $x = 6$.
We note that, for any list of integers, a common multiple, m, of all of the integers in this list is always itself a multiple of their least common multiple l. This is because if p is a prime number and a is a positive integer for which p^a is a factor of l, then there must be an integer in the list that is a multiple of p^a. For m to be a common multiple of every number in the list, p^a must also be a factor of m. Since this is true for every prime power p^a that is a factor of l, then m is a multiple of l.

Step 2: Connection between m and $L(n, x)$
We show that n is a Nella number with corresponding x exactly when $L(n, x)$ is not divisible by x or $x + 1$.
Suppose that n is a Nella number with corresponding x and m.
Then m must be divisible by each of $1, 2, 3, \ldots, x-2, x-1, x+2, x+3, \ldots, n-1, n$.
From Step 1, m is a multiple of $L(n, x)$.
Since m is a multiple of $L(n, x)$ and m is not divisible by x or $x+1$, then $L(n, x)$ is not either.
Since $L(n, x)$ is divisible by every other positive integer from between 1 and n, inclusive, then $L(n, x)$ satisfies the required conditions for m in the definition of a Nella number.
Also, if n and x are positive integers with $n \geq 3$ and $x < n$ and $L(n, x)$ has the two required conditions in the definition of a Nella number, then n is indeed a Nella number.
Putting this together, n is a Nella number with corresponding x exactly when $L(n, x)$ is not divisible by x or $x + 1$.

Step 3: Re-statement of problem
Based on Step 2, we now want to find all positive integers n with $50 \leq n \leq 2017$ for which there exists a positive integer x with $x < n$ with the property that $L(n, x)$ is not divisible by x and $x + 1$.

Step 4: If n is a Nella number with corresponding x, then x and $x + 1$ are both prime powers
Suppose that n is a Nella number with corresponding x.
Further, suppose that x is not a prime power.
Then $x = p^a b$ for some prime number p, positive integer a and positive integer $b > 1$ that is not divisible by p.

In this case, $p^a < x$ and $b < x$ and so p^a and b are both in the list $1, 2, 3, \ldots, x-2, x-1$, which means that $L(n, x)$ is a multiple of both p^a and b and so is a multiple of $p^a b = x$. (It is important here that p^a and b have no common prime factors.) This is a contradiction.
This means that if n is a Nella number, then x is a prime power.
Similarly, $x + 1$ must also be a prime power. To see this, we use the same argument with the additional observation that, because $x+1$ and x are consecutive, they cannot have any common divisor larger than 1 and so if $x + 1 = p^c d$, then x cannot equal p^c or d and therefore both p^c and d are indeed in the list $1, 2, 3, \ldots, x-1$.

Step 5: Further analysis of x and $x + 1$
Suppose that n is a Nella number with corresponding x.
From Step 4, both x and $x + 1$ are prime powers.
Since x and $x + 1$ are consecutive, then one is even and one is odd.
In other words, one of x and $x + 1$ is a power of 2 and the other is a power of an odd prime.
Furthermore, we know that $L(n, x)$ is not divisible by x or $x + 1$.
This means that the list $x + 2, x + 3, \ldots, n - 1, n$ cannot contain a multiple of x or $x + 1$.
This means that $x < n < 2x$ and $x + 1 \leq n < 2(x + 1)$, because the "next" multiple of x is $2x$ and the next multiple of $x + 1$ is $2(x + 1)$.
Since one of x and $x + 1$ is a power of 2, then 2 times this power of 2 (that is, $2x$ or $2(x + 1)$) is the next power of 2, and so the inequalities tell us that n is smaller than the next power of 2, and so x or $x + 1$ has to be the largest power of 2 less than (or less than or equal to, respectively) n.

Step 6: Second re-statement of problem
We want to find all positive integers n with $50 \leq n \leq 2017$ for which there exists a positive integer x with $x < n$ with the property that $L(n, x)$ is not divisible by x and $x + 1$.
From Step 5, we know that either x is the largest power of 2 less than n or $x + 1$ is the largest power of 2 less than or equal to n.
This means that we want to find all positive integers n with $50 \leq n \leq 2017$ for which at least one of the following two statements is true for some positive integer $x < n$:

- If x is the largest power of 2 less than n, then $x + 1$ is also a prime power.
- If $x + 1$ is the largest power of 2 less than or equal to n, then x is also a prime power.

Step 7: Determining Nella numbers
We make two separate tables, one where $x < n$ is a power of 2 and one where $x + 1 \leq n$ is a power of 2. In each case, we determine whether $x + 1$ or x is also a prime power.

Range of n	Largest power of 2 less than n	x	$x+1$	Prime power?	Nella?
$50 \leq n \leq 64$	32	32	33	No ($33 = 3 \cdot 11$)	No
$65 \leq n \leq 128$	64	64	65	No ($65 = 5 \cdot 13$)	No
$129 \leq n \leq 256$	128	128	129	No ($129 = 3 \cdot 43$)	No
$257 \leq n \leq 512$	256	256	257	Yes	See below
$513 \leq n \leq 1024$	512	512	513	No ($513 = 3 \cdot 171$)	No
$1025 \leq n \leq 2017$	1024	1024	1025	No ($1025 = 5 \cdot 205$)	No

Note that 257 is a prime number since it is not divisible by any prime number less than $\sqrt{257} \approx 16.03$. (These primes are $2, 3, 5, 7, 11, 13$.)
For each n with $257 \leq n \leq 511$, $L(n, 256)$ is not divisible by 256 or 257, so each of these n

(there are $511 - 257 + 1 = 255$ of them) is a Nella number.
For $n = 512$, $L(n, 256)$ is divisible by 256 (since 512 is divisible by 256), so $n = 512$ is not a Nella number as this is the only possible candidate for x in this case.

Range of n	Largest power of 2 at most n	$x + 1$	x	Prime power?	Nella?
$50 \leq n \leq 63$	32	32	31	Yes	See below
$64 \leq n \leq 127$	64	64	63	No ($63 = 3 \cdot 21$)	No
$128 \leq n \leq 255$	128	128	127	Yes	See below
$256 \leq n \leq 511$	256	256	255	No ($255 = 5 \cdot 51$)	No
$512 \leq n \leq 1023$	512	512	511	No ($511 = 7 \cdot 73$)	No
$1024 \leq n \leq 2017$	1024	1024	1023	No ($1023 = 3 \cdot 341$)	No

Note that 31 and 127 are prime.
For each n with $50 \leq n \leq 61$, $L(n, 31)$ is not divisible by 31 or 32, so each of these n (there are $61 - 50 + 1 = 12$ of them) is a Nella number.
For $n = 62, 63$, $L(n, 31)$ is divisible by 31 (since 62 is divisible by 31), so neither is a Nella number as this is the only possible candidate for x in this case.
For each n with $128 \leq n \leq 253$, $L(n, 127)$ is not divisible by 127 or 128, so each of these n (there are $253 - 128 + 1 = 126$ of them) is a Nella number.
For $n = 254, 255$, $L(n, 127)$ is divisible by 127 (since 254 is divisible by 127), so neither is a Nella number as this is the only possible candidate for x in this case.

Step 8: Final tally
From the work above, there are $255 + 12 + 126 = 393$ Nella numbers n with $50 \leq n \leq 2017$.

Answer: (A)

The CENTRE for EDUCATION
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2016 Cayley Contest

(Grade 10)

Wednesday, February 24, 2016
(in North America and South America)

Thursday, February 25, 2016
(outside of North America and South America)

Solutions

1. Evaluating, $(3+2)-(2+1)=5-3=2$.

   Answer: (E)

2. According to the graph, 7 of the 20 teachers picked "Square" as their favourite shape.
   Thus, $20-7=13$ teachers did not pick "Square" as their favourite shape.
   (We can also note that the numbers of teachers who chose "Triangle", "Circle" and "Hexagon" were 3, 4 and 6. The sum of these totals is indeed $3+4+6=13$.)

   Answer: (E)

3. Evaluating, $\sqrt{5^2-4^2}=\sqrt{25-16}=\sqrt{9}=3$.

   Answer: (C)

4. Since each of Bill's steps is $\frac{1}{2}$ metre long, then 2 of Bill's steps measure 1 m.
   To walk 12 m, Bill thus takes $12\times 2=24$ steps.

   Answer: (D)

5. *Solution 1*
   Since $\angle PQS$ is an exterior angle of $\triangle QSR$, then $\angle PQS=\angle QSR+\angle SRQ$.
   Thus, $2x^\circ = x^\circ + 50^\circ$ and so $2x=x+50$ or $x=50$.

   Solution 2
   Since $\angle PQS$ and $\angle SQR$ form a straight angle, then $\angle PQS$ and $\angle SQR$ are supplementary.
   Therefore, $\angle SQR = 180^\circ - \angle PQS = 180^\circ - 2x^\circ$.
   Since the sum of the angles in $\triangle SQR$ is 180°, then

   $$\begin{aligned} \angle SQR+\angle QSR+\angle QRS &= 180^\circ \\ (180^\circ-2x^\circ)+x^\circ+50^\circ &= 180^\circ \\ 50^\circ - x^\circ &= 0^\circ \\ x^\circ &= 50^\circ \end{aligned}$$

   Therefore, $x=50$.

   Answer: (A)

6. Since the slope of the line through points $(2,7)$ and $(a,3a)$ is 2, then $\dfrac{3a-7}{a-2}=2$.
   From this, $3a-7=2(a-2)$ and so $3a-7=2a-4$ which gives $a=3$.

   Answer: (C)

7. When Team A played Team B, if Team B won, then Team B scored more goals than Team A, and if the game ended in a tie, then Team A and Team B scored the same number of goals.
   Therefore, if a team has 0 wins, 1 loss, and 2 ties, then it scored fewer goals than its opponent once (the 1 loss) and the same number of goals as its oppponent twice (the 2 ties).
   Combining this information, we see that the team must have scored fewer goals than were scored against them.
   In other words, it is not possible for a team to have 0 wins, 1 loss, and 2 ties, and to have scored more goals than were scored against them.

   We can also examine choices (A), (B), (D), (E) to see that, in each case, it is possible that the team scored more goals than it allowed.
   This will eliminate each of these choices, and allow us to conclude that (C) must be correct.

(A): If the team won 2-0 and 3-0 and tied 1-1, then it scored 6 goals and allowed 1 goal.
(B): If the team won 4-0 and lost 1-2 and 2-3, then it scored 7 goals and allowed 5 goals.
(D): If the team won 4-0, lost 1-2, and tied 1-1, then it scored 6 goals and allowed 3 goals.
(E): If the team won 2-0, and tied 1-1 and 2-2, then it scored 5 goals and allowed 3 goals.
Therefore, it is only the case of 0 wins, 1 loss, and 2 ties where it is not possible for the team to score more goals than it allows.

ANSWER: (C)

8. *Solution 1*
We calculate the value of each of the five words as follows:

- The value of BAD is $2 + 1 + 4 = 7$
- The value of CAB is $3 + 1 + 2 = 6$
- The value of DAD is $4 + 1 + 4 = 9$
- The value of BEE is $2 + 5 + 5 = 12$
- The value of BED is $2 + 5 + 4 = 11$

Of these, the word with the largest value is BEE.

Solution 2
We determine the word with the largest value by comparing the given words.
Since BAD and CAB share the common letters A and B, then the value of BAD is larger than the value of CAB because the value of D is larger than the value of C.
Similarly, the value of DAD is larger than the value of BAD (whose value is larger than the value of CAB), and the value of BEE is larger than the value of BED.
Therefore, the two possibilities for the word with the largest value are DAD and BEE.
The value of DAD is $4 + 1 + 4 = 9$ and the value of BEE is $2 + 5 + 5 = 12$.
Thus, the word with the largest value is BEE.
(Alternatively, we could note that 2 E's have a larger value than 2 D's, and B has a larger value than A, so BEE has a larger value than DAD.)

ANSWER: (D)

9. *Solution 1*
We write out the numbers in the sequence until we obtain a negative number:

$$43, 39, 35, 31, 27, 23, 19, 15, 11, 7, 3, -1$$

Since each number is 4 less than the number before it, then once a negative number is reached, every following number will be negative.
Thus, Grace writes 11 positive numbers in the sequence.

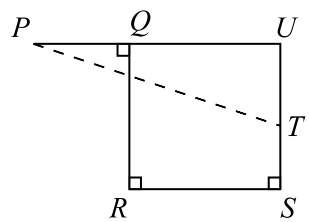
Solution 2
The nth number in the sequence is $4(n - 1)$ smaller than the first number, since we will have subtracted 4 a total of $n - 1$ times to obtain the nth number.
Therefore, the nth number is $43 - 4(n - 1) = 47 - 4n$.
The first positive integer n for which this is expression negative is $n = 12$.
Therefore, Grace writes 11 positive numbers in the sequence.

ANSWER: (A)

10. *Solution 1*
We label the players as A, B, C, D, and E.
The total number of matches played will be equal to the number of pairs of players that can be formed times the number of matches that each pair plays.
The possible pairs of players are AB, AC, AD, AE, BC, BD, BE, CD, CE, and DE. There are 10 such pairs.
Thus, the total number of matches played is $10 \times 3 = 30$.

Solution 2
Each student plays each of the other 4 students 3 times and so plays in $4 \times 3 = 12$ matches.
Since there are 5 students, then the students play in a total of $5 \times 12 = 60$ matches.
But each match includes 2 students, so each match is counted twice in the number of matches in which students play (60). Thus, the total number of matches player is $60 \div 2 = 30$.
Answer: (C)

11. Extend PQ and ST to meet at U.



Since $QUSR$ has three right angles, then it must have four right angles and so is a rectangle.
Thus, $\triangle PUT$ is right-angled at U.
By the Pythagorean Theorem, $PT^2 = PU^2 + UT^2$.
Now $PU = PQ + QU$ and $QU = RS$ so $PU = 4 + 8 = 12$.
Also, $UT = US - ST$ and $US = QR$ so $UT = 8 - 3 = 5$.
Therefore, $PT^2 = 12^2 + 5^2 = 144 + 25 = 169$.
Since $PT > 0$, then $PT = \sqrt{169} = 13$.
Answer: (E)

12. Since Alejandro selects one ball out of a box that contains 30 balls, then the possibility that is the most likely is the one that is satisfied by the largest number of balls in the box.
There are 3 balls in the box whose number is a multiple of 10. (These are 10, 20, 30.)
There are 15 balls in the box whose number is odd. (These are the numbers whose ones digits are 1, 3, 5, 7, or 9.)
There are 4 balls in the box whose number includes the digit 3. (These are 3, 13, 23, and 30.)
There are 6 balls in the box whose number is a multiple of 5. (These are 5, 10, 15, 20, 25, 30.)
There are 12 balls in the box whose number includes the digit 2. (These are 2, 12, and the then integers from 20 to 29, inclusive.)
The most likely of these outcomes is that he selects a ball whose number is odd.
Answer: (B)

13. We note that $\frac{1}{6} = \frac{5}{30}$ and that $\frac{1}{4} = \frac{5}{20}$.
If we compare two fractions with equal positive numerators, the fraction with the smaller positive denominator will be largest of the two fractions.
Therefore, $\frac{5}{30} < \frac{5}{24}$ and $\frac{5}{24} < \frac{5}{20}$, or $\frac{1}{6} < \frac{5}{24} < \frac{1}{4}$.
(Alternatively, we could note that since $\frac{1}{6} = \frac{6}{24}$ and that $\frac{1}{4} = \frac{6}{24}$, then $\frac{5}{24}$ must be the one of the given choices that is between these.)
Answer: (C)

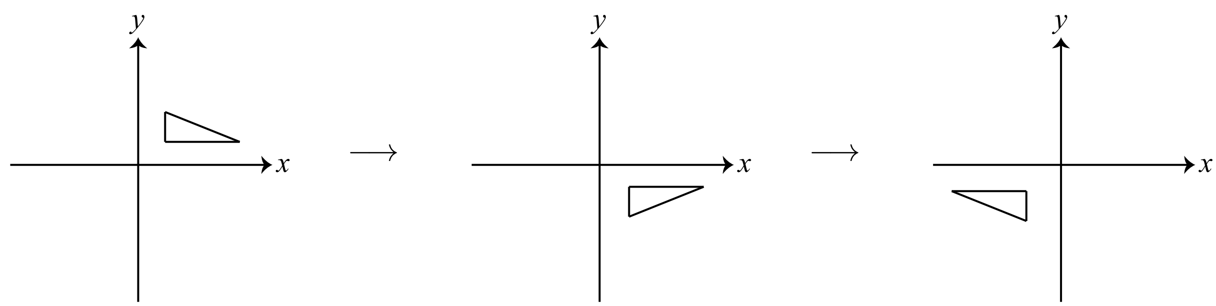
14. Since $100 = 10^2$, then $100^{10} = (10^2)^{10} = 10^{20}$.
Therefore, $(10^{100}) \times (100^{10}) = (10^{100}) \times (10^{20}) = 10^{120}$.
When written out, this integer consists of a 1 followed by 120 zeros.

Answer: (A)

15. We note that $20 = 2^2 \cdot 5$ and $16 = 2^4$ and $2016 = 16 \cdot 126 = 2^5 \cdot 3^2 \cdot 7$.
For an integer to be divisible by each of $2^2 \cdot 5$ and 2^4 and $2^5 \cdot 3^2 \cdot 7$, it must include at least 5 factors of 2, at least 2 factors of 3, at least 1 factor of 5, and at least 1 factor of 7.
The smallest such positive integer is $2^5 \cdot 3^2 \cdot 5^1 \cdot 7^1 = 10\,080$. The tens digit of this integer is 8.

Answer: (E)

16. The three diagrams below show first the original position of the triangle, the position after it is reflected in the x-axis, and then the position when this result is reflected in the y-axis:



The final position is seen in (D).

Answer: (D)

17. Since the perimeter of square $PQRS$ is 120, then its side length is $\frac{1}{4}(120) = 30$.
Therefore, $PQ = QR = RS = SP = 30$.
Since the perimeter of $\triangle PZS$ is $2x$, then $PZ + ZS + SP = 2x$.
Since $PS = 30$, then $PZ + ZS = 2x - PS = 2x - 30$.
Therefore, the perimeter of pentagon $PQRSZ$ is

$$PQ + QR + RS + ZS + PZ = 30 + 30 + 30 + (PZ + ZS) = 30 + 30 + 30 + (2x - 30) = 2x + 60$$

which equals $60 + 2x$.

Answer: (C)

18. *Solution 1*
Suppose that the three integers are x, y and z where $x + y = 998$ and $x + z = 1050$ and $y + z = 1234$.
From the first two equations, $(x + z) - (x + y) = 1050 - 998$ or $z - y = 52$.
Since $z + y = 1234$ and $z - y = 52$, then $(z + y) + (z - y) = 1234 + 52$ or $2z = 1286$ and so $z = 643$.
Since $z = 643$ and $z - y = 52$, then $y = z - 52 = 643 - 52 = 591$.
Since $x + y = 998$ and $y = 591$, then $x = 998 - y = 998 - 591 = 407$.
The three original numbers are 407, 591 and 643.
The difference between the largest and smallest of these integers is $643 - 407 = 236$.

Solution 2
Suppose that the three numbers are x, y and z, and that $x \leq y \leq z$.
Since $y \leq z$, then $x + y \leq x + z$.
Since $x \leq y$, then $x + z \leq y + z$.
Therefore, $x + y \leq x + z \leq y + z$.
This tells us that $x + y = 998$, $x + z = 1050$, and $y + z = 1234$.
Since z is the largest and x is the smallest of the three original integers, we want to determine the value of $z - x$.
But $z - x = (y + z) - (x + y) = 1234 - 998 = 236$.

Answer: (E)

19. The number of points on the circle equals the number of spaces between the points around the circle.
Moving from the point labelled 7 to the point labelled 35 requires moving $35 - 7 = 28$ points and so 28 spaces around the circle.
Since the points labelled 7 and 35 are diametrically opposite, then moving along the circle from 7 to 35 results in travelling halfway around the circle.
Since 28 spaces makes half of the circle, then $2 \cdot 28 = 56$ spaces make the whole circle.
Thus, there are 56 points on the circle, and so $n = 56$.

Answer: (C)

20. *Solution 1*
Suppose that, when the n students are put in groups of 2, there are g complete groups and 1 incomplete group.
Since the students are being put in groups of 2, an incomplete group must have exactly 1 student in it.
Therefore, $n = 2g + 1$.
Since the number of complete groups of 2 is 5 more than the number of complete groups of 3, then there were $g - 5$ complete groups of 3.
Since there was still an incomplete group, this incomplete group must have had exactly 1 or 2 students in it.
Therefore, $n = 3(g - 5) + 1$ or $n = 3(g - 5) + 2$.
If $n = 2g+1$ and $n = 3(g-5)+1$, then $2g+1 = 3(g-5)+1$ or $2g+1 = 3g-14$ and so $g = 15$.
In this case, $n = 2g + 1 = 31$ and there were 15 complete groups of 2 and 10 complete groups of 3.
If $n = 2g+1$ and $n = 3(g-5)+2$, then $2g+1 = 3(g-5)+2$ or $2g+1 = 3g-13$ and so $g = 14$.
In this case, $n = 2g + 1 = 29$ and there were 14 complete groups of 2 and 9 complete groups of 3.
If $n = 31$, dividing the students into groups of 4 would give 7 complete groups of 4 and 1 incomplete group.
If $n = 29$, dividing the students into groups of 4 would give 7 complete groups of 4 and 1 incomplete group.
Since the difference between the number of complete groups of 3 and the number of complete groups of 4 is given to be 3, then it must be the case that $n = 31$.
In this case, $n^2 - n = 31^2 - 31 = 930$; the sum of the digits of $n^2 - n$ is 12.

Solution 2
Since the n students cannot be divided exactly into groups of 2, 3 or 4, then n is not a multiple of 2, 3 or 4.

The first few integers larger than 1 that are not divisible by 2, 3 or 4 are 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, and 35.
In each case, we determine the number of complete groups of each size:

n	5	7	11	13	17	19	23	25	29	31	35
# of complete groups of 2	2	3	5	6	8	9	11	12	14	15	17
# of complete groups of 3	1	2	3	4	5	6	7	8	9	10	11
# of complete groups of 4	1	1	2	3	4	4	5	6	7	7	8

Since the number of complete groups of 2 is 5 more than the number of complete groups of 3 which is 3 more than the number of complete groups of 4, then of these possibilities, $n = 31$ works.
In this case, $n^2 - n = 31^2 - 31 = 930$; the sum of the digits of $n^2 - n$ is 12.
(Since the problem is a multiple choice problem and we have found a value of n that satisfies the given conditions and for which an answer is present, then this answer must be correct. Solution 1 shows why $n = 31$ is the only value of n that satisfies the given conditions.)

ANSWER: (B)

21. Suppose that Jackie had played n games before her last game.
Since she scored an average of 20 points per game over these n games, then she scored $20n$ points over these n games.
In her last game, she scored 36 points and so she has now scored $20n + 36$ points in total.
But, after her last game, she has now played $n+1$ games and has an average of 21 points scored per game.
Therefore, we can also say that her total number of points scored is $21(n + 1)$.
Thus, $21(n + 1) = 20n + 36$ or $21n + 21 = 20n + 36$ and so $n = 15$.
This tells us that after 16 games, Jackie has scored $20(15) + 36 = 336$ points.
For her average to be 22 points per game after 17 games, she must have scored a total of $17 \cdot 22 = 374$ points.
This would mean that she must score $374 - 336 = 38$ points in her next game.

ANSWER: (A)

22. Since the track is circular with radius 25 km, then its circumference is $2\pi(25) = 50\pi$ km.
In the 15 minutes that Alain drives at 80 km/h, he drives a distance of $\frac{1}{4}(80) = 20$ km (because 15 minutes is one-quarter of an hour).
When Louise starts driving, she drives in the opposite direction to Alain.
Suppose that Alain and Louise meet for the first time after Louise has been driving for t hours.
During this time, Louise drives at 100 km/h, and so drives $100t$ km.
During this time, Alain drives at 80 km/h, and so drives $80t$ km.
Since they start $50\pi - 20$ km apart along the track (the entire circumference minus the 20 km that Alain drove initially), then the sum of the distances that they travel is $50\pi - 20$ km.
Therefore, $100t + 80t = 50\pi - 20$ and so $180t = 50\pi - 20$ or $t = \frac{5\pi-2}{18}$.
Suppose that Alain and Louise meet for the next time after an additional T hours.
During this time, Louise drives $100T$ km and Alain drives $80T$ km.
In this case, the sum of the distances that they drive is the complete circumference of the track, or 50π km. Thus, $180T = 50\pi$ or $T = \frac{5\pi}{18}$.
The length of time between the first and second meetings will be the same as the amount of time between the second and third, and between the third and fourth meetings.
Therefore, the total time that Louise has been driving when she and Alain meet for the fourth time will be $t + 3T = \frac{5\pi-2}{18} + 3 \cdot \frac{5\pi}{18} = \frac{20\pi-2}{18} = \frac{10\pi-1}{9}$ hours.

ANSWER: (C)

23. If the four sides that are chosen are adjacent, then when these four sides are extended, they will not form a quadrilateral that encloses the octagon. (See Figure 1.)
If the four sides are chosen so that there are exactly three adjacent sides that are not chosen and one other side not chosen, then when these four sides are extended, they will not form a quadrilateral that encloses the octagon. (See Figures 2 and 3.)

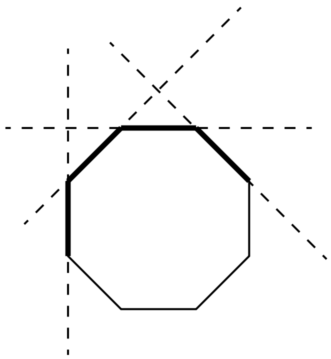


Figure 1

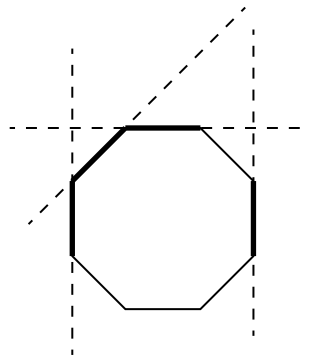


Figure 2

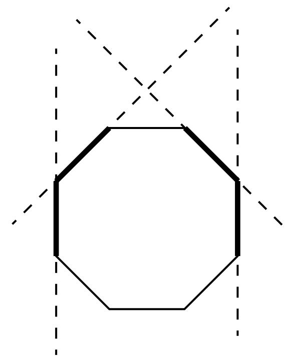


Figure 3

Any other set of four sides that are chosen will form a quadrilateral that encloses the octagon. This is true based on the following argument:

Suppose that side s_1 is chosen and that s_2 is the next side chosen in a clockwise direction.
These two sides will either be adjacent (Figure 4) or have one unchosen side between them (Figure 5) or will have two unchosen sides between them (Figure 6). (They cannot have three or four adjacent unchosen sides between them from the previous argument. They cannot have more than four adjacent unchosen sides between them since four sides have to be chosen.)

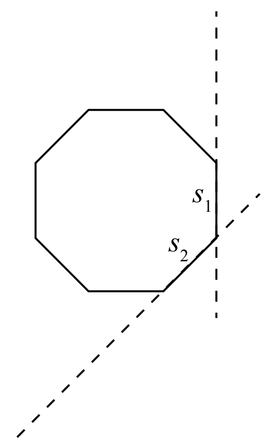


Figure 4

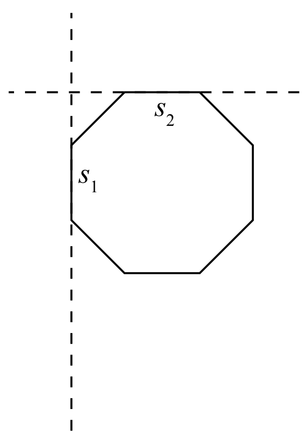


Figure 5

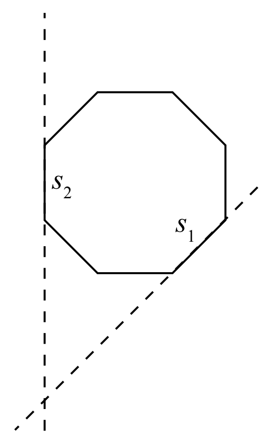


Figure 6

In each of these cases, sides s_1 and s_2 when extended meet on or outside the octagon and "between" s_1 and s_2.
Continue the process to choose s_3 and s_4 in such a way as there are not three or four adjacent unchosen sides between any consecutive pair of chosen sides.
Since each consecutive pair of chosen sides meets between the two sides and on or outside the octagon, then these four meeting points (between s_1 and s_2, s_2 and s_3, s_3

and s_4, and s_4 and s_1) and the line segments joining them will form a quadrilateral that includes the octagon. (An example is shown in Figure 7.)

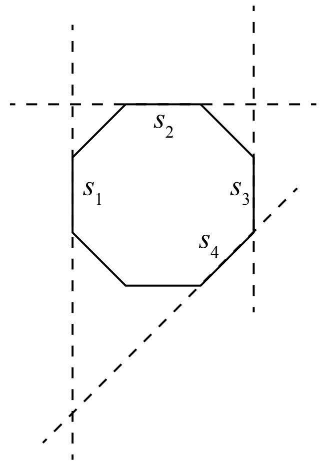


Figure 7

We are told that there is a total of 70 ways in which four sides can be chosen.
We will count the number of ways in which four sides can be chosen that do not result in the desired quadrilateral, and subtract this from 70 to determine the number of ways in which the desired quadrilateral does result.
There are 8 ways in which to choose four adjacent sides: choose one side to start (there are 8 ways to choose one side) and then choose the next three adjacent sides in clockwise order (there is 1 way to do this).
There are 8 ways to choose adjacent sides to be not chosen (after picking one such set, there are 7 additional positions into which these sides can be rotated). For each of these choices, there are 3 possible choices for the remaining unchosen sides. (Figures 2 and 3 give two of these choices; the third choice is the reflection of Figure 2 through a vertical line.) Therefore, there are $8 \times 3 = 24$ ways to choose four sides so that there are 3 adjacent sides unchosen.
Therefore, of the 70 ways of choosing four sides, exactly $8 + 24 = 32$ of them do not give the desired quadrilateral, and so $70 - 32 = 38$ ways do.
Thus, the probability that four sides are chosen so that the desired quadrilateral is formed is $\frac{38}{70} = \frac{19}{35}$.

Answer: (B)

24. Suppose that a number q has the property that there are exactly 19 integers n with $\sqrt{q} < n < q$.
Suppose that these 19 integers are $m, m+1, m+2, \dots, m+17, m+18$.
Then $\sqrt{q} < m < m+1 < m+2 < \cdots < m+17 < m+18 < q$.
This tells us that $q - \sqrt{q} > (m+18) - m = 18$ because $q - \sqrt{q}$ is as small as possible when q is as small as possible and $\sqrt{q}$ is as large as possible.
Also, since this is exactly the list of integers that is included strictly between $\sqrt{q}$ and q, then we must have $m - 1 \leq \sqrt{q} < m < m+1 < m+2 < \cdots < m+17 < m+18 < q \leq m+19$.
In other words, neither $m-1$ nor $m+19$ can satisfy $\sqrt{q} < n < q$.
This tell us that $q - \sqrt{q} \leq (m+19) - (m-1) = 20$.
Therefore, we have that $18 < q - \sqrt{q} \leq 20$.

Next, we use $18 < q - \sqrt{q} \leq 20$ to get a restriction on q itself.
To have $q - \sqrt{q} > 18$, we certainly need $q > 18$.
But if $q > 18$, then $\sqrt{q} > \sqrt{18} > 4$.

Furthermore, $q - \sqrt{q} > 18$ and $\sqrt{q} > 4$ give $q - 4 > q - \sqrt{q} > 18$ and so $q > 22$.
Next, note that $q - \sqrt{q} = \sqrt{q}(\sqrt{q} - 1)$.
When q is larger than 1 and increases, each factor $\sqrt{q}$ and $\sqrt{q} - 1$ increases, so the product $q - \sqrt{q}$ increases.
When $q = 25$, $q - \sqrt{q} = 25 - 5 = 20$.
Since we need $q - \sqrt{q} \leq 20$ and since $q - \sqrt{q} = 20$ when $q = 25$ and since $q - \sqrt{q}$ is increasing, then for $q - \sqrt{q} \leq 20$, we must have $q \leq 25$.
Therefore, $18 < q - \sqrt{q} \leq 20$ tells us that $22 < q \leq 25$.
So we limit our search for q to this range.

When $q = 22$, $\sqrt{q} \approx 4.69$, and so the integers n that satisfy $\sqrt{q} < n < q$ are $5, 6, 7, \ldots, 20, 21$, of which there are 17.
When $22 < q \leq 23$, we have $4 < \sqrt{q} < 5$ and $22 < q \leq 23$, which means that the integers n that satisfy $\sqrt{q} < n < q$ are $5, 6, 7, \ldots, 20, 21, 22$, of which there are 18.
When $23 < q \leq 24$, we have $4 < \sqrt{q} < 5$ and $23 < q \leq 24$, which means that the integers n that satisfy $\sqrt{q} < n < q$ are $5, 6, 7, \ldots, 20, 21, 22, 23$, of which there are 19.
When $24 < q < 25$, we have $4 < \sqrt{q} < 5$ and $24 < q < 25$, which means that the integers n that satisfy $\sqrt{q} < n < q$ are $5, 6, 7, \ldots, 20, 21, 22, 23, 24$, of which there are 20.
When $q = 25$, $\sqrt{q} = 5$ and so the integers that satisfy $\sqrt{q} < n < q$ are $6, 7, \ldots, 20, 21, 22, 23, 24$, of which there are 19.
Therefore, the numbers q for which there are exactly 19 integers n that satisfy $\sqrt{q} < n < q$ are $q = 25$ and those q that satisfy $23 < q \leq 24$.

Finally, we must determine the sum of all such q that are of the form $q = \dfrac{a}{b}$ where a and b are positive integers with $b \leq 10$.
The integers $q = 24$ and $q = 25$ are of this form with $a = 24$ and $a = 25$, respectively, and $b = 1$.
The q between 23 and 24 that are of this form with $b \leq 4$ are

$$23\tfrac{1}{2} = \tfrac{47}{2}, 23\tfrac{1}{3} = \tfrac{70}{3}, 23\tfrac{2}{3} = \tfrac{71}{3}, 23\tfrac{1}{4} = \tfrac{93}{4}, 23\tfrac{3}{4} = \tfrac{95}{4}$$

Notice that we don't include $23\frac{2}{4}$ since this is the same as the number $23\frac{1}{2}$.
We continue by including those satisfying $5 \leq b \leq 10$ and not including equivalent numbers that have already been included with smaller denominators, we obtain

$$23\tfrac{1}{2}, 23\tfrac{1}{3}, 23\tfrac{2}{3}, 23\tfrac{1}{4}, 23\tfrac{3}{4}, 23\tfrac{1}{5}, 23\tfrac{2}{5}, 23\tfrac{3}{5}, 23\tfrac{4}{5}, 23\tfrac{1}{6}, 23\tfrac{5}{6}, 23\tfrac{1}{7}, 23\tfrac{2}{7}, 23\tfrac{3}{7}, 23\tfrac{4}{7}, 23\tfrac{5}{7}, 23\tfrac{6}{7},$$

$$23\tfrac{1}{8}, 23\tfrac{3}{8}, 23\tfrac{5}{8}, 23\tfrac{7}{8}, 23\tfrac{1}{9}, 23\tfrac{2}{9}, 23\tfrac{4}{9}, 23\tfrac{5}{9}, 23\tfrac{7}{9}, 23\tfrac{8}{9}, 23\tfrac{1}{10}, 23\tfrac{3}{10}, 23\tfrac{7}{10}, 23\tfrac{9}{10}$$

There are 31 numbers in this list.
Each of these 31 numbers equals 23 plus a fraction between 0 and 1.
With the exception of the one number with denominator 2, each of the fractions can be paired with another fraction with the same denominator to obtain a sum of 1. (For example, $\frac{1}{5} + \frac{4}{5} = 1$ and $\frac{2}{5} + \frac{3}{5} = 1$.)
Therefore, the sum of all of these q between 23 and 24 is $31(23) + \frac{1}{2} + 15(1) = 728\frac{1}{2}$, because there are 31 contributions of 23 plus the fraction $\frac{1}{2}$ plus 15 pairs of fractions with a sum of 1.
Finally, the sum of all q of the proper form for which there are exactly 19 integers that satisfy $\sqrt{q} < n < q$ is $728\frac{1}{2} + 25 + 24 = 777\frac{1}{2}$.

ANSWER: (C)

25. Using the given rules, the words that are 1 letter long are A, B, C, D, E.
Using the given rules, the words that are 2 letters long are AB, AC, AD, BA, BC, BD, BE, CA, CB, CD, CE, DA, DB, DC, DE, EB, EC, ED.
Let v_n be the number of words that are n letters long and that begin with a vowel. Note that $v_1 = 2$ and $v_2 = 6$.
Let c_n be the number of words that are n letters long and that begin with a consonant. Note that $c_1 = 3$ and $c_2 = 12$.
Suppose $n \geq 2$.
Consider a word of length n that begins with a vowel (that is, with A or E).
Since two vowels cannot occur in a row, the second letter of this word must be B, C or D.
This means that every word of length n that begins with a vowel can be transformed into a word of length $n-1$ that begins with a consonant by removing the first letter.
Also, each word of length $n-1$ that begins with a consonant can form two different words of length n that begin with a vowel.
Therefore, $v_n = 2c_{n-1}$.

Consider a word of length n that begins with a consonant.
Since the same letter cannot occur twice in a row, then the second letter of this word is either a vowel or a different consonant than the first letter of the word.
Each word of length $n-1$ that begins with a vowel can form 3 words of length n that begin with a consonant, obtained by adding B, C, D to the start of the word.
Each word of length $n-1$ that begins with a consonant can form 2 words of length n that begin with a consonant, obtained by adding each of the consonants other than the one with which the word of length $n-1$ starts.
Therefore, $c_n = 3v_{n-1} + 2c_{n-1}$.

We note from above that $v_1 = 2$ and $c_1 = 3$.
The equations $v_2 = 2c_1$ and $c_2 = 3v_1 + 2c_1$ are consistent with the information that $v_2 = 6$ and $c_2 = 12$.
Since $v_2 = 6$ and $c_2 = 12$, then $v_3 = 2c_2 = 24$ and $c_3 = 3v_2 + 2c_2 = 3(6) + 2(12) = 42$.
We want to determine v_{10}.
We continue these calculations and make a table:

n	v_n	c_n
1	2	3
2	6	12
3	24	42
4	84	156
5	312	564
6	1128	2064
7	4128	7512
8	15 024	27 408
9	54 816	99 888
10	199 776	364 224

Therefore, there are 199 776 words of length 10 that begin with a vowel.

ANSWER: (E)

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2015 Cayley Contest

(Grade 10)

Tuesday, February 24, 2015
(in North America and South America)

Wednesday, February 25, 2015
(outside of North America and South America)

Solutions

1. Evaluating, $2 \times 2015 - 2015 = 4030 - 2015 = 2015$.
   Alternatively, $2 \times 2015 - 2015 = 2 \times 2015 - 1 \times 2015 = 1 \times 2015 = 2015$.

   Answer: (A)

2. Evaluating, $\sqrt{1} + \sqrt{9} = 1 + 3 = 4$.

   Answer: (D)

3. The volume of a rectangular box equals the area of its base times its height.
   Thus, the height equals the volume divided by the area of the base.
   The area of the base of the given box is $2 \cdot 5 = 10 \text{ cm}^2$.
   Therefore, the height of the given box is $\dfrac{30}{10} = 3$ cm.

   Answer: (C)

4. *Solution 1*
   $\angle SRQ$ is an exterior angle of $\triangle PQR$.
   Thus, $\angle SRQ = \angle RPQ + \angle PQR = 50^\circ + 90^\circ = 140^\circ$.
   Therefore, $x^\circ = 140^\circ$ and so $x = 140$.

   Solution 2
   The sum of the angles of $\triangle PQR$ is 180°, and so

   $$\angle PRQ = 180^\circ - \angle RPQ - \angle PQR = 180^\circ - 50^\circ - 90^\circ = 40^\circ$$

   Since $\angle PRQ$ and $\angle SRQ$ are supplementary, then $x^\circ + 40^\circ = 180^\circ$, and so $x = 180 - 40 = 140$.

   Answer: (D)

5. From the given graph, 3 provinces and territories joined Confederation between 1890 and 1929, and 1 between 1930 and 1969.
   Thus, between 1890 and 1969, a total of 4 provinces and territories joined Confederation.
   Therefore, if one of the 13 provinces and territories is chosen at random, the probability that it joined Confederation between 1890 and 1969 is $\frac{4}{13}$.

   Answer: (B)

6. Since $a^2 = 9$, then $a^4 = (a^2)^2 = 9^2 = 81$.
   Alternatively, we note that since $a^2 = 9$, then $a = \pm 3$. If $a = 3$, then $a^4 = 3^4 = 81$ and if $a = -3$, then $a^4 = (-3)^4 = 3^4 = 81$.

   Answer: (B)

7. First, we note that $3 + \frac{1}{10} + \frac{4}{100} = 3 + \frac{10}{100} + \frac{4}{100} = 3 + \frac{14}{100} = 3\frac{14}{100}$.
   Since $\frac{14}{100} = 0.14$, then the given expression also equals 3.14.
   Since $\frac{14}{100} = \frac{7}{50}$, then the given expression also equals $3\frac{7}{50}$.
   We also see that $3\frac{7}{50} = 3 + \frac{7}{50} = \frac{150}{50} + \frac{7}{50} = \frac{157}{50}$.
   Therefore, the only remaining expression is $3\frac{5}{110}$.
   We note further that $3\frac{5}{110} = 3.0\overline{45}$, which is not equal to 3.14.

   Answer: (C)

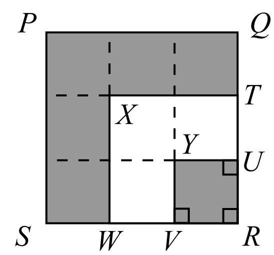
8. Violet starts with one-half of the money that she needed to buy the necklace.
After her sister gives her money, she has three-quarters of the amount that she needs.
This means that her sister gave her $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ of the total amount that she needs.
Since she now has three-quarters of the amount that she needs, then she still needs one-quarter of the total cost.
In other words, her father will give her the same amount that her sister gave her, or \$30.

Answer: (D)

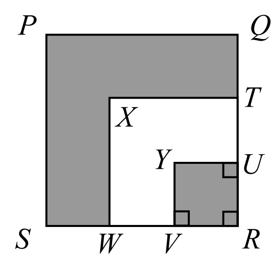
9. Since January 5 is a Monday and Mondays are 7 days apart, then January 12, 19 and 26 are also Mondays.
Since John goes for a run every 3 days, the dates in January on which he runs are January 5, 8, 11, 14, 17, 20, 23, 26, and 29.
The first of the Mondays on which John goes for a run after January 5 is January 26.

Answer: (C)

10. *Solution 1*
Since $PQRS$ is a square and TX and UY are perpendicular to QR, then TX and UY are parallel to PQ and SR.
Similarly, VY and WX are parallel to PS and QR.
Therefore, if we extend WX and VY to meet PQ and extend TX and UY to meet PS, then square $PQRS$ is divided into 9 rectangles.
Since $QT = TU = UR = 1$ and $RV = VW = WS = 1$, then in fact $PQRS$ is divided into 9 squares, each of which is 1 by 1.
Of these 9 squares, 6 are shaded and 3 are unshaded.
Therefore, the ratio of the shaded area to the unshaded area is $6 : 3$, which equals $2 : 1$.



Solution 2
Consider quadrilateral $YURV$.
$YURV$ has three right angles: at U and V because UY and VY are perpendicular to QR and RS, respectively, and at R because $PQRS$ is a square. Since $YURV$ has three right angles, then it has four right angles and so is a rectangle.
Since $RV = UR = 1$, then $YURV$ is actually a square and has side length 1, and so has area 1^2, or 1.
Similarly, $XTRW$ is a square of side length 2, and so has area 2^2, or 4.
Since square $PQRS$ is 3×3, then its area is 3^2, or 9.
The area of the unshaded region is equal to the difference between the areas of square $XTRW$ and square $YURV$, or $4 - 1 = 3$.
Since square $PQRS$ has area 9 and the area of the unshaded region is 3, then the area of the shaded region is $9 - 3 = 6$.
Finally, the ratio of the shaded area to the unshaded area is $6 : 3$, which equals $2 : 1$.



Answer: (A)

11. From the given definition,

$$4 \otimes 8 = \frac{4}{8} + \frac{8}{4} = \frac{1}{2} + 2 = 2\frac{1}{2} = \frac{5}{2}$$

Answer: (E)

12. The line with equation $y = \frac{3}{2}x + 1$ has slope $\frac{3}{2}$.
Since the line segment joining $(-1, q)$ and $(-3, r)$ is parallel to the line with equation $y = \frac{3}{2}x+1$, then the slope of this line segment is $\frac{3}{2}$.
Therefore, $\dfrac{r-q}{(-3)-(-1)} = \dfrac{3}{2}$ or $\dfrac{r-q}{-2} = \dfrac{3}{2}$.
Thus, $r - q = (-2) \cdot \dfrac{3}{2} = -3$.

Answer: (E)

13. *Solution 1*
The two teams include a total of $25 + 19 = 44$ players.
There are exactly 36 students who are at least one team.
Thus, there are $44 - 36 = 8$ students who are counted twice.
Therefore, there are 8 students who play both baseball and hockey.

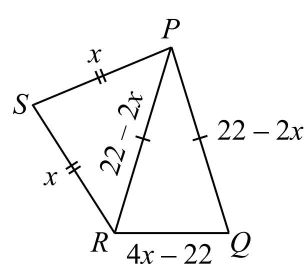
Solution 2
Suppose that there are x students who play both baseball and hockey.
Since there are 25 students who play baseball, then $25-x$ of these play baseball and not hockey.
Since there are 19 students who play hockey, then $19-x$ of these play hockey and not baseball.
Since 36 students play either baseball or hockey or both, then

$$(25 - x) + (19 - x) + x = 36$$

(The left side is the sum of the numbers of those who play baseball and not hockey, those who play hockey and not baseball, and those who play both.)
Therefore, $44 - x = 36$ and so $x = 44 - 36 = 8$.
Thus, 8 students play both baseball and hockey.

Answer: (B)

14. Since $PS = SR = x$ and the perimeter of $\triangle PRS$ is 22, then $PR = 22 - PS - SR = 22 - 2x$.
Since $PQ = PR$ and $PR = 22 - 2x$, then $PQ = 22 - 2x$.
Since $\triangle PQR$ has perimeter 22, then $RQ = 22-PR-PQ = 22-(22-2x)-(22-2x) = 4x-22$.



Since the perimeter of $PQRS$ is 24, then

$$\begin{aligned} PQ + RQ + SR + PS &= 24 \\ (22-2x) + (4x-22) + x + x &= 24 \\ 4x &= 24 \\ x &= 6 \end{aligned}$$

Therefore, $x = 6$.

Answer: (D)

15. We note that

$$1! = 1 \qquad 2! = (1)(2) = 2 \qquad 3! = (1)(2)(3) = 6$$

$$4! = (1)(2)(3)(4) = 24 \qquad 5! = (1)(2)(3)(4)(5) = 120$$

Thus, $1! + 2! + 3! + 4! + 5! = 1 + 2 + 6 + 24 + 120 = 153$.
Now for each positive integer $n \geq 5$, the ones digit of $n!$ is 0:

> One way to see this is to note that we obtain each successive factorial by multiplying the previous factorial by an integer. (For example, $6! = 6(5!)$.)
> Thus, if one factorial ends in a 0, then all subsequent factorials will also end in a 0. Since the ones digit of $5!$ is 0, then the ones digit of each $n!$ with $n > 5$ will also be 0.
> Alternatively, we note that for each positive integer n, the factorial $n!$ is the product of the positive integers from 1 to n. When $n \geq 5$, the product represented by $n!$ includes factors of both 2 and 5, and so has a factor of 10, thus has a ones digit of 0.

Therefore, the ones digit of each of $6!$, $7!$, $8!$, $9!$, and $10!$ is 0, and so the ones digit of $6! + 7! + 8! + 9! + 10!$ is 0.
Since the ones digit of $1! + 2! + 3! + 4! + 5!$ is 3 and the ones digit of $6! + 7! + 8! + 9! + 10!$ is 0, then the ones digit of $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10!$ is $3 + 0$ or 3.
(We can verify, using a calculator, that $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10! = 4\,037\,913$.)

ANSWER: (B)

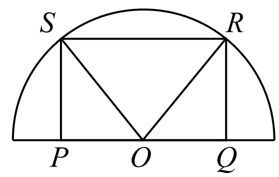
16. In a magic square, the numbers in each row, the numbers in each column, and numbers on each diagonal have the same sum.
Since the sum of the numbers in the first row equals the sum of the numbers in the first column, then $a + 13 + b = a + 19 + 12$ or $b = 19 + 12 - 13 = 18$.
Therefore, the sum of the numbers in any row, in any column, or along either diagonal equals the sum of the numbers in the third column, which is $18 + 11 + 16 = 45$.
Using the first column, $a + 19 + 12 = 45$ or $a = 14$.
Using the second row, $19 + c + 11 = 45$ or $c = 15$.
Thus, $a + b + c = 14 + 18 + 15 = 47$.

ANSWER: (C)

17. Suppose that Deanna drove at v km/h for the first 30 minutes.
Since 30 minutes equals one-half of an hour, then in these 30 minutes, she drove $\frac{1}{2}v$ km.
In the second 30 minutes, she drove at $(v + 20)$ km/h.
Thus, in these second 30 minutes, she drove $\frac{1}{2}(v + 20)$ km.
Since she drove 100 km in total, then $\frac{1}{2}v + \frac{1}{2}(v + 20) = 100$ or $\frac{1}{2}v + \frac{1}{2}v + 10 = 100$.
Thus, $v + 10 = 100$ or $v = 90$.
Therefore, Deanna drove 90 km/h for the first 30 minutes.

ANSWER: (B)

18. Let O be the centre of the circle. Join OS and OR.



Since the diameter of the semicircle is 20, then its radius is half of this, or 10.
Since OS and OR are radii, then $OS = OR = 10$.
Consider $\triangle OPS$ and $\triangle OQR$.
Since $PQRS$ is a rectangle, both triangles are right-angled (at P and Q).
Also, $PS = QR$ (equal sides of the rectangle) and $OS = OR$ (since they are radii of the circle).
Therefore, $\triangle OPS$ is congruent to $\triangle OQR$. (Right-angled triangles with equal hypotenuses and one other pair of equal corresponding sides are congruent.)
Since $\triangle OPS$ and $\triangle OQR$ are congruent, then $OP = OQ$.
Since $PQ = 16$, then $OP = \frac{1}{2}PQ = 8$.
Finally, since $\triangle OPS$ is right-angled at P, then we can apply the Pythagorean Theorem to conclude that $PS = \sqrt{OS^2 - OP^2} = \sqrt{10^2 - 8^2} = \sqrt{100 - 64} = 6$.

Answer: (A)

19. Consider a stack of bills with a total value of \$1000 that includes x \$20 bills and y \$50 bills.
The \$20 bills are worth \$$20x$ and the \$50 bills are worth \$$50y$, and so $20x + 50y = 1000$ or $2x + 5y = 100$.
Determining the number of possible stacks that the teller could have is equivalent to determining the numbers of pairs (x, y) of integers with $x \geq 1$ and $y \geq 1$ and $2x + 5y = 100$.
(We must have $x \geq 1$ and $y \geq 1$ because each stack includes at least one \$20 bill and at least one \$50 bill.)
Since $x \geq 1$, then $2x \geq 2$, so $5y = 100 - 2x \leq 98$.
This means that $y \leq \frac{98}{5} = 19.6$.
Since y is an integer, then $y \leq 19$.
Also, since $5y = 100 - 2x$, then the right side is the difference between two even integers, so $5y$ is itself even, which means that y must be even.
Therefore, the possible values of y are $2, 4, 6, 8, 10, 12, 14, 16, 18$.
Each of these values gives a pair (x, y) that satisfies the equation $2x + 5y = 100$:

$$(x, y) = (45, 2), (40, 4), (35, 6), (30, 8), (25, 10), (20, 12), (15, 14), (10, 16), (5, 18)$$

Translating back to the original context, we see that the maximum number of stacks that the teller could have is 9.

Answer: (A)

20. First, we calculate the value of $72\left(\frac{3}{2}\right)^n$ for each integer from $n=-3$ to $n=4$, inclusive:

$$72\left(\tfrac{3}{2}\right)^{-3} = 72\cdot\tfrac{2^3}{3^3} = 72\cdot\tfrac{8}{27} = \tfrac{64}{3} \qquad 72\left(\tfrac{3}{2}\right)^{-2} = 72\cdot\tfrac{2^2}{3^2} = 72\cdot\tfrac{4}{9} = 32$$
$$72\left(\tfrac{3}{2}\right)^{-1} = 72\cdot\tfrac{2^1}{3^1} = 72\cdot\tfrac{2}{3} = 48 \qquad 72\left(\tfrac{3}{2}\right)^{0} = 72\cdot 1 = 72$$
$$72\left(\tfrac{3}{2}\right)^{1} = 72\cdot\tfrac{3^1}{2^1} = 72\cdot\tfrac{3}{2} = 108 \qquad 72\left(\tfrac{3}{2}\right)^{2} = 72\cdot\tfrac{3^2}{2^2} = 72\cdot\tfrac{9}{4} = 162$$
$$72\left(\tfrac{3}{2}\right)^{3} = 72\cdot\tfrac{3^3}{2^3} = 72\cdot\tfrac{27}{8} = 243 \qquad 72\left(\tfrac{3}{2}\right)^{4} = 72\cdot\tfrac{3^4}{2^4} = 72\cdot\tfrac{81}{16} = \tfrac{729}{2}$$

Therefore, there are at least 6 integer values of n for which $72\left(\frac{3}{2}\right)^n$ is an integer, namely $n=-2,-1,0,1,2,3$.
Since 6 is the largest possible choice given, then it must be the correct answer (that is, it must be the case that there are no more values of n that work).

We can justify this statement informally by noting that if we start with $72\left(\frac{3}{2}\right)^4 = \frac{729}{2}$, then making n larger has the effect of continuing to multiply by $\frac{3}{2}$ which keeps the numerator odd and the denominator even, and so $72\left(\frac{3}{2}\right)^n$ is never an integer when $n>3$. A similar argument holds when $n<-2$.
We could justify the statement more formally by re-writing

$$72\left(\tfrac{3}{2}\right)^n = 3^2\cdot 2^3\cdot 3^n\cdot 2^{-n} = 3^2 3^n 2^3 2^{-n} = 3^{2+n}2^{3-n}$$

For this product to be an integer, it must be the case that each of 3^{2+n} and 2^{3-n} is an integer. (Each of 3^{2+n} and 2^{3-n} is either an integer or a fraction with numerator 1 and denominator equal to a power of 2 or 3. If each is such a fraction, then their product is less than 1 and so is not an integer. If exactly one is an integer, then their product equals a power of 2 divided by a power of 3 or vice versa. Such a fraction cannot be an integer since powers of 3 cannot be "divided out" of powers of 2 and vice versa.)
This means that $2+n\geq 0$ (and so $n\geq -2$) and $3-n\geq 0$ (and so $n\leq 3$).
Therefore, $-2\leq n\leq 3$. The integers in this range are the six integers listed above.

ANSWER: (E)

21. We are given that three consecutive odd integers have an average of 7.
These three integers must be 5, 7 and 9.
One way to see this is to let the three integers be $a-2, a, a+2$. (Consecutive odd integers differ by 2.)
Since the average of these three integers is 7, then their sum is $3\cdot 7 = 21$.
Thus, $(a-2)+a+(a+2)=21$ or $3a=21$ and so $a=7$.
When m is included, the average of the four integers equals their sum divided by 4, or $\dfrac{21+m}{4}$.
This average is an integer whenever $21+m$ is divisible by 4.
Since 21 is 1 more than a multiple of 4, then m must be 1 less than a multiple of 4 for the sum $21+m$ to be a multiple of 4.
The smallest positive integers m that are 1 less than a multiple of 4 are 3, 7, 11, 15, 19.
Since m cannot be equal to any of the original three integers 5, 7 and 9, then the three smallest possible values of m are 3, 11 and 15.
The sum of these possible values is $3+11+15=29$.

ANSWER: (D)

22. We label the players P, Q, R, S, T, U.
Each player plays 2 games against each of the other 5 players, and so each player plays 10 games.
Thus, each player earns between 0 and 10 points, inclusive.
We show that a player must have at least $9\frac{1}{2}$ points to guarantee that he has more points than every other player.
We do this by showing that it is possible to have two players with 9 points, and that if one player has $9\frac{1}{2}$ or 10 points, then every other player has at most 9 points.
Suppose that P and Q each win both of their games against each of R, S, T, and U and tie each of their games against each other.
Then P and Q each have a record of 8 wins, 2 ties, 0 losses, giving them each $8 \cdot 1 + 2 \cdot \frac{1}{2} + 0 \cdot 0$ or 9 points.
We note also that R, S, T, U each have 4 losses (2 against each of P and Q), so have at most 6 points.
Therefore, if a player has 9 points, it does not guarantee that he has more points than every other player, since in the scenario above both P and Q have 9 points.
Suppose next that P has $9\frac{1}{2}$ or 10 points.
If P has 10 points, then P won every game that he played, which means that every other player lost at least 2 games, and so can have at most 8 points.
If P has $9\frac{1}{2}$ points, then P must have 9 wins, 1 tie and 0 losses. (With $9\frac{1}{2}$ points, P has only "lost" $\frac{1}{2}$ point and so cannot have lost any games and can only have tied 1 game.)
Since P has 9 wins, then P must have beaten each of the other players at least once. (If there was a player that P had not beaten, then P would have at most $4 \cdot 2 = 8$ wins.)
Since every other player has at least 1 loss, then every other player has at most 9 points.
Therefore, if P has $9\frac{1}{2}$ or 10 points, then P has more points than every other player.
In summary, if a player has $9\frac{1}{2}$ or 10 points, then he is guaranteed to have more points than every other player, while if he has 9 points, it is possible to have the same number of points as another player.
Thus, the minimum number of points necessary to guarantee having more points than every other player is $9\frac{1}{2}$.

Answer: (D)

23. Nylah's lights come on randomly at one of the times 7:00 p.m., 7:30 p.m., 8:00 p.m., 8:30 p.m., or 9:00 p.m., each with probability $\frac{1}{5}$.
What is the probability that the lights come on at 7:00 p.m. and are on for t hours with $4 < t < 5$?
If the lights come on at 7:00 p.m. and are on for between 4 and 5 hours, then they go off between 11:00 p.m. and 12:00 a.m.
Since the length of this interval is 1 hour and the length of the total interval of time in which the lights randomly go off is 2 hours (11:00 p.m. to 1:00 a.m.), then the probability that they go off between 11:00 p.m. and 12:00 a.m. is $\frac{1}{2}$.
Therefore, the probability that the lights come on at 7:00 p.m. and are on for t hours with $4 < t < 5$ is $\frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}$.
Similarly, if the lights come on at 7:30 p.m., they can go off between 11:30 p.m. and 12:30 a.m., and the probability of this combination of events is also $\frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}$.
Similarly again, the probability of the lights coming on at 8:00 p.m. and going off between 12:00 a.m. and 1:00 a.m. is also $\frac{1}{10}$.
If the lights come on at 8:30 p.m., then to be on for between 4 and 5 hours, they must go off between 12:30 a.m. and 1:00 a.m. (They cannot stay on past 1:00 a.m.)
The probability of this combination is $\frac{1}{5} \cdot \frac{1/2}{2} = \frac{1}{5} \cdot \frac{1}{4} = \frac{1}{20}$.

If the lights come on at 9:00 p.m., they cannot be on for more than 4 hours, since the latest that they can go off is 1:00 a.m.
Therefore, the probability that the lights are on for between 4 and 5 hours is $3 \cdot \frac{1}{10} + \frac{1}{20} = \frac{7}{20}$.
(We note that we can safely ignore the question of whether the lights coming on at 7:30 p.m. and going off at exactly 11:30 p.m., for example, affects the probability calculation, because 11:30 p.m. is a single point in an interval containing an infinite number of points, and so does not affect the probability.)

Answer: (E)

24. We call an arrangement of tiles of a specific region that satisfies any given conditions a *tiling*.
Since no tile can cross the line TU, we can consider tilings of the regions $PTUS$ and $TQRU$ separately.
We proceed without including the units of metres on each dimension.
First, we determine the number of tilings of the 2×4 region $PTUS$.
To be able to discuss this effectively, we split $PTUS$ into 8 squares measuring 1×1 and label these squares as shown in Figure 1.

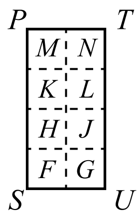


Figure 1

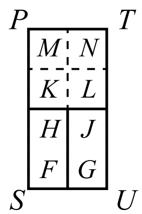


Figure 2

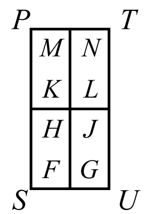


Figure 3

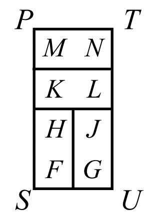


Figure 4

Consider square F. It must be covered with either a horizontal tile (covering FG) or a vertical tile (covering FH).
If F is covered with a vertical tile FH, then G must also be covered with a vertical tile GJ, since G is covered and its tile cannot overlap TU.
This gives the configuration in Figure 2.
The remaining 2×2 square can be covered in the two ways shown in Figure 3 and Figure 4.
This gives 2 tilings of $PTUS$ so far.
If F is covered with a horizontal tile FG, then we focus on H and J.
Either H and J are covered by one horizontal tile HJ (again leaving a 2×2 square that can be covered in 2 ways as above (see Figures 5 and 6)) or H and J are each covered by vertical tiles HK and JL, which means that MN is covered with 1 horizontal tile (see Figure 7).

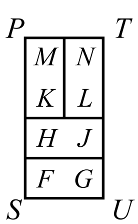


Figure 5

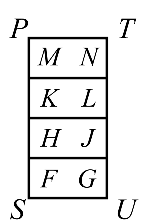


Figure 6

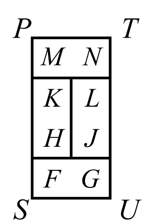


Figure 7

So if F is covered with a horizontal tile, there are $2 + 1 = 3$ tilings.
In total, there are $2 + 3 = 5$ possible tilings of the 2×4 region $PTUS$.

Consider now region $TQRU$.
Suppose that the number of tilings of the 4×4 region $TQRU$ is t.
Then for each of the 5 tilings of $PTUS$, there are t tilings of $TQRU$, so there will be $5t$ tilings of the entire region $PQRS$.

Divide $TQRU$ into 1×1 squares and label them as shown in Figure 8. We define V and W to be the midpoints of TQ and UR, respectively.

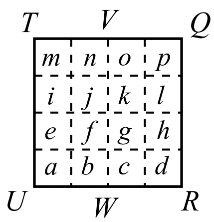


Figure 8

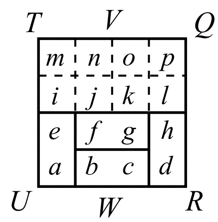


Figure 9

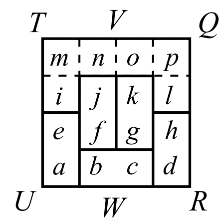


Figure 10

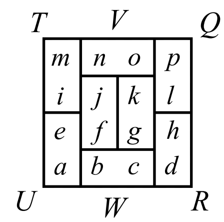


Figure 11

We consider two cases – either the line VW is overlapped by a tile, or it isn't.
If VW is not overlapped by a tile, then each of $TVWU$ and $VQRW$ is a 2×4 region to be tiled, and so can be tiled in 5 ways, as we saw with $PTUS$.
In this case, the number of tilings of $TQRU$ is $5 \times 5 = 25$.

Suppose that VW is overlapped by at least one tile.
If bc is covered by a horizontal tile, then ae and dh are covered by vertical tiles.
In this case, either fg is covered by a horizontal tile (Figure 9), or each of f and g is covered by a vertical tile (Figure 10).
In the first case (Figure 9), the upper 4×2 region needs to be tiled and there are 5 ways to do this, as above.
In the second case (Figure 10), the remaining tiling is forced to be as shown in Figure 11. Can you see why?
Therefore, if bc is covered by a horizontal tile, there are $5 + 1 = 6$ tilings.

Suppose that bc is not covered by a horizontal tile, but fg is covered by a horizontal tile.
Then ab and cd are each covered by horizontal tiles and so ei and hl are each covered by vertical tiles and so mn and op are each covered by horizontal tiles, and so jk must be covered by a horizontal tile.

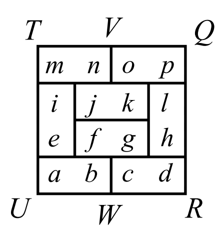


Figure 12

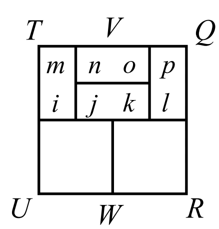


Figure 13

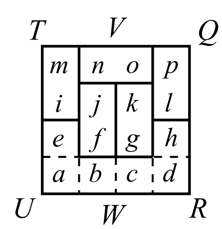


Figure 14

In other words, there is only 1 tiling in this case, shown in Figure 12.

Suppose now that bc and fg are not covered by a horizontal tile, but jk is.
In this case, each of the bottom 2×2 squares is tiled in a self-contained way. There are 2 ways to tile each, and so 4 ways to tile the pair of squares. (These tilings are each self-contained because if either ei or hl is covered by a vertical tile, then the remaining three tiles in the corresponding 2×2 square cannot be covered with 1×2 tiles.)

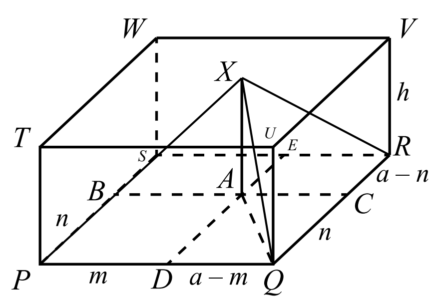
Furthermore, im and lp must be covered by vertical tiles, meaning that no is tiled with a horizontal tile, as shown in Figure 13, and so there is only one tiling of the upper rectangle.
Thus, there are $2 \times 2 \times 1 = 4$ tilings of $TQRU$ in this case, since the rest of the tiling is determined without choice.

Suppose finally that none of bc, fg, or jk is covered by a horizontal tile, but no is.
Then im and lp are covered with vertical tiles, which means that fj and gk are covered by vertical tiles, giving the diagram in Figure 14.
There is no way to complete this tiling without using a horizontal tile bc.
Therefore, there are no tilings in this case.

Finally, we can now say that there are $t = 25 + 6 + 4 + 1$ tilings of $TQRU$.
This means that there are $5 \times 36 = 180$ ways of tiling the entire 6×4 region with the given conditions.

Answer: (A)

25. Suppose that the square base $PQRS$ of the prism has side length a, and that the prism has height h.
We are asked to find the maximum possible area for rectangle $PQUT$.
The area of rectangle $PQUT$ is equal to ah, since $PQ = a$ and $QU = h$.
Let A be the point on $PQRS$ directly below X (that is, XA is perpendicular to the plane of $PQRS$). Note that $AX = h$.
Draw line segment BC through A with B on PS and C on QR so that BC is parallel to PQ.
Draw line segment DE through A with D on PQ and E on SR so that DE is parallel to QR.
Then segments BC and DE divide square $PQRS$ into four rectangles.
Let $PD = m$ and $PB = n$.
Then $SE = m$, $QC = n$, $DQ = ER = a - m$, and $CR = BS = a - n$.



$\triangle XAQ$ is right-angled at A so $QX^2 = AX^2 + AQ^2$.
But AQ is the hypotenuse of right-angled $\triangle ADQ$, so $AQ^2 = AD^2 + DQ^2$.
Thus, $QX^2 = AX^2 + AD^2 + DQ^2$.
Since $QX = 10$, $AX = h$, $AD = CQ = n$, and $DQ = a - m$, then $10^2 = h^2 + n^2 + (a - m)^2$.
Similarly, using $PX = 12$, we find that $12^2 = h^2 + n^2 + m^2$ and using $RX = 8$, we find that $8^2 = h^2 + (a - n)^2 + (a - m)^2$.
Subtracting $10^2 = h^2+n^2+(a-m)^2$ from $12^2 = h^2+n^2+m^2$, we obtain $144-100 = m^2-(a-m)^2$ or $44 = m^2 - (a^2 - 2am + m^2)$ which gives $44 = 2am - a^2$ or $m = \dfrac{44 + a^2}{2a}$.
Similarly, subtracting $8^2 = h^2 + (a - n)^2 + (a - m)^2$ from $10^2 = h^2 + n^2 + (a - m)^2$ gives $100 - 64 = n^2 - (a - n)^2$ or $36 = n^2 - (a^2 - 2an + n^2)$ which gives $36 = 2an - a^2$ or $n = \dfrac{36 + a^2}{2a}$.

Substituting these expressions for m and n into $12^2 = h^2 + n^2 + m^2$ gives

$$\begin{aligned} h^2 &= 144 - m^2 - n^2 \\ h^2 &= 144 - \left(\frac{44+a^2}{2a}\right)^2 - \left(\frac{36+a^2}{2a}\right)^2 \end{aligned}$$

Recall that we want to maximize ah.
Since $ah > 0$, then maximizing ah is equivalent to maximizing $(ah)^2 = a^2h^2$, which is equivalent to maximizing $4a^2h^2$.
From above,

$$\begin{aligned} 4a^2h^2 &= 4a^2\left(144 - \left(\frac{44+a^2}{2a}\right)^2 - \left(\frac{36+a^2}{2a}\right)^2\right) \\ &= 4a^2\left(144 - \frac{(44+a^2)^2}{4a^2} - \frac{(36+a^2)^2}{4a^2}\right) \\ &= 576a^2 - (44+a^2)^2 - (36+a^2)^2 \\ &= 576a^2 - (1936 + 88a^2 + a^4) - (1296 + 72a^2 + a^4) \\ &= -2a^4 + 416a^2 - 3232 \\ &= -2(a^4 - 208a^2 + 1616) \\ &= -2((a^2-104)^2 + 1616 - 104^2) \qquad \text{(completing the square)} \\ &= -2(a^2-104)^2 - 2(1616 - 104^2) \\ &= -2(a^2-104)^2 + 18400 \end{aligned}$$

Since $(a^2 - 104)^2 \geq 0$, then $4a^2h^2 \leq 18400$ (with equality when $a = \sqrt{104}$).
Therefore, $a^2h^2 \leq 4600$ and so $ah \leq \sqrt{4600}$.
This means that maximum possible area of $PQUT$ is $\sqrt{4600} = 10\sqrt{46} \approx 67.823$.
Of the given answers, this is closest to 67.82.

Answer: (B)

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2014 Cayley Contest

(Grade 10)

Thursday, February 20, 2014
(in North America and South America)

Friday, February 21, 2014
(outside of North America and South America)

Solutions

1. We rearrange the given expression to obtain $2000 + 200 - 80 - 120$.
Since $200 - 80 - 120 = 0$, then $2000 + 200 - 80 - 120 = 2000$.
Alternatively, we could have evaluated each operation in order to obtain
$$2000 - 80 + 200 - 120 = 1920 + 200 - 120 = 2120 - 120 = 2000$$
Answer: (A)

2. Since $(2)(3)(4) = 6x$, then $6(4) = 6x$. Dividing both sides by 6, we obtain $x = 4$.
Answer: (E)

3. The unlabelled angle inside the triangle equals its vertically opposite angle, or 40°.
Since the sum of the angles in a triangle is 180°, then $40^\circ + 60^\circ + x^\circ = 180^\circ$ or $100 + x = 180$.
Thus, $x = 80$.
Answer: (C)

4. The line representing a temperature of 3° is the horizontal line passing halfway between 2° and 4° on the vertical axis.
There are two data points on this line: one at 2 p.m. and one at 9 p.m.
The required time is 9 p.m.
Answer: (A)

5. Since $2n + 5 = 16$, then $2n - 3 = (2n + 5) - 8 = 16 - 8 = 8$.
Alternatively, we could solve the equation $2n + 5 = 16$ to obtain $2n = 11$ or $n = \frac{11}{2}$.
From this, we see that $2n - 3 = 2(\frac{11}{2}) - 3 = 11 - 3 = 8$.
Answer: (A)

6. Since $3 = \frac{6}{2}$ and $\frac{5}{2} < \frac{6}{2}$, then $\frac{5}{2} < 3$.
Since $3 = \sqrt{9}$ and $\sqrt{9} < \sqrt{10}$, then $3 < \sqrt{10}$.
Thus, $\frac{5}{2} < 3 < \sqrt{10}$, and so the list of the three numbers in order from smallest to largest is $\frac{5}{2}, 3, \sqrt{10}$.
Answer: (B)

7. 20% of the number 100 is 20, so when 100 is increased by 20%, it becomes $100 + 20 = 120$.
50% of a number is half of that number, so 50% of 120 is 60.
Thus, when 120 is increased by 50%, it becomes $120 + 60 = 180$.
Therefore, Meg's final result is 180.
Answer: (E)

8. Since $\triangle PQR$ is right-angled at P, we can use the Pythagorean Theorem.
We obtain $PQ^2 + PR^2 = QR^2$ or $10^2 + PR^2 = 26^2$.
This gives $PR^2 = 26^2 - 10^2 = 676 - 100 = 576$ and so $PR = \sqrt{576} = 24$, since $PR > 0$.
Since $\triangle PQR$ is right-angled at P, its area equals $\frac{1}{2}(PR)(PQ) = \frac{1}{2}(24)(10) = 120$.
Answer: (B)

9. We use A, B, C, D, E to represent Amy, Bob, Carla, Dan, and Eric, respectively.
We use the greater than symbol ($>$) to represent "is taller than" and the less than symbol ($<$) to represent "is shorter than".
From the first bullet, $A > C$.
From the second bullet, $D < E$ and $D > B$ so $E > D > B$.
From the third bullet, $E < C$ or $C > E$.
Since $A > C$ and $C > E$ and $E > D > B$, then $A > C > E > D > B$, which means that Bob is the shortest.
Answer: (B)

10. *Solution 1*

We start from the `OUTPUT` and work back to the `INPUT`.

Since the `OUTPUT` 32 is obtained from adding 16 to the previous number, then the previous number is $32 - 16 = 16$.

$$\boxed{\texttt{INPUT}} \rightarrow \text{Subtract } 8 \rightarrow \boxed{} \rightarrow \text{Divide by } 2 \rightarrow \boxed{16} \rightarrow \text{Add } 16 \rightarrow \boxed{32}$$

Since 16 is obtained by dividing the previous number by 2, then the previous number is 2×16 or 32.

$$\boxed{\texttt{INPUT}} \rightarrow \text{Subtract } 8 \rightarrow \boxed{32} \rightarrow \text{Divide by } 2 \rightarrow \boxed{16} \rightarrow \text{Add } 16 \rightarrow \boxed{32}$$

Since 32 is obtained by subtracting 8 from the `INPUT`, then the `INPUT` must have been $32+8 = 40$.

$$\boxed{40} \rightarrow \text{Subtract } 8 \rightarrow \boxed{32} \rightarrow \text{Divide by } 2 \rightarrow \boxed{16} \rightarrow \text{Add } 16 \rightarrow \boxed{32}$$

Solution 2

Suppose that the `INPUT` is x.

Subtracting 8 gives $x - 8$.

Dividing this result by 2 gives $\frac{1}{2}(x-8)$ or $\frac{1}{2}x - 4$.

Adding 16 to this result gives $(\frac{1}{2}x - 4) + 16 = \frac{1}{2}x + 12$, which is the `OUTPUT`.

$$\boxed{x} \rightarrow \text{Subtract } 8 \rightarrow \boxed{x-8} \rightarrow \text{Divide by } 2 \rightarrow \boxed{\tfrac{1}{2}x - 4} \rightarrow \text{Add } 16 \rightarrow \boxed{\tfrac{1}{2}x + 12}$$

If the `OUTPUT` is 32, then $\frac{1}{2}x + 12 = 32$ or $\frac{1}{2}x = 20$ and so $x = 40$.

Therefore, the `INPUT` must have been 40.

Answer: (D)

11. We consider the equation of the line shown in the form $y = mx + b$.

The slope, m, of the line shown is negative.

The y-intercept, b, of the line shown is positive.

Of the given choices only $y = -2x + 3$ has $m < 0$ and $b > 0$.

Therefore, a possible equation for the line is $y = -2x + 3$.

Answer: (E)

12. Since $x = 2y$, then $(x - y)(2x + y) = (2y - y)(2(2y) + y) = (y)(5y) = 5y^2$.

Answer: (A)

13. Erika assembling 9 calculators is the same as assembling three groups of 3 calculators.

Since Erika assembles 3 calculators in the same amount of time that Nick assembles 2 calculators, then he assembles three groups of 2 calculators (that is, 6 calculators) in this time.

Since Nick assembles 1 calculator in the same amount of time that Sam assembles 3 calculators, then Sam assembles 18 calculators while Nick assembles 6 calculators.

Thus, the three workers assemble $9+6+18 = 33$ calculators while Erika assembles 9 calculators.

Answer: (E)

14. Since 1 GB = 1024 MB, then Julia's 300 GB hard drive has $300 \times 1024 = 307\,200$ MB of storage space.

When Julia puts 300 000 MB of data on the empty hard drive, the amount of empty space remaining is $307\,200 - 300\,000 = 7200$ MB.

Answer: (C)

15. From the second row, $\triangle + \triangle + \triangle + \triangle = 24$ or $4\triangle = 24$, and so $\triangle = 6$.
From the first row, $\heartsuit + \triangle + \triangle + \heartsuit = 26$ or $2\heartsuit + 2\triangle = 26$.
Since $\triangle = 6$, then $2\heartsuit = 26 - 12 = 14$, and so $\heartsuit = 7$.
From the fourth row, $\square + \heartsuit + \square + \triangle = 33$.
Since $\triangle = 6$ and $\heartsuit = 7$, then $2\square + 7 + 6 = 33$, and so $2\square = 20$ or $\square = 10$.
Finally, from the third row, $\square + \blacklozenge + \heartsuit + \blacklozenge = 27$.
Since $\square = 10$ and $\heartsuit = 7$, then $2\blacklozenge = 27 - 10 - 7 = 10$.
Thus, $\blacklozenge = 5$.

Answer: (A)

16. The mean number of hamburgers eaten per student equals the total number of hamburgers eaten divided by the total number of students.
12 students each eat 0 hamburgers. This is a total of 0 hamburgers eaten.
14 students each eat 1 hamburger. This is a total of 14 hamburgers eaten.
8 students each eat 2 hamburgers. This is a total of 16 hamburgers eaten.
4 students each eat 3 hamburgers. This is a total of 12 hamburgers eaten.
2 students each eat 4 hamburgers. This is a total of 8 hamburgers eaten.
Thus, a total of $0 + 14 + 16 + 12 + 8 = 50$ hamburgers are eaten.
The total number of students is $12 + 14 + 8 + 4 + 2 = 40$.
Therefore, the mean number of hamburgers eaten is $\frac{50}{40} = 1.25$.

Answer: (C)

17. A circle with area 36π has radius 6, since the the area of a circle with radius r equals πr^2 and $\pi(6^2) = 36\pi$.
The circumference of a circle with radius 6 equals $2\pi(6) = 12\pi$.
Therefore, each quarter-circle contributes $\frac{1}{4}(12\pi) = 3\pi$ to the circumference.
The perimeter of the given figure consists of three quarter-circle sections and two radii from the circle.
Thus, its perimeter is $3(3\pi) + 2(6) = 9\pi + 12$.

Answer: (B)

18. Suppose that the number of 2¢ stamps that Sonita buys is x.
Then the number of 1¢ stamps that she buys is $10x$.
The total value of the 2¢ and 1¢ stamps that she buys is $2(x) + 1(10x) = 12x$¢.
Since she buys some 5¢ stamps as well and the total value of the stamps that she buys is 100¢, then the value of the 5¢ stamps that she buys is $(100 - 12x)$¢.
Thus, $100 - 12x$ must be a multiple of 5. Since 100 is a multiple of 5, then $12x$ must be a multiple of 5, and so x is a multiple of 5 (since 12 has no divisors larger than 1 in common with 5).
Note that $x > 0$ (since she buys some 2¢ stamps) and $x < 9$ (since $12x$ is less than 100). The only multiple of 5 between 0 and 9 is 5, so $x = 5$.
When $x = 5$, the value of 5¢ stamps is $100 - 12x = 100 - 12(5) = 40$¢ and so she buys $\frac{40}{5} = 8$ 5¢ stamps.
Finally, she buys 5 2¢ stamps, 50 1¢ stamps, and 8 5¢ stamps, for a total of $5 + 50 + 8 = 63$ stamps.
(Checking, these stamps are worth $5(2) + 50(1) + 8(5) = 10 + 50 + 40 = 100$¢ in total, as required.)

Answer: (D)

19. There are ten possible pairs of numbers that can be chosen: -3 and -1; -3 and 0; -3 and 2; -3 and 4; -1 and 0; -1 and 2; -1 and 4; 0 and 2; 0 and 4; 2 and 4. Each pair is equally likely to be chosen.
Pairs that include 0 (4 pairs) have a product of 0; pairs that do not include 0 (6 of them) do not have a product of 0.
Therefore, the probability that a randomly chosen pair has a product of 0 is $\frac{4}{10}$ or $\frac{2}{5}$.
Answer: (D)

20. The layer sum of $wxyz$ equals 2014.
This means that the sum of the integer with digits $wxyz$, the integer with digits xyz, the integer with digits yz, and the integer z is 2014.
Note that the integer with digits $wxyz$ equals $1000w + 100x + 10y + z$, the integer with digits xyz equals $100x + 10y + z$, and the integer with digits yz equals $10y + z$.
Therefore, we have

$$(1000w + 100x + 10y + z) + (100x + 10y + z) + (10y + z) + z = 2014$$

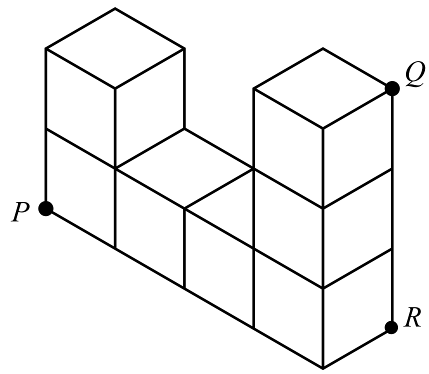
or

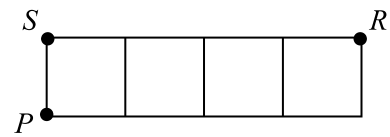
$$1000w + 200x + 30y + 4z = 2014 \quad (*)$$

Each of w, x, y, z is a single digit and $w \neq 0$.
Now w cannot be 3 or greater, or the left side of $(*)$ would be at least 3000, which is too large.
Thus, $w = 1$ or $w = 2$.
If $w = 2$, then $2000+200x+30y+4z = 2014$ and so $200x+30y+4z = 14$ or $100x+15y+2z = 7$. This would mean that $x = y = 0$ (since otherwise the terms $100x + 15y$ would contribute more than 7), which gives $2z = 7$ which has no integer solutions. Thus, $w \neq 2$.
Therefore, $w = 1$.
This gives $1000+200x+30y+4z = 2014$ and so $200x+30y+4z = 1014$ or $100x+15y+2z = 507$.
Since $0 \le y \le 9$ and $0 \le z \le 9$, then $0 \le 15y + 2z \le 15(9) + 2(9) = 153$.
Since $100x$ is a multiple of 100 and $0 \le 15y + 2z \le 153$, then $100x = 400$ or $100x = 500$ so $15y + 2z = 507 - 400 = 107$ or $15y + 2z = 507 - 500 = 7$. From above, we saw that $15y + 2z$ cannot equal 7, so $15y + 2z = 107$, which means that $100x = 400$ or $x = 4$.
Thus, $15y + 2z = 107$.
Since $2z$ is even, then $15y$ must be odd to make $15y + 2z$ odd.
The odd multiples of 15 less than 107 are $15, 45, 75, 105$.
Since $0 \le 2z \le 18$, then we must have $15y = 105$ or $y = 7$. This gives $2z = 2$ or $z = 1$.
Therefore, the integer $wxyz$ is 1471. (Checking, $1471 + 471 + 71 + 1 = 2014$.)
Finally, $w + x + y + z = 1 + 4 + 7 + 1 = 13$.

Answer: (D)

21. Suppose that R is the point at the bottom of the solid directly under Q and S is the back left bottom corner of the solid (unseen in the problem's diagram).
Since QR is perpendicular to the bottom surface of the solid, then $\triangle PRQ$ is right-angled at R and so $PQ^2 = PR^2 + RQ^2$.
We note also that $\triangle PSR$ is right-angled at S, since the solid is made up of cubes.
Therefore, $PR^2 = PS^2 + SR^2$.





This tells us that $PQ^2 = PS^2 + SR^2 + RQ^2$.
Since the edge length of each cube in the diagram is 1, then $PS = 1$, $SR = 4$, and $RQ = 3$.
Therefore, $PQ^2 = 1^2 + 4^2 + 3^2 = 26$.
Since $PQ > 0$, then $PQ = \sqrt{26}$.

ANSWER: (B)

22. We write such a five-digit positive integer with digits $VWXYZ$.
We want to count the number of ways of assigning 1, 3, 5, 7, 9 to the digits V, W, X, Y, Z in such a way that the given properties are obeyed.
From the given conditions, $W > X$, $W > V$, $Y > X$, and $Y > Z$.
The digits 1 and 3 cannot be placed as W or Y, since W and Y are larger than both of their neighbouring digits, while 1 is smaller than all of the other digits and 3 is only larger than one of the other possible digits.
The digit 9 cannot be placed as V, X or Z since it is the largest possible digit and so cannot be smaller than W or Y. Thus, 9 is placed as W or as Y.
Therefore, the digits W and Y are 9 and either 5 or 7.

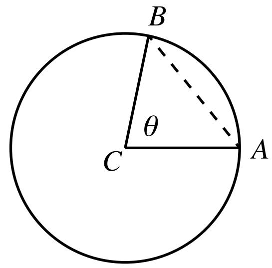
Suppose that $W = 9$ and $Y = 5$. The number is thus $V9X5Z$.
Neither X or Z can equal 7 since $7 > 5$, so $V = 7$. X and Z are then 1 and 3 or 3 and 1.
There are 2 possible integers in this case.
Similarly, if $Y = 9$ and $W = 5$, there are 2 possible integers.

Suppose that $W = 9$ and $Y = 7$. The number is thus $V9X7Z$.
The digits 1, 3, 5 can be placed in any of the remaining spots. There are 3 choices for the digit V. For each of these choices, there are 2 choices for X and then 1 choice for Z.
There are thus $3 \times 2 \times 1 = 6$ possible integers in this case.
Similarly, if $Y = 9$ and $W = 7$, there are 6 possible integers.

Overall, there are thus $2 + 2 + 6 + 6 = 16$ possible integers.

ANSWER: (C)

23. We call Clarise's spot C and Abe's spot A.
Consider a circle centred at C with radius 10 m. Since A is 10 m from C, then A is on this circle.
Bob starts at C and picks a direction to walk, with every direction being equally likely to be chosen. We model this by having Bob choose an angle θ between 0° and 360° and walk 10 m along a segment that makes this angle when measured counterclockwise from CA.
Bob ends at point B, which is also on the circle.



We need to determine the probability that $AB < AC$.
Since the circle is symmetric above and below the diameter implied by CA, we can assume that θ is between 0° and 180° as the probability will be the same below the diameter.
Consider $\triangle CAB$ and note that $CA = CB = 10$ m.
It will be true that $AB < AC$ whenever AB is the shortest side of $\triangle ABC$.
AB will be the shortest side of $\triangle ABC$ whenever it is opposite the smallest angle of $\triangle ABC$. (In any triangle, the shortest side is opposite the smallest angle and the longest side is opposite the largest angle.)
Since $\triangle ABC$ is isosceles with $CA = CB$, then $\angle CAB = \angle CBA$.
We know that $\theta = \angle ACB$ is opposite AB and $\angle ACB + \angle CAB + \angle CBA = 180^\circ$.
Since $\angle CAB = \angle CBA$, then $\angle ACB + 2\angle CAB = 180^\circ$ or $\angle CAB = 90^\circ - \frac{1}{2}\angle ACB$.
If $\theta = \angle ACB$ is smaller than 60°, then $\angle CAB = 90^\circ - \frac{1}{2}\theta$ will be greater than 60°.
Similarly, if $\angle ACB$ is greater than 60°, then $\angle CAB = 90^\circ - \frac{1}{2}\theta$ will be smaller than 60°.
Therefore, AB is the shortest side of $\triangle ABC$ whenever θ is between 0° and 60°.
Since θ is uniformly chosen in the range 0° to 180° and $60^\circ = \frac{1}{3}(180^\circ)$, then the probability that θ is in the desired range is $\frac{1}{3}$.
Therefore, the probability that Bob is closer to Abe than Clarise is to Abe is $\frac{1}{3}$.
(Note that we can ignore the cases $\theta = 0^\circ$, $\theta = 60^\circ$ and $\theta = 180^\circ$ because these are only three specific cases out of an infinite number of possible values for θ.)

Answer: (B)

24. For each positive integer n, $S(n)$ is defined to be the smallest positive integer divisible by each of $1, 2, 3, \ldots, n$. In other words, $S(n)$ is the least common multiple (lcm) of $1, 2, 3, \ldots, n$.
To calculate the lcm of a set of numbers, we

- determine the prime factorization of each number in the set,
- determine the list of prime numbers that occur in these prime factorizations,
- determine the highest power of each prime number from this list that occurs in the prime factorizations, and
- multiply these highest powers together.

For example, to calculate $S(8)$, we determine the lcm of 1, 2, 3, 4, 5, 6, 7, 8.
The prime factorizations of the numbers 2, 3, 4, 5, 6, 7, 8 are $2, 3, 2^2, 5, 2 \cdot 3, 7, 2^3$.

The primes used in this list are 2, 3, 5, 7, with highest powers $2^3, 3^1, 5^1, 7^1$.
Therefore, $S(8) = 2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1$.

Since $S(n)$ is the lcm of $1, 2, 3, \ldots, n$ and $S(n+4)$ is the lcm of $1, 2, 3, \ldots, n, n+1, n+2, n+3, n+4$, then $S(n) \neq S(n+4)$ if either (i) there are prime factors that occur in $n+1, n+2, n+3, n+4$ that don't occur in $1, 2, 3, \ldots, n$ or (ii) there is a higher power of a prime that occurs in the factorizations of one of $n+1, n+2, n+3, n+4$ that doesn't occur in any of $1, 2, 3, \ldots, n$.
For (i) to occur, consider a prime number p that is a divisor of one of $n+1, n+2, n+3, n+4$ and none of $1, 2, 3, \ldots, n$. This means that the smallest positive integer that has p as a divisor is one of the integers $n+1, n+2, n+3, n+4$, which in fact means that this integer equals p. (The smallest multiple of a prime p is $1 \cdot p$, or p itself.)
Thus, for (i) to occur, one of $n+1, n+2, n+3, n+4$ is a prime number.
For (ii) to occur, consider a prime power p^k (with $k > 1$) that is a divisor of one of $n+1$, $n+2, n+3, n+4$ and none of $1, 2, 3, \ldots, n$. Using a similar argument to condition (i), one of $n+1, n+2, n+3, n+4$ must equal that prime power p^k.
Therefore, $S(n) \neq S(n+4)$ whenever one of $n+1, n+2, n+3, n+4$ is a prime number or a prime power.
In other words, $S(n) = S(n+4)$ whenever none of $n+1, n+2, n+3, n+4$ is a prime number or a prime power.
Therefore, we want to determine the positive integers n with $1 \leq n \leq 100$ for which none of $n+1, n+2, n+3, n+4$ is a prime number or a prime power.
The prime numbers less than or equal to 104 are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103$$

(We go up to 104 since n can be as large as 100 so $n+4$ can be as large as 104.)
The prime powers (with exponent at least 2) less than or equal to 100 are

$$4, 8, 16, 32, 64, 9, 27, 81, 25, 49$$

There are 5 powers of 2, 3 powers of 3, 1 power of 5, and 1 power of 7 in this list. No primes larger than 7 have a power less than 100.
Therefore, we want to count the positive integers n with $1 \leq n \leq 100$ for which none of $n+1, n+2, n+3, n+4$ appear in the list

$$2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, 37, 41, 43, 47, 49, 53, 59, 61, 64,$$

$$67, 71, 73, 79, 81, 83, 89, 97, 101, 103$$

For four consecutive integers not to occur in this list, we need a difference between adjacent numbers to be at least 5.
The values of n that satisfy this condition are $n = 32, 53, 54, 73, 74, 83, 84, 89, 90, 91, 92$.
(For example, 54 is a value of n that works since none of $55, 56, 57, 58$ appears in the list.)
Therefore, there are 11 values of n with $1 \leq n \leq 100$ for which $S(n) = S(n+4)$.

Answer: (C)

25. Suppose that P has coordinates $P(0, 2a)$ for some real number a.
Since P has y-coordinate greater than 0 and less than 100, then $0 < 2a < 100$ or $0 < a < 50$.
We determine an expression for the radius of the circle in terms of a and then determine how many values of a give an integer radius.
We determine the desired expression by first finding the coordinates of the centre, C, of the

circle in terms of a, and then calculating the distance from C to one of the points O, P, Q.
If a circle passes through the three vertices O, P and Q of a triangle, then its centre is the point of intersection of the perpendicular bisectors of the sides OP, OQ, and PQ of the triangle.
We determine the centre of the circle by finding the point of intersection of the perpendicular bisectors of OP and OQ. (We could use PQ instead, but this would be more complicated algebraically.)
Since O has coordinates $(0,0)$ and P has coordinates $(0,2a)$, then OP is vertical so its perpendicular bisector is horizontal.
The midpoint of OP is $(\frac{1}{2}(0+0), \frac{1}{2}(0+2a)) = (0,a)$.
Therefore, the perpendicular bisector of OP is the horizontal line through $(0,a)$, and so has equation $y = a$.
Since O has coordinates $(0,0)$ and Q has coordinates $(4,4)$, then OQ has slope $\frac{4-0}{4-0} = 1$.
Therefore, a line perpendicular to OQ has slope -1.
The midpoint of OQ is $(\frac{1}{2}(0+4), \frac{1}{2}(0+4)) = (2,2)$.
Therefore, the perpendicular bisector of OQ has slope -1 and passes through $(2,2)$, so has equation $y - 2 = (-1)(x-2)$ or $y = -x + 4$.
The centre of the desired circle is thus the point of intersection of the lines with equations $y = a$ and $y = -x + 4$.
The y-coordinate of this point is a and the x-coordinate is obtained by solving $a = -x + 4$ and obtaining $x = 4 - a$.
Therefore, the coordinates of C are $(4-a, a)$.
The radius, r, of the circle is the distance from C to any of the three points O, P and Q. It is easiest to find the distance from O to C, which is

$$r = \sqrt{(4-a)^2 + a^2} = \sqrt{a^2 - 8a + 16 + a^2} = \sqrt{2a^2 - 8a + 16}$$

We rewrite this as

$$r = \sqrt{2(a^2 - 4a + 8)} = \sqrt{2(a^2 - 4a + 4 + 4)} = \sqrt{2((a-2)^2 + 4)} = \sqrt{2(a-2)^2 + 8}$$

Since $(a-2)^2 \geq 0$ and $(a-2)^2 = 0$ only when $a = 2$, then the minimum value of $2(a-2)^2 + 8$ is 8 and this occurs when $a = 2$. Thus, $r \geq \sqrt{8}$.
The expression $\sqrt{2(a-2)^2 + 8}$ is decreasing from $a = 0$ to $a = 2$ and then increasing from $a = 2$ to $a = 50$.
When $a = 0$, $r = \sqrt{2(a-2)^2 + 8} = \sqrt{2(-2)^2 + 8} = 4$.
When $a = 2$, $r = \sqrt{2(a-2)^2 + 8} = \sqrt{2(0)^2 + 8} = \sqrt{8} \approx 2.83$.
When $a = 50$, $r = \sqrt{2(a-2)^2 + 8} = \sqrt{2(48)^2 + 8} = \sqrt{4616} \approx 67.94$.
Therefore, when $0 < a \leq 2$, we have $\sqrt{8} \leq r < 4$ and when $2 \leq a < 50$, we have $\sqrt{8} \leq r < \sqrt{4616}$.
The expression $r = \sqrt{2(a-2)^2 + 8}$ will take every real number value in each of these ranges, because $b = 2(a-2)^2 + 8$ represents the equation of a parabola which is a "smooth" curve.
Between $\sqrt{8} \approx 2.83$ and 4, there is one integer value (namely, 3) which is achieved by the expression. (We do not count 4 since it is an endpoint that is not included.)
Between $\sqrt{8} \approx 2.83$ and $\sqrt{4616} \approx 67.94$, there are 65 integer values (namely, 3 to 67, inclusive) which are achieved by the expression.
In total, there are $1 + 65 = 66$ integer values achieved by the expression in the allowable range for a, so there are 66 positions of P for which the radius is an integer.

Answer: (C)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2013 Cayley Contest

(Grade 10)

Thursday, February 21, 2013
(in North America and South America)

Friday, February 22, 2013
(outside of North America and South America)

Solutions

1. Simplifying, $\dfrac{8+4}{8-4} = \dfrac{12}{4} = 3$.

   Answer: (B)

2. Since $2^1 = 2$ and $2^2 = 2 \times 2 = 4$ and $2^3 = 2 \times 2 \times 2 = 8$, then $2^3 + 2^2 + 2^1 = 8 + 4 + 2 = 14$.

   Answer: (C)

3. If $x + \sqrt{81} = 25$, then $x + 9 = 25$ or $x = 16$.

   Answer: (A)

4. We could use a calculator to divide each of the four given numbers by 3 to see which calculations give an integer answer.
   Alternatively, we could use the fact that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3.
   The sums of the digits of 222, 2222, 22 222, and 222 222 are 6, 8, 10, and 12, respectively.
   Two of these sums are divisible by 3 (namely, 6 and 12) so two of the four integers (namely, 222 and 222 222) are divisible by 3.

   Answer: (C)

5. Since the field originally has length 20 m and width 5 m, then its area is $20 \times 5 = 100$ m^2.
   The new length of the field is $20 + 10 = 30$ m, so the new area is $30 \times 5 = 150$ m^2.
   The increase in area is $150 - 100 = 50$ m^2.
   (Alternatively, we could note that since the length increases by 10 m and the width stays constant at 5 m, then the increase in area is $10 \times 5 = 50$ m^2.)

   Answer: (C)

6. Since the tick marks divide the cylinder into four parts of equal volume, then the level of the milk shown is a bit less than $\frac{3}{4}$ of the total volume of the cylinder.
   Three-quarters of the total volume of the cylinder is $\frac{3}{4} \times 50 = 37.5$ L.
   Of the five given choices, the one that is slightly less than 37.5 L is 36 L, or (D).

   Answer: (D)

7. Since $\triangle PQR$ is equilateral, then $PQ = QR = RP$.
   Therefore, $4x = x + 12$ or $3x = 12$ and so $x = 4$.

   Answer: (C)

8. Using the definition of the symbol, $3 \diamond 6 = \dfrac{3+6}{3 \times 6} = \dfrac{9}{18} = \dfrac{1}{2}$.

   Answer: (E)

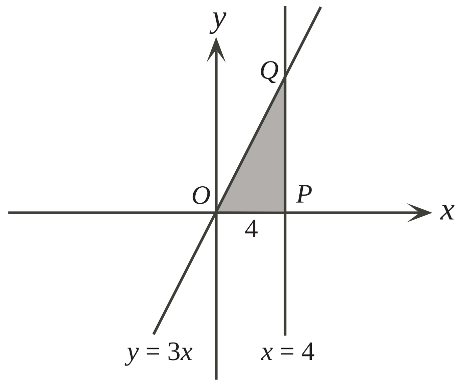
9. One way to phrase the Pythagorean Theorem is that the area of the square formed on the hypotenuse of a right-angled triangle equals the sum of the areas of the squares formed on the other two sides.
   Therefore, the area of the square on PQ equals the area of the square on PR minus the area of the square on QR, which equals $169 - 144$ or 25.

   Answer: (E)

10. Since the average age of the three sisters is 27, then the sum of their ages is $3 \times 27 = 81$.
    When Barry is included the average age of the four people is 28, so the sum of the ages of the four people is $4 \times 28 = 112$.
    Barry's age is the difference between the sum of the ages of all four people and the sum of the ages of the three sisters, which equals $112 - 81$ or 31.

    Answer: (E)

11. Let O be the origin (where the line with equation $y = 3x$ intersects the x-axis).
Let P be the point where the line with equation $x = 4$ intersects the x-axis, and let Q be the point where the lines with equations $x = 4$ and $y = 3x$ intersect.



The line $x = 4$ is perpendicular to the x-axis, so the given triangle is right-angled at P.
Therefore, the area of the triangle equals $\frac{1}{2}(OP)(PQ)$.
Now P lies on the x-axis and on the line $x = 4$, so has coordinates $(4, 0)$.
Thus, $OP = 4$.
Point Q also has x-coordinate 4. Since Q lies on $y = 3x$, then its y-coordinate is $y = 3(4) = 12$.
Since P has coordinates $(4, 0)$ and Q has coordinates $(4, 12)$, then $PQ = 12$.
Therefore, the area of the triangle is $\frac{1}{2}(4)(12) = 24$.

Answer: (B)

12. *Solution 1*
Since $a(x + b) = 3x + 12$ for all x, then $ax + ab = 3x + 12$ for all x.
Since the equation is true for all x, then the coefficients on the left side must match the coefficients on the right side.
Therefore, $a = 3$ and $ab = 12$, which gives $3b = 12$ or $b = 4$.
Finally, $a + b = 3 + 4 = 7$.

Solution 2
Since $a(x + b) = 3x + 12$ for all x, then the equation is true for $x = 0$ and $x = 1$.
When $x = 0$, we obtain $a(0 + b) = 3(0) + 12$ or $ab = 12$.
When $x = 1$, we obtain $a(1 + b) = 3(1) + 12$ or $a + ab = 15$.
Since $ab = 12$, then $a + 12 = 15$ or $a = 3$.
Since $ab = 12$ and $a = 3$, then $b = 4$.
Finally, $a + b = 3 + 4 = 7$.

Answer: (D)

13. *Solution 1*
If $x = 1$, then $3x + 1 = 4$, which is an even integer.
In this case, the four given choices are

(A) $x + 3 = 4$ (B) $x - 3 = -2$ (C) $2x = 2$ (D) $7x + 4 = 11$ (E) $5x + 3 = 8$

Of these, the only odd integer is (D). Therefore, (D) must be the correct answer as the result must be true no matter what integer value of x is chosen that makes $3x + 1$ even.

Solution 2

If x is an integer for which $3x+1$ is even, then $3x$ is odd, since it is 1 less than an even integer.
If $3x$ is odd, then x must be odd (since if x is even, then $3x$ would be even).
If x is odd, then $x+3$ is even (odd plus odd equals even), so (A) cannot be correct.
If x is odd, then $x-3$ is even (odd minus odd equals even), so (B) cannot be correct.
If x is odd, then $2x$ is even (even times odd equals even), so (C) cannot be correct.
If x is odd, then $7x$ is odd (odd times odd equals odd) and so $7x+4$ is odd (odd plus even equals odd).
If x is odd, then $5x$ is odd (odd times odd equals odd) and so $5x+3$ is even (odd plus odd equals even), so (E) cannot be correct.
Therefore, the one expression which must be odd is $7x+4$.

ANSWER: (D)

14. With a given set of four digits, the largest possible integer that can be formed puts the largest digit in the thousands place, the second largest digit in the hundreds place, the third largest digit in the tens place, and the smallest digit in the units place. This is because the largest digit can make the largest contribution in the place with the most value.
Thus, the largest integer that can be formed with the digits 2, 0, 1, 3 is 3210.
With a given set of digits, the smallest possible integer comes from listing the numbers in increasing order from the thousands place to the units place.
Here, there is an added wrinkle that the integer must be at least 1000. Therefore, the thousands digit is at least 1. The smallest integer of this type that can be made uses a thousands digit of 1, and then lists the remaining digits in increasing order; this integer is 1023.
The difference between these integers is $3210 - 1023 = 2187$.

ANSWER: (A)

15. Since 40% of the songs on the updated playlist are Country, then the remaining $100\% - 40\%$ or 60% must be Hip Hop and Pop songs.
Since the ratio of Hip Hop song to Pop songs does not change, then 65% of this remaining 60% must be Hip Hop songs.
Overall, this is $65\% \times 60\% = 0.65 \times 0.6 = 0.39 = 39\%$ of the total number of songs on the playlist.

ANSWER: (E)

16. First, we note that $5^{35} - 6^{21}$ is a positive integer, since

$$5^{35} - 6^{21} = (5^5)^7 - (6^3)^7 = 3125^7 - 216^7$$

and $3125 > 216$.
Second, we note that any positive integer power of 5 has a units digit of 5. Since $5 \times 5 = 25$ and this product has a units digit of 5, then the units digit of 5^3 is obtained by multiplying 5 by the units digit 5 of 25. Thus, the units digit of 5^3 is 5. Similarly, each successive power of 5 has a units digit of 5.
Similarly, each power of 6 has a units digit of 6.
Therefore, 5^{35} has a units digit of 5 and 6^{21} has a units digit of 6. When a positive integer with units digit 6 is subtracted from a larger positive integer whose units digit is 5, the difference has a units digit of 9.
Therefore, $5^{35} - 6^{21}$ has a units digit of 9.

ANSWER: (B)

17. We have

$$\begin{aligned}
\text{Perimeter of } \triangle PST &= PS + ST + PT \\
&= PS + (SU + UT) + PT \\
&= PS + SQ + TR + PT \qquad \text{(since } SU = SQ \text{ and } UT = TR\text{)} \\
&= (PS + SQ) + (PT + TR) \\
&= PQ + PR \\
&= 19 + 17 \\
&= 36
\end{aligned}$$

Therefore, the perimeter of $\triangle PQR$ is 36.

ANSWER: (A)

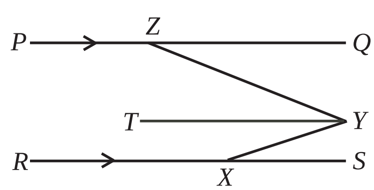
18. Suppose that the quotient of the division of 109 by x is q.
Since the remainder is 4, this is equivalent to $109 = qx + 4$ or $qx = 105$.
Put another way, x must be a positive integer divisor of 105.
Since $105 = 5 \times 21 = 5 \times 3 \times 7$, its positive integer divisors are

$$1, 3, 5, 7, 15, 21, 35, 105$$

Of these, 15, 21 and 35 are two-digit positive integers so are the possible values of x.
The sum of these values is $15 + 21 + 35 = 71$.

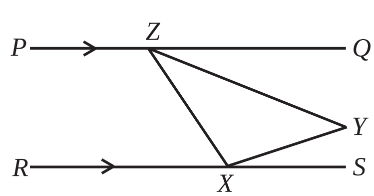
ANSWER: (D)

19. *Solution 1*
Draw a line segment TY through Y parallel to PQ and RS, as shown, and delete line segment ZX.



Since TY is parallel to RS, then $\angle TYX = \angle YXS = 20^\circ$.
Thus, $\angle ZYT = \angle ZYX - \angle TYX = 50^\circ - 20^\circ = 30^\circ$.
Since PQ is parallel to TY, then $\angle QZY = \angle ZYT = 30^\circ$.

Solution 2
Since QZ and XS are parallel, then $\angle QZX + \angle ZXS = 180^\circ$.



Now $\angle QZX = \angle QZY + \angle YZX$ and $\angle ZXS = \angle ZXY + \angle YXS$.
We know that $\angle YXS = 20^\circ$.
Also, the sum of the angles in $\triangle XYZ$ is 180°, so $\angle YZX + \angle ZXY + \angle ZYX = 180^\circ$, or $\angle YZX + \angle ZXY = 180^\circ - \angle ZYX = 180^\circ - 50^\circ = 130^\circ$.
Combining all of these facts, we obtain $\angle QZY + \angle YZX + \angle ZXY + \angle YXS = 180^\circ$ or $\angle QZY + 130^\circ + 20^\circ = 180^\circ$.
From this, we obtain $\angle QZY = 180^\circ - 130^\circ - 20^\circ = 30^\circ$.

ANSWER: (A)

20. Suppose that the length of the route is d km.
Then Jill jogs $\frac{d}{2}$ km at 6 km/h and runs $\frac{d}{2}$ km at 12 km/h.
Note that time equals distance divided by speed.
Since her total time was x hours, then $x = \frac{d/2}{6} + \frac{d/2}{12} = \frac{d}{12} + \frac{d}{24} = \frac{2d}{24} + \frac{d}{24} = \frac{3d}{24} = \frac{d}{8}$.
Also, Jack walks $\frac{d}{3}$ km at 5 km/h and runs $\frac{2d}{3}$ km at 15 km/h.
Since his total time is y hours, then $y = \frac{d/3}{5} + \frac{2d/3}{15} = \frac{d}{15} + \frac{2d}{45} = \frac{3d}{45} + \frac{2d}{45} = \frac{5d}{45} = \frac{d}{9}$.
Finally, $\frac{x}{y} = \frac{d/8}{d/9} = \frac{9}{8}$.

ANSWER: (A)

21. We start by analyzing the given sum as if we were performing the addition by hand. Doing this, we would start with the units column.
Here, we see that the units digit of the sum $X + Y + Z$ is X.
Thus, the units digit of $Y + Z$ must be 0.
Because none of the digits is zero, then there must be a carry to the tens column.
Because $Y + Z$ are different digits between 1 and 9, then $Y + Z$ is at most 17. Since the units digit of $Y + Z$ is 0, then $Y + Z = 10$ and there is a carry of 1 to the tens column.
Comparing the sums and digits in the tens and units columns, we see that $Y = X + 1$ (since Y cannot be 0).

$$\begin{array}{cccc} & & {\scriptstyle 1} & \\ & X & X & X \\ & Y & Y & Y \\ + & Z & Z & Z \\ \hline Z & Y & Y & X \end{array}$$

Here are two ways that we could finish the solution.

<u>Method 1</u>
Since $Y + Z = 10$, then $YYY + ZZZ = 1110$.
In other words, each column gives a carry of 1 that is added to the next column to the left.
Alternatively, we could notice that $YYY = Y \times 111$ and $ZZZ = Z \times 111$.
Thus, $YYY + ZZZ = (Y + Z) \times 111 = 10 \times 111 = 1110$.
Therefore, the given sum simplifies to $1110 + XXX = ZYYX$.
If $X = 9$, then the sum would be $1110 + 999 = 2009$, which has $Y = 0$, which can't be the case.
Therefore, $X \leq 8$.
This tells us that $1110 + XXX$ is at most $1110 + 888 = 1998$, and so Z must equal 1, regardless of the value of X.
Since $Y + Z = 10$, then $Y = 9$.
This means that we have $1110 + XXX = 199X$.
Since X is a digit, then $X + 1 = 9$, and so $X = 8$, which is consistent with the above.
(Checking, if $X = 8$, $Y = 9$, $Z = 1$, we have $888 + 999 + 111 = 1998$.)

Method 2

Consider the tens column.

The sum in this column is $1 + X + Y + Z$ (we add 1 as the "carry" from the units column).

Since $Y + Z = 10$, then $1 + X + Y + Z = 11 + X$.

Since X is between 1 and 9, then $11 + X$ is at least 12 and at most 20.

In fact, $11 + X$ cannot equal 20, since this would mean entering a 0 in the sum, and we know that none of the digits is 0.

$$\begin{array}{cccc} & {\scriptstyle 1} & {\scriptstyle 1} & \\ & X & X & X \\ & Y & Y & Y \\ + & Z & Z & Z \\ \hline 1 & Y & Y & X \end{array}$$

Thus, $11 + X$ is less than 20, and so must equal $10 + Y$ (giving a digit of Y in the tens column of the sum and a carry of 1 to the hundreds column).

Further, since the tens column and the hundreds column are the same, then the carry to the thousands column is also 1. In other words, $Z = 1$.

Since $Y + Z = 10$ and $Z = 1$, then $Y = 9$.

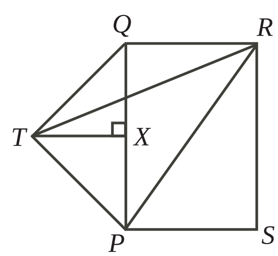
Since $Y = X + 1$ and $Y = 9$, then $X = 8$.

As in Method 1, we can check that these values satisfy the given sum.

ANSWER: (D)

22. *Solution 1*

Let X be the point on QP so that TX is perpendicular to QP.



Since $\triangle QTP$ is isosceles, then X is the midpoint of QP.

Since $QP = 4$, then $QX = XP = 2$.

Since $\angle TQP = 45^\circ$ and $\angle QXT = 90^\circ$, then $\triangle QXT$ is also isosceles and right-angled.

Therefore, $TX = QX = 2$.

We calculate the area of $\triangle PTR$ by adding the areas of $\triangle QRP$ and $\triangle QTP$ and subtracting the area of $\triangle QRT$.

Since $QR = 3$, $PQ = 4$ and $\angle PQR = 90^\circ$, then the area of $\triangle QRP$ is $\frac{1}{2}(3)(4) = 6$.

Since $QP = 4$, $TX = 2$ and TX is perpendicular to QP, then the area of $\triangle QTP$ is $\frac{1}{2}(4)(2) = 4$.

We can view $\triangle QRT$ as having base QR with its height being the perpendicular distance from QR to T, which equals the length of QX. Thus, the area of $\triangle QRT$ is $\frac{1}{2}(3)(2) = 3$.

Therefore, the area of $\triangle PTR$ is $6 + 4 - 3 = 7$.

Solution 2

Let X be the point on QP so that TX is perpendicular to QP.

Since $\triangle QTP$ is isosceles, then X is the midpoint of QP.

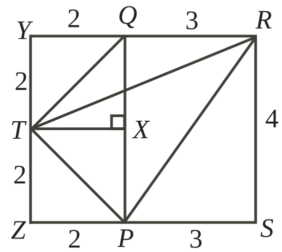
Since $QP = 4$, then $QX = XP = 2$.

Since $\angle TQP = 45^\circ$ and $\angle QXT = 90^\circ$, then $\triangle QXT$ is also isosceles and right-angled.

Therefore, $TX = QX = 2$.

Extend RQ to Y and SP to Z so that YZ is perpendicular to each of YR and ZS and so that YZ passes through T.

Each of $YQXT$ and $TXPZ$ has three right angles (at Y, Q and X, and X, P and Z, respectively), so each of these is a rectangle.
Since $QX = TX = XP = 2$, then each of $YQXT$ and $TXPZ$ is a square with side length 2.
Now $YRSZ$ is a rectangle with $YR = YQ + QR = 2 + 3 = 5$ and $RS = 4$.



The area of $\triangle PTR$ equals the area of rectangle $YRSZ$ minus the areas of $\triangle TYR$, $\triangle RSP$ and $\triangle PZT$.
Rectangle $YRSZ$ is 5 by 4 and so has area $5 \times 4 = 20$.
Since $TY = 2$ and $YR = 5$ and TY is perpendicular to YR, then the area of $\triangle TYR$ is $\frac{1}{2}(TY)(YR) = 5$.
Since $RS = 4$ and $SP = 3$ and RS is perpendicular to SP, then the area of $\triangle RSP$ is $\frac{1}{2}(RS)(SP) = 6$.
Since $PZ = ZT = 2$ and PZ is perpendicular to ZT, then the area of $\triangle PZT$ is $\frac{1}{2}(PZ)(ZT) = 2$.
Therefore, the area of $\triangle PTR$ is $20 - 5 - 6 - 2 = 7$.

Answer: (C)

23. First, we consider the first bag, which contains a total of $2 + 2 = 4$ marbles.
There are 4 possible marbles that can be drawn first, leaving 3 possible marbles that can be drawn second. This gives a total of $4 \times 3 = 12$ ways of drawing two marbles.
For both marbles to be red, there are 2 possible marbles (either red marble) that can be drawn first, and 1 marble that must be drawn second (the remaining red marble). This gives a total of $2 \times 1 = 2$ ways of drawing two red marbles.
For both marbles to be blue, there are 2 possible marbles that can be drawn first, and 1 marble that must be drawn second. This gives a total of $2 \times 1 = 2$ ways of drawing two blue marbles.
Therefore, the probability of drawing two marbles of the same colour from the first bag is the total number of ways of drawing two marbles of the same colour $(2 + 2 = 4)$ divided by the total number of ways of drawing two marbles (12), or $\dfrac{4}{12} = \dfrac{1}{3}$.
Second, we consider the second bag, which contains a total of $2 + 2 + g = g + 4$ marbles.
There are $g + 4$ possible marbles that can be drawn first, leaving $g + 3$ possible marbles that can be drawn second. This gives a total of $(g + 4)(g + 3)$ ways of drawing two marbles.
As with the first bag, there are $2 \times 1 = 2$ ways of drawing two red marbles.
As with the first bag, there are $2 \times 1 = 2$ ways of drawing two blue marbles.
For both marbles to be green, there are g possible marbles that can be drawn first, and $g - 1$ marbles that must be drawn second. This gives a total of $g(g - 1)$ ways of drawing two green marbles.
Therefore, the probability of drawing two marbles of the same colour from the second bag is the total number of ways of drawing two marbles of the same colour $(2+2+g(g-1) = g^2 - g + 4)$ divided by the total number of ways of drawing two marbles $((g+4)(g+3))$, or $\dfrac{g^2 - g + 4}{(g + 4)(g + 3)}$.

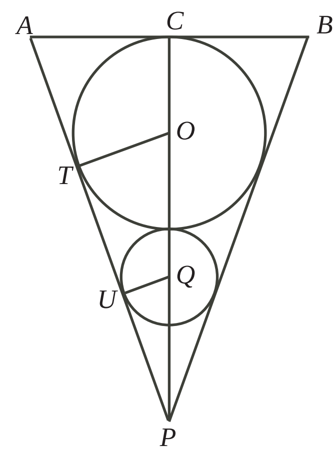
Since the two probabilities that we have calculated are to be equal and $g \neq 0$, then

$$\begin{aligned} \frac{1}{3} &= \frac{g^2 - g + 4}{(g+4)(g+3)} \\ (g+4)(g+3) &= 3g^2 - 3g + 12 \\ g^2 + 7g + 12 &= 3g^2 - 3g + 12 \\ 10g &= 2g^2 \\ g &= 5 \qquad (\text{since } g \neq 0) \end{aligned}$$

Therefore, $g = 5$.

Answer: (B)

24. Let the radius of the smaller sphere be r.
Thus, the radius of the larger sphere is $2r$.
We determine expressions for the height and radius of the cone in terms of r and use these to help solve the problem.
By symmetry, the centres of the two spheres (Q of the smaller sphere and O of the larger sphere) lie on the line joining the centre of the circular top of the cone (C) to the tip of the cone (P).
Draw a vertical cross-section of the cone through the centre of the circular top of the cone and through the tip of the cone.
Each such cross-section will be an identical triangle.
Because the centres of the spheres lie on a line which is in the plane of this cross-section, the cross-section of each sphere will be a "great" circle (that is, the largest possible circular cross-section of the sphere).
Because the top of the larger sphere is just level with the top of the cone, then the sphere "touches" the circular top of the cone at its centre C.
Finally, because the spheres touch the cone all the way around, then the circles will be tangent to the triangle in the cross-section.



We label the triangular cross section as ABP.
Note that CP is perpendicular to AB at C.
Draw radii from O and Q to the points T and U, respectively, on AP where the circles with centre O and Q are tangent to AP.
Note that OT and QU are perpendicular to AP, with $OT = 2r$ and $QU = r$.
Also, since the two circles are just touching, then the line segment joining their centres, OQ, passes through this point of tangency, and so $OQ = 2r + r = 3r$.
Now $\triangle OTP$ is similar to $\triangle QUP$, since each is right-angled and they share a common angle at P.
Since $\dfrac{OT}{QU} = \dfrac{2r}{r} = 2$, then $\dfrac{OP}{QP} = 2$ or $OP = 2QP$.
Since $OP = OQ + QP = 3r + QP$, then $3r + QP = 2QP$ or $QP = 3r$.
Thus, the height of the cone is $CP = CO + OQ + QP = 2r + 3r + 3r = 8r$.
(Note that CO is a radius of the larger circle, so $CO = 2r$.)
We also see that $\triangle ACP$ is similar to $\triangle QUP$, since $\triangle ACP$ is also right-angled (at C) and shares the angle at P.
Thus, $\dfrac{AC}{CP} = \dfrac{QU}{UP}$.

We know that $CP = 8r$ and $QU = r$.
To calculate UP, we use the Pythagorean Theorem to get

$$UP = \sqrt{QP^2 - QU^2} = \sqrt{(3r)^2 - r^2} = \sqrt{8r^2} = \sqrt{8}r$$

since $r > 0$.
Therefore, $AC = \dfrac{CP \cdot QU}{UP} = \dfrac{8r \cdot r}{\sqrt{8}r} = \sqrt{8}r$.
Finally, we can use the given information. We are told that the volume of water remaining after the full cone has the two spheres added is 2016π. This is equivalent to saying that the difference between the volume of the cone and the combined volumes of the spheres is 2016π.
Using the given volume formulae, we obtain

$$\begin{aligned}
\tfrac{1}{3}\pi(AC)^2(CP) - \tfrac{4}{3}\pi(QU)^3 - \tfrac{4}{3}\pi(OT)^3 &= 2016\pi \\
\tfrac{1}{3}\pi(\sqrt{8}r)^2(8r) - \tfrac{4}{3}\pi r^3 - \tfrac{4}{3}\pi(2r)^3 &= 2016\pi \\
64\pi r^3 - 4\pi r^3 - 32\pi r^3 &= 6048\pi \\
28r^3 &= 6048 \\
r^3 &= 216 \\
r &= 6
\end{aligned}$$

Therefore, the radius of the smaller sphere is 6.

Answer: (B)

25. We define $L(n) = n - Z(n!)$ to be the nth number in Lloyd's list.
We note that the number of trailing zeros in any positive integer m (which is $Z(m)$) equals the number of factors of 10 that m has. For example, 2400 has two trailing zeros since $2400 = 24 \times 10 \times 10$. Further, since $10 = 2 \times 5$, the number of factors of 10 in any positive integer m is determined by the number of factors of 2 and 5.
Consider $n! = n(n-1)(n-2)\cdots(3)(2)(1)$.
Since $5 > 2$, then $n!$ will always contain more factors of 2 than factors of 5. This is because if we make a list of the multiples of 2 and a list of the multiples of 5, then there will be more numbers in the first list than in the second list that are less than or equal to a given positive integer n (and so numbers in the first list that contribute more than one factor of 2 will occur before numbers in the second list that contribute more than one factor of 5, and so on).
In other words, the value of $Z(n!)$ will equal the number of factors of 5 that $n!$ has.
We use the notation $V(m)$ to represent the number of factors of 5 in the integer m.
Thus, $Z(n!) = V(n!)$ and so $L(n) = n - V(n!)$.
Since $(n+1)! = (n+1) \times n!$, then $V((n+1)!) = V(n+1) + V(n!)$. (This is because any additional factors of 5 in $(n+1)!$ that are not in $n!$ come from $n+1$.)
Therefore, if $n+1$ is not a multiple of 5, then $V(n+1) = 0$ and so $V((n+1)!) = V(n!)$.
If $n+1$ is a multiple of 5, then $V(n+1) > 0$ and so $V((n+1)!) > V(n!)$.
Note that

$$\begin{aligned}
L(n+1) - L(n) &= ((n+1) - V((n+1)!)) - (n - V(n!)) \\
&= ((n+1) - n) - (V((n+1)!) - V(n!)) \\
&= 1 - V(n+1)
\end{aligned}$$

If $n+1$ is not a multiple of 5, then $V(n+1) = 0$ and so $L(n+1) - L(n) = 1$.
This tells us that when $n+1$ is not a multiple of 5, the corresponding term in the list is one

larger than the previous term; thus, the terms in the list increase by 1 for four terms in a row whenever there is not a multiple of 5 in this list (since multiples of 5 occur every fifth integer). When $n+1$ is a multiple of 5, the corresponding term will be the same as the previous one (if $n+1$ includes only one factor of 5) or will be smaller if $n+1$ includes more than one factor of 5.
After a bit of experimentation, it begins to appear that, in order to get an integer to appear three times in the list, there needs to be an integer n that contains at least five factors of 5.
We explicitly show that there are six integers that appear three times in the list $L(100)$ to $L(10\,000)$. Since 6 is the largest of the answer choices, then 6 must be the correct answer.

Let $N = 5^5k = 3125k$ for some positive integer k. If $N \leq 10\,000$, then k can equal 1, 2 or 3. Also, define $a = L(N)$.
We make a table of the values of $L(N-6)$ to $L(N+6)$. We note that since N contains five factors of 5, then $N-5$ and $N+5$ are each divisible by 5 (containing only one factor of 5 each), and none of the other integers in the list is divisible by 5. Also, we note that $L(m+1) - L(m) = 1 - V(m+1)$ as seen above.

m	$N-6$	$N-5$	$N-4$	$N-3$	$N-2$	$N-1$	N	$N+1$	$N+2$	$N+3$	$N+4$	$N+5$	$N+6$
$V(m)$	0	1	0	0	0	0	5	0	0	0	0	1	0
$L(m)$	a	a	$a+1$	$a+2$	$a+3$	$a+4$	a	$a+1$	$a+2$	$a+3$	$a+4$	$a+4$	$a+5$

Therefore, if $N = 5^5k$, then the integers $L(N) = a$ and $L(N) + 4 = a + 4$ each appear in the list three times.
Since there are three values of k that place N in the range $100 \leq N \leq 10\,000$, then there are six integers in Lloyd's list that appear at least three times.

In order to prove that there are no other integers that appear at least three times in the list (rather than relying on the multiple choice nature of the problem), we would need to prove some additional facts. One way to do this would be to prove:

> If n and k are positive integers with $k \geq 7$ and $n \leq 10\,000$ and $n + k \leq 10\,000$, then $L(n+k) \neq L(n)$.

This allows us to say that if two integers in the list are equal, then they must come from $L(n)$ to $L(n+6)$ inclusive, for some n. This then would allow us to say that if three integers in the list are equal, then they must come from $L(n-6)$ to $L(n+6)$, inclusive, for some n. Finally, we could then prove:

> If n is a positive integer with $n \leq 10\,000$ with three of the integers from $L(n-6)$ to $L(n+6)$, inclusive, equal, then one of the integers in the list $n-6$ to $n+6$ must be divisible by 3125.

These facts together allow us to reach the desired conclusion.

ANSWER: (E)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
www.cemc.uwaterloo.ca

2012 Cayley Contest

(Grade 10)

Thursday, February 23, 2012
(in North America and South America)

Friday, February 24, 2012
(outside of North America and South America)

Solutions

1. Simplifying, $\dfrac{5-2}{2+1} = \dfrac{3}{3} = 1$.

    Answer: (B)

2. Since the average of three numbers equals 3, then their sum is $3 \times 3 = 9$.
   Therefore, $1 + 3 + x = 9$ and so $x = 9 - 4 = 5$.

    Answer: (B)

3. When the given figure is rotated 90° clockwise, the top edge becomes the right edge, so the two outer shaded triangles are along the right edge of the resulting figure.
   Also, the bottom left shaded triangle moves to the top left.
   Therefore, the resulting figure is the one in (A).

    Answer: (A)

4. Since -1 raised to an even exponent equals 1 and -1 raised to an odd exponent equals -1, then $(-1)^3 + (-1)^2 + (-1) = -1 + 1 - 1 = -1$.
   Alternatively, we write

   $$(-1)^3 + (-1)^2 + (-1) = (-1)(-1)(-1) + (-1)(-1) + (-1) = 1(-1) + 1 - 1 = -1 + 1 - 1 = -1$$

    Answer: (D)

5. Since $\sqrt{100 - x} = 9$, then $100 - x = 9^2 = 81$, and so $x = 100 - 81 = 19$.

    Answer: (E)

6. When 3 bananas are added to the basket, there are 12 apples and 18 bananas in the basket.
   Therefore, the fraction of the fruit in the basket that is bananas is $\frac{18}{12+18} = \frac{18}{30} = \frac{3}{5}$.

    Answer: (C)

7. Since 20% of the students chose pizza and 38% of the students chose Thai food, then the percentage of students that chose Greek food is $100\% - 20\% - 38\% = 42\%$.
   Since there were 150 students surveyed, then $42\% \times 150 = \frac{42}{100} \times 150 = 63$ chose Greek food.

    Answer: (E)

8. Simplifying, $\left(1 - \dfrac{1}{3}\right)\left(1 - \dfrac{1}{4}\right)\left(1 - \dfrac{1}{5}\right) = \left(\dfrac{2}{3}\right)\left(\dfrac{3}{4}\right)\left(\dfrac{4}{5}\right)$.
   We can simplify further by dividing equal numerators and denominators to obtain a final value of $\dfrac{2}{5}$.

    Answer: (A)

9. Since 20 students went skating and 5 students went both skating and skiing, then $20 - 5 = 15$ students went skating only.
   Since 9 students went skiing and 5 students went both skating and skiing, then $9 - 5 = 4$ students went skiing only.
   The number of students who went skating or skiing or both equals the sum of the number who went skating only, the number who went skiing only, and the number who went both skating and skiing, or $15 + 4 + 5 = 24$.
   Therefore, $30 - 24 = 6$ students did not go skating or skiing.

    Answer: (B)

10. The original prism has four faces that are 4 by 2 rectangles, and two faces that are 2 by 2 rectangles. Thus, the surface area of the original prism is $4(4 \cdot 2) + 2(2 \cdot 2) = 32 + 8 = 40$.
When a 1 by 1 by 1 cube is cut out, a 1 by 1 square is removed from each of three faces of the prism, but three new 1 by 1 squares become part of the surface area.
In other words, there is no change to the total surface area.
Therefore, the surface area of the new solid is also 40.

Answer: (C)

11. During a 3 hour shift, Matilda will deliver $3 \times 30 = 90$ newspapers.
Therefore, she earns a total of $3 \times \$6.00 + 90 \times \$0.25 = \$18.00 + \$22.50 = \$40.50$ during her 3 hour shift.

Answer: (A)

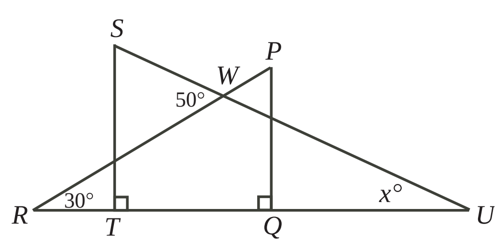
12. Since (p, q) lies on the line $y = \frac{2}{5}x$, then $q = \frac{2}{5}p$.
The given rectangle has two sides on the axes, so has width p and height q.
Therefore, the area of the rectangle equals $pq = p \cdot \frac{2}{5}p = \frac{2}{5}p^2$.
Since we are told that the area of the rectangle is 90, then $\frac{2}{5}p^2 = 90$ or $p^2 = \frac{5}{2}(90) = 225$.
Since $p > 0$, then $p = \sqrt{225} = 15$.

Answer: (D)

13. If N is divisible by both 5 and 11, then N is divisible by $5 \times 11 = 55$.
This is because 5 and 11 have no common divisor larger than 1.
Therefore, we are looking for a multiple of 55 between 400 and 600 that is odd.
One way to find such a multiple is to start with a known multiple of 55, such as 550, which is even.
We can add or subtract 55 from this multiple and still obtain multiples of 55.
Note that $550 + 55 = 605$, which is too large.
Now $550 - 55 = 495$ which is in the correct range and is odd.
Since we are told that there is only such such integer, then it must be the case that $N = 495$.
The sum of the digits of N is $4 + 9 + 5 = 18$.

Answer: (E)

14. We label the point of intersection of RP and SU as W.



Now $\angle SWR$ is an exterior angle for $\triangle RWU$.
Therefore, $\angle SWR = \angle WRU + \angle WUR$, and so $50° = 30° + x°$ or $x = 50 - 30 = 20$.
Alternatively, we could see that $\angle RWU = 180° - \angle SWR = 180° - 50° = 130°$.
Since the angles in $\triangle RWU$ add to $180°$, then $30° + 130° + x° = 180°$, or $x = 180 - 130 - 30 = 20$.

Answer: (B)

15. Since the radius of the larger circle is 9, then $OQ = 9$ and the area of the larger circle is $\pi 9^2 = 81\pi$.
Since $OP : PQ = 1 : 2$ and $OQ = 9$, then $OP = \frac{1}{3}OQ = 3$.
Thus, the radius of the smaller circle is 3 and so the area of the smaller circle is $\pi 3^2 = 9\pi$.
The area of the shaded region equals the area of the large circle minus the area of the small circle, or $81\pi - 9\pi = 72\pi$.
ANSWER: (D)

16. From the table of values, when $x = 0$, $y = 8$, and so $8 = a \cdot 0^2 + b \cdot 0 + c$ or $c = 8$.
From the table of values, when $x = 1$, $y = 9$, and so $9 = a \cdot 1^2 + b \cdot 1 + c$ or $a + b + c = 9$.
Since $a + b + c = 9$ and $c = 8$, then $a + b + 8 = 9$ or $a + b = 1$.
ANSWER: (B)

17. Let L be the length of the string.
If x is the length of the shortest piece, then since each of the other pieces is twice the length of the next smaller piece, then the lengths of the remaining pieces are $2x$, $4x$, and $8x$.
Since these four pieces make up the full length of the string, then $x + 2x + 4x + 8x = L$ or $15x = L$ and so $x = \frac{1}{15}L$.
Thus, the longest piece has length $8x = \frac{8}{15}L$, which is $\frac{8}{15}$ of the length of the string.
ANSWER: (A)

18. *Solution 1*
After one of the six integers is erased, there are five integers remaining which add to 2012.
Since the original six integers are consecutive, then we can treat them as roughly equal.
Since there are five roughly equal integers that add to 2012, then each is roughly equal to $\frac{2012}{5}$, which is roughly 400.
We finish our solution by trial and error.
Suppose that the original six integers were 400, 401, 402, 403, 404, 405.
The sum of these integers is 2415. If one of the integers is to be removed to obtain a total of 2012, then the integer removed must be $2415 - 2012 = 403$.
Is there another possible answer?
Suppose that the original six integers were larger, say 401, 402, 403, 404, 405, 406. In this case, the smallest that the sum of five of these could be is $401 + 402 + 403 + 404 + 405 = 2015$, which is too large for the given sum. Any larger set of integers only makes the smallest possible sum of five integers larger.
Suppose that the original six integers were smaller, say 399, 400, 401, 402, 403, 404. In this case, the largest that the sum of five of these could be is $400 + 401 + 402 + 403 + 404 = 2010$, which is too small for the given sum. Any smaller set of integers only makes the largest possible sum of five integers smaller.
Therefore, the possibility found above is the only possibility, and so the sum of the digits of the integer that was erased is $4 + 0 + 3 = 7$.
(Note that, since this is a multiple choice problem, once we had found an answer that works, it must be the correct answer.)

Solution 2
Suppose that the original six integers are x, $x + 1$, $x + 2$, $x + 3$, $x + 4$, and $x + 5$.
Suppose also that the integer that was erased is $x + a$, where a is 0, 1, 2, 3, 4, or 5.

The sum of the integers left is $(x + (x + 1) + (x + 2) + (x + 3) + (x + 4) + (x + 5)) - (x + a)$. Therefore,

$$\begin{aligned} (x + (x + 1) + (x + 2) + (x + 3) + (x + 4) + (x + 5)) - (x + a) &= 2012 \\ (6x + 15) - (x + a) &= 2012 \\ 5x + 15 &= 2012 + a \\ 5(x + 3) &= 2012 + a \end{aligned}$$

Since the left side is an integer that is divisible by 5, then the right side is an integer that is divisible by 5.
Since a is 0, 1, 2, 3, 4, or 5 and $2012 + a$ is divisible by 5, then a must equal 3.
Thus, $5(x + 3) = 2015$ or $x + 3 = 403$ and so $x = 400$.
Finally, the integer that was erased is $x+a = 400+3 = 403$. The sum of its digits is $4+0+3 = 7$.

Answer: (C)

19. Suppose that the sum of the four integers along each straight line equals S.
Then $S = 9 + p + q + 7 = 3 + p + u + 15 = 3 + q + r + 11 = 9 + u + s + 11 = 15 + s + r + 7$.
Thus,

$$\begin{aligned} 5S &= (9 + p + q + 7) + (3 + p + u + 15) + (3 + q + r + 11) + (9 + u + s + 11) + (15 + s + r + 7) \\ &= 2p + 2q + 2r + 2s + 2u + 90 \end{aligned}$$

Since p, q, r, s, and u are the numbers 19, 21, 23, 25, and 27 in some order, then

$$p + q + r + s + u = 19 + 21 + 23 + 25 + 27 = 115$$

and so $5S = 2(115) + 90 = 320$ or $S = 64$.
Since $S = 64$, then $3 + p + u + 15 = 64$ or $p + u = 46$.
Since $S = 64$, then $15 + s + r + 7 = 64$ or $s + r = 42$.
Therefore, $q = (p + q + r + s + u) - (p + u) - (s + r) = 115 - 46 - 42 = 27$.

Answer: (D)

20. In order to find N, which is the smallest possible integer whose digits have a fixed product, we must first find the minimum possible number of digits with this product. (This is because if the integer a has more digits than the integer b, then $a > b$.)
Once we have determined the digits that form N, then the integer N itself is formed by writing the digits in increasing order. (Given a fixed set of digits, the leading digit of N will contribute to the largest place value, and so should be the smallest digit. The next largest place value should get the next smallest digit, and so on.)
Note that the digits of N cannot include 0, or else the product of its digits would be 0.
Also, the digits of N cannot include 1, otherwise we could remove the 1s and obtain an integer with fewer digits (thus, a smaller integer) with the same product of digits.

Since the product of the digits of N is 2700, we find the prime factorization of 2700 to help us determine what the digits are:

$$2700 = 27 \times 100 = 3^3 \times 10^2 = 3^3 \times 2^2 \times 5^2$$

In order for a non-zero digit to have a factor of 5, then the digit must equal 5.
Since 2700 has two factors of 5, then the digits of N includes two 5s.

The remaining digits have a product of $3^3 \times 2^2 = 108$.
Therefore, we must try to find a combination of the smallest number of possible digits whose product is 108.
We cannot have 1 digit with a product of 108. We also cannot have a 2 digits with a product of 108, as the product of 2 digits is at most $9 \times 9 = 81$.
We can have a product of 3 digits with a product of 108 (for example, $2 \times 6 \times 9$ or $3 \times 6 \times 6$).
Therefore, the number N has 5 digits (two 5s and three other digits with a product of 108).
In order for N to be as small as possible, its leading digit (that is, its ten thousands digit) must be as small as possible. Recall that N cannot include the digit 1.
The next smallest possible leading digit is 2. In this case, 2 must be one of the three digits whose product is 108. Thus, the remaining two of these three digits have a product of $108 \div 2 = 54$, and so must be 6 and 9.
Therefore, the digits of N must be 2, 6, 9, 5, 5. The smallest possible number formed by these digits is when the digits are placed in increasing order, and so $N = 25\,569$.
The sum of the digits of N is $2 + 5 + 5 + 6 + 9 = 27$.

Answer: (E)

21. Since $x + xy = 391$, then $x(1 + y) = 391$.
We note that $391 = 17 \cdot 23$.
Since 17 and 23 are both prime, then if 391 is written as the product of two positive integers, it must be 1×391 or 17×23 or 23×17 or 391×1.
Matching x and $1+y$ to these possible factors, we obtain $(x, y) = (1, 390)$ or $(17, 22)$ or $(23, 16)$ or $(391, 0)$.
Since y is a positive integer, the fourth pair is not possible.
Since $x > y$, the first two pairs are not possible.
Therefore, $(x, y) = (23, 16)$ and so $x + y = 39$.

Answer: (B)

22. Since the five monkeys are randomly numbered, then the probability that any given monkey is numbered Monkey 1 is $\frac{1}{5}$. There are five possibilities to consider.

- Case 1: If the monkey in seat P is numbered Monkey 1, then it stays in its seat, so the monkey in seat R cannot move to seat P.
- Case 2: If the monkey in seat Q is numbered Monkey 1, then after the monkeys have moved, Monkey 2 will be in seat R, Monkey 3 in seat S, Monkey 4 in seat T, and Monkey 5 in seat P. Thus, if Monkey 1 is in seat Q, then the monkey in seat R moves to seat P if it was numbered Monkey 5.
- Case 3: If the monkey in seat R is numbered Monkey 1, then it stays in its seat and so cannot move to seat P.
- Case 4: If the monkey in seat S is numbered Monkey 1, then after the monkeys have moved, Monkey 2 will be in seat T, Monkey 3 in seat P, Monkey 4 in seat Q, and Monkey 5 in seat R. Thus, if Monkey 1 is in seat S, then the monkey in seat R moves to seat P if it was numbered Monkey 3.
- Case 5: If the monkey in seat T is numbered Monkey 1, then after the monkeys have moved, Monkey 2 will be in seat P, Monkey 3 in seat Q, Monkey 4 in seat R, and Monkey 5 in seat S. Thus, if Monkey 1 is in seat T, then the monkey in seat R moves to seat P if it was numbered Monkey 2.

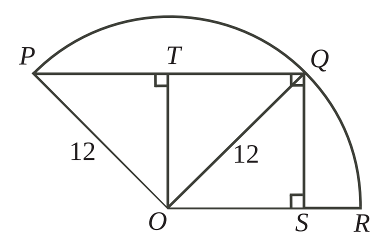
The possible cases that we have to consider are Cases 2, 4 and 5.

- From Case 2, what is the probability that the monkey in seat Q is numbered Monkey 1 and the monkey in seat R is numbered Monkey 5? We know that the probability that the monkey in seat Q is numbered Monkey 1 is $\frac{1}{5}$. Given this and the fact that the remaining four monkeys are numbered randomly, the probability is $\frac{1}{4}$ that the monkey in seat R is numbered Monkey 5. Therefore, the probability of this combined event happening is $\frac{1}{5} \cdot \frac{1}{4} = \frac{1}{20}$.
- Similarly, from Case 4 the probability that the monkey in seat S is numbered Monkey 1 and the monkey in seat R is numbered Monkey 3 is also $\frac{1}{20}$.
- Also, from Case 5 the probability that the monkey in seat T is numbered Monkey 1 and the monkey in seat R is numbered Monkey 2 is $\frac{1}{20}$.

Therefore, the probability that the monkey who was in seat R moves to seat P is $3 \cdot \frac{1}{20} = \frac{3}{20}$.

ANSWER: (C)

23. Join O to Q and draw a perpendicular from O to T on PQ.
Since the radius of the circle is 12, then $OP = OQ = 12$.
Consider $\triangle OTP$ and $\triangle OTQ$. Each is right-angled, they share side OT, and they have hypotenuses (OP and OQ) of equal length. Therefore, these triangles are congruent.
Consider quadrilateral $TQSO$. Since the quadrilateral has three right angles, then it must be a rectangle so its fourth angle, $\angle TOS$, is 90°.



Thus, $\angle TOP = 135^\circ - 90^\circ = 45^\circ$.
Since the angles in $\triangle OTP$ add to 180°, then $\angle OPT = 180^\circ - 90^\circ - 45^\circ = 45^\circ$.
Therefore, $\triangle OTP$ is isosceles and right-angled with hypotenuse 12.
Since $\triangle OTQ$ is congruent to $\triangle OTP$, it is also isosceles and right-angled with hypotenuse 12.
Since $\angle TOP = \angle TOQ = 45^\circ$, then $\angle QOS = 135^\circ - 45^\circ - 45^\circ = 45^\circ$, which tells us that $\triangle OQS$, which is right-angled, has one 45° angle and so must have a second. Therefore, $\triangle OQS$ is also isosceles and right-angled, and also has hypotenuse $OQ = 12$.
So $\triangle OQS$ is congruent to $\triangle OTQ$.
Therefore, the area of trapezoid $OPQS$ equals three times the area of an isosceles right-angled triangle with hypotenuse 12.
We calculate the area of $\triangle OTP$, which is one of these triangles.
Suppose that $OT = TP = a$.
Since $\triangle OTP$ is right-angled and isosceles, then $OP = \sqrt{2}a$.
(We can see this by using the Pythagorean Theorem to obtain

$$OP = \sqrt{OT^2 + TP^2} = \sqrt{a^2 + a^2} = \sqrt{2a^2} = \sqrt{2}a$$

since $a > 0$.)
Since $OP = 12$, then $\sqrt{2}a = 12$ or $a = \dfrac{12}{\sqrt{2}}$.
The area of $\triangle OTP$ is $\dfrac{1}{2}(OT)(TP) = \dfrac{1}{2}a^2 = \dfrac{1}{2}\left(\dfrac{12}{\sqrt{2}}\right)^2 = \dfrac{1}{2}\left(\dfrac{144}{2}\right) = 36$.
Thus, the area of trapezoid $OPQS$ is $3 \times 36 = 108$.

ANSWER: (C)

24. We solve this problem by first determining the possible structures of the exchange within the book club. Then, we count the number of ways the friends can be fit into the structures.
Suppose that the six friends exchange books as described.
Consider one person, who we call A.
A must give a book to a second person, who we call B ($A \to B$).
B cannot give a book back to A, so must give a book to a third person, who we call C ($A \to B \to C$).
C cannot give a book to B since B already receives a book from A. Therefore, C can give a book to A ($A \to B \to C \to A$) or C can give a book to a fourth person, who we call D ($A \to B \to C \to D$).

   - In the first case ($A \to B \to C \to A$), each of A, B and C already both give and receive a book. Therefore, the fourth person, who we call D, must give a book to a fifth person, who we call E ($A \to B \to C \to A$; $D \to E$).
   E cannot give a book back to D and cannot give to any of A, B or C, so must give a book to the sixth (and final) person, who we call F ($A \to B \to C \to A$; $D \to E \to F$).
   The only person that F can give a book to is D, since everyone else already is receiving a book. This completes this case as ($A \to B \to C \to A$; $D \to E \to F \to D$).
   - The other option from above is $A \to B \to C \to D$.
   Here, D cannot give a book to B or C who each already receive a book. Thus, D gives a book to A or to one of the two remaining people.
   If D gives a book to A, then each of A, B, C and D both gives and receives, so the final two people are left to exchange books with each other, which is not possible.
   Therefore, D gives a book to one of the remaining people, who we call E (giving $A \to B \to C \to D \to E$).
   Only two people have not received a book now: A and the sixth person, who we call F.
   E must give a book to F, otherwise F does not give or receive a book at all. Thus, F must give a book to A.
   This gives the structure $A \to B \to C \to D \to E \to F \to A$.

Therefore, there are two possible structures for the exchanges: $A \to B \to C \to A$ with $D \to E \to F \to D$, and $A \to B \to C \to D \to E \to F \to A$.
(We can think of the first structure as two cycles of three people, and the second as one cycle of six people.)

Suppose the six friends are P, Q, R, S, T, and U. (We could call them Peter, Quinn, Rad, Steve, Troy, and Ursula, but will use abbreviations to save space.) We must count the number of ways that these six friends can be assigned to the positions A through F in the two structures above. We have to be careful because assigning

$$\begin{array}{ccccccc} A & \to & B & \to & C & \to & A \\ P & & Q & & R & & P \end{array} \qquad\qquad \begin{array}{ccccccc} D & \to & E & \to & F & \to & D \\ S & & T & & U & & S \end{array}$$

in the first structure is the same as assigning

$$\begin{array}{ccccccc} A & \to & B & \to & C & \to & A \\ U & & S & & T & & U \end{array} \qquad\qquad \begin{array}{ccccccc} D & \to & E & \to & F & \to & D \\ Q & & P & & R & & Q \end{array}$$

but is different from assigning

$$\begin{array}{ccccccc} A & \to & B & \to & C & \to & A \\ P & & R & & Q & & P \end{array} \qquad \begin{array}{ccccccc} D & \to & E & \to & F & \to & D \\ S & & T & & U & & S \end{array}$$

- Case 1: $A \to B \to C \to A$ with $D \to E \to F \to D$
  Because this case consists of two cycles of three people each, each person serves exactly the same function, so we can assign P to position A.
  There are then 5 possible friends for position B (all but P).
  For each of these, there are 4 possible friends for position C (all put P and the friend in position B). This completes the first cycle.
  Now, the remaining three friends must complete the second cycle.
  Suppose without loss of generality that these three friends are S, T and U.
  Since each of the three positions in the second cycle serves exactly the same function, we can assign S to position D.
  There are then 2 possible friends for position E (either T or U).
  For each of these, there is 1 possible friend left to go in position F.
  This gives $5 \times 4 \times 2 \times 1 = 40$ ways of assigning the friends to this structure.
- Case 2: $A \to B \to C \to D \to E \to F \to A$
  Because this case is a single cycle of six people, each person serves exactly the same function, so we can assign P to position A.
  There are then 5 possible friends for position B (all but P).
  For each of these, there are 4 possible friends for position C (all put P and the friend in position B).
  Similarly, there are 3 possibilities for position D, 2 for position E, and 1 for position F.
  This gives $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways of assigning the friends to this structure.

Therefore, there are $40 + 120 = 160$ ways in total in which the books can be exchanged.

ANSWER: (E)

25. Let $S(n)$ represent the sum of the digits of n and let $S(2n)$ represent the sum of the digits of $2n$. In the table below, we make a claim about how each digit of n contributes to $S(2n)$. We use the data in the table to answer the question, following which we justify the data in the table:

Digit in n	$2\times$Digit	Contribution to $S(2n)$
0	0	0
1	2	2
2	4	4
3	6	6
4	8	8
5	10	$1+0=1$
6	12	$1+2=3$
7	14	$1+4=5$
8	16	$1+6=7$

We know that the digits of n include no 9s, exactly four 8s, exactly three 7s, and exactly two 6s. These digits contribute $4 \cdot 8 + 3 \cdot 7 + 2 \cdot 6 = 65$ to $S(n)$, leaving a sum of $104 - 65 = 39$ for the remaining digits.
Suppose that n includes m 5s. These 5s contribute $5m$ to $S(n)$, so the remaining $39 - 5m$ come from the digits 0 to 4.

Consider now the digits of $2n$.
The table above shows that the 9s, 8s, 7s and 6s each contribute 9, 7, 5 and 3, respectively, to $S(2n)$. Since the digits of n include no 9s, four 8s, three 7s, and two 6s, then these digits contribute $4 \cdot 7 + 3 \cdot 5 + 2 \cdot 3 = 49$ to $S(2n)$.
Each 5 in n contributes 1 to $S(2n)$, so the m 5s in n contribute $m \cdot 1 = m$ to $S(2n)$.
The digits 0 to 4 each contribute twice as much to $S(2n)$ as they do to $S(n)$, so the sum of the their contributions to $S(2n)$ is twice the sum of their contributions to $S(n)$.
Since the sum of their contributions to $S(n)$ is $39 - 5m$, then the sum of their contributions to $S(2n)$ is $2(39 - 5m)$.
Since $S(2n) = 100$, then $49 + m + 2(39 - 5m) = 100$.
Therefore, $127 - 9m = 100$ and so $9m = 27$ or $m = 3$.
That is, the digits of n include three 5s.

We must still justify the data in the table above.
Suppose that n ends with the digits $dcba$. That is, $n = \cdots dcba$.
Then we can write $n = \cdots + 1000d + 100c + 10b + a$.
Then $2n = \cdots + 1000(2d) + 100(2c) + 10(2b) + (2a)$. The difficulty in determining the digits of $2n$ is that each of $2d$, $2c$, $2b$ and $2a$ may not be a single digit.
We use the notation $u(2a)$ and $t(2a)$ to represent the units digit and tens digit of $2a$, respectively.
Note that $u(2a)$ is one of 0, 2, 4, 6, or 8, and $t(2a)$ is 0 or 1.
We define $u(2b), t(2b), u(2c), t(2c), u(2d), t(2d)$ similarly.
Note that $2a = 10 \cdot t(2a) + u(2a)$ and $2b = 10 \cdot t(2b) + u(2b)$ and $2c = 10 \cdot t(2c) + u(2c)$ and $2d = 10 \cdot t(2d) + u(2d)$.
Thus,

$$\begin{aligned} 2n &= \cdots + 1000(10 \cdot t(2d) + u(2d)) + 100(10 \cdot t(2c) + u(2c)) \\ &\qquad + 10(10 \cdot t(2b) + u(2b)) + (10 \cdot t(2a) + u(2a)) \\ &= \cdots + 1000(u(2d) + t(2c)) + 100(u(2c) + t(2b)) + 10(u(2b) + t(2a)) + u(2a) \end{aligned}$$

Since $u(2a), u(2b), u(2c), u(2d) \leq 8$ and $t(2a), t(2b), t(2c), t(2d) \leq 1$, then each of $u(2d) + t(2c)$ and $u(2c) + t(2b)$ and $u(2b) + t(2a)$ and $u(2a)$ is a single digit, so these are the thousands, hundreds, tens and units digits, respectively, of $2n$.
Thus, the sum of the digits of $2n$ is

$$\begin{aligned} u(2a) + (u(2b) + t(2a)) + (u(2c) + t(2b)) + (u(2d) + t(2c)) + \cdots &= \\ (t(2a) + u(2a)) + (t(2b) + u(2b)) + (t(2c) + u(2c)) + \cdots & \end{aligned}$$

The above argument extends to the remaining digits of n.
In other words, if r is a digit in n, then its contribution to the sum of the digits of $2n$ is the sum of the tens and units digits of $2r$.
Therefore, the digits of n contribute to the sum of the digits of $2n$ as outlined in the table above.

Answer: (D)

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

2011 Cayley Contest

(Grade 10)

Thursday, February 24, 2011

Solutions

1. We regroup $(5 + 2) + (8 + 6) + (4 + 7) + (3 + 2)$ as $5 + (2 + 8) + (6 + 4) + (7 + 3) + 2$, which equals $5 + 10 + 10 + 10 + 2$ or 37.
   We could add the numbers directly instead.

   Answer: (B)

2. Since $(-1)(2)(x)(4) = 24$, then $-8x = 24$ or $x = \dfrac{24}{-8} = -3$.

   Answer: (B)

3. *Solution 1*
   Since $\angle PRS$ is an exterior angle to $\triangle PQR$, then $\angle PQR + \angle QPR = \angle PRS$ or $x^\circ + 75^\circ = 125^\circ$.
   Thus, $x + 75 = 125$ or $x = 50$.

   Solution 2
   Since QRS is a straight line, then $\angle PRQ = 180^\circ - \angle PRS = 180^\circ - 125^\circ = 55^\circ$.
   Since the sum of the angles in a triangle is 180°, then $x^\circ + 75^\circ + 55^\circ = 180^\circ$ or $x + 130 = 180$ or $x = 50$.

   Answer: (A)

4. *Solution 1*
   To get back to the original number, we undo the given operations.
   We add 5 to 16 to obtain 21 and then divide by 3 to obtain 7.
   These are the "inverse" operations of decreasing by 5 and multiplying by 3.

   Solution 2
   Suppose that the original number is x.
   After tripling and decreasing by 5, the result is $3x - 5$.
   Therefore, $3x - 5 = 16$ or $3x = 21$ or $x = 7$.

   Answer: (C)

5. We evaluate from the inside towards the outside:

$$\sqrt{13 + \sqrt{7 + \sqrt{4}}} = \sqrt{13 + \sqrt{7 + 2}} = \sqrt{13 + \sqrt{9}} = \sqrt{13 + 3} = \sqrt{16} = 4$$

   Answer: (D)

6. A graph that is linear with a slope of 0 is a horizontal straight line. This is Graph Q.

   Answer: (B)

7. With a fair die that has faces numbered from 1 to 6, the probability of rolling each of 1 to 6 is $\frac{1}{6}$.
   We calculate the probability for each of the five choices.
   There are 4 values of x that satisfy $x > 2$, so the probability is $\frac{4}{6} = \frac{2}{3}$.
   There are 2 values of x that satisfy $x = 4$ or $x = 5$, so the probability is $\frac{2}{6} = \frac{1}{3}$.
   There are 3 values of x that are even, so the probability is $\frac{3}{6} = \frac{1}{2}$.
   There are 2 values of x that satisfy $x < 3$, so the probability is $\frac{2}{6} = \frac{1}{3}$.
   There is 1 value of x that satisfies $x = 3$, so the probability is $\frac{1}{6}$.
   Therefore, the most likely of the five choices is that x is greater than 2.

   Answer: (A)

8. When 2.4×10^8 is doubled, the result is $2 \times 2.4 \times 10^8 = 4.8 \times 10^8$.

ANSWER: (C)

9. Since the face of a foonie has area 5 cm^2 and its thickness is 0.5 cm, then the volume of one foonie is $5 \times 0.5 = 2.5 \text{ cm}^3$.
If a stack of foonies has a volume of 50 cm^3 and each foonie has a volume of 2.5 cm^3, then there are $50 \div 2.5 = 20$ foonies in the stack.

ANSWER: (D)

10. In order to make the playoffs, the Athenas must win at least 60% of their 44 games. That is, they must win at least $0.6 \times 44 = 26.4$ games.
Since they must win an integer number of games, then the smallest number of games that they can win to make the playoffs is the smallest integer larger than 26.4, or 27.
Since they have won 20 games so far, then they must win $27 - 20 = 7$ of their remaining games to make the playoffs.

ANSWER: (E)

11. From the definition, $(3, 1)\nabla(4, 2) = (3)(4) + (1)(2) = 12 + 2 = 14$.

ANSWER: (D)

12. Since the angle in the sector representing cookies is 90°, then this sector represents $\frac{1}{4}$ of the total circle.
Therefore, 25% of the students chose cookies as their favourite food.
Thus, the percentage of students who chose sandwiches was $100\% - 30\% - 25\% - 35\% = 10\%$.
Since there are 200 students in total, then $200 \times \frac{10}{100} = 20$ students said that their favourite food was sandwiches.

ANSWER: (B)

13. *Solution 1*
We work from right to left as we would if doing this calculation by hand.
In the units column, we have $L - 1$ giving 1. Thus, $L = 2$. (There is no borrowing required.)
In the tens column, we have $3 - N$ giving 5.
Since 5 is larger than 3, we must borrow from the hundreds column. Thus, $13 - N$ gives 5, which means $N = 8$. This gives

$$\begin{array}{rcccc} & & {\scriptstyle K-1} & {\scriptstyle 13} & \\ & 5 & \cancel{K} & \cancel{3} & 2 \\ - & M & 4 & 8 & 1 \\ \hline & 4 & 4 & 5 & 1 \end{array}$$

In the hundreds column, we have $(K - 1) - 4$ giving 4, which means $K = 9$. This gives

$$\begin{array}{rcccc} & 5 & 9 & 3 & 2 \\ - & M & 4 & 8 & 1 \\ \hline & 4 & 4 & 5 & 1 \end{array}$$

In the thousands column, we have 5 (with nothing borrowed) minus M giving 4.
Thus, $5 - M = 4$ or $M = 1$.
This gives $5932 - 1481 = 4451$, which is correct.
Finally, $K + L + M + N = 9 + 2 + 1 + 8 = 20$.

Solution 2

Since $5K3L - M4N1 = 4451$, then

$$\begin{array}{ccccc} & M & 4 & N & 1 \\ + & 4 & 4 & 5 & 1 \\ \hline & 5 & K & 3 & L \end{array}$$

We start from the units column and work towards the left.

Considering the units column, the sum $1+1$ has a units digit of L. Thus, $L = 2$. (There is no carry to the tens column.)

Considering the tens column, the sum $N+5$ has a units digit of 3. Thus, $N = 8$. (There is a carry of 1 to the hundreds column.) This gives

$$\begin{array}{ccccc} & & {}^{1} & & \\ & M & 4 & 8 & 1 \\ + & 4 & 4 & 5 & 1 \\ \hline & 5 & K & 3 & 2 \end{array}$$

Considering the hundreds column, the sum $4+4$ plus the carry of 1 from the tens column has a units digit of K. Therefore, $K = 4+4+1 = 9$. There is no carry from the hundreds column to the thousands column.

Considering the thousands column, the sum $M+4$ equals 5. Therefore, $M = 1$. This gives

$$\begin{array}{ccccc} & 1 & 4 & 8 & 1 \\ + & 4 & 4 & 5 & 1 \\ \hline & 5 & 9 & 3 & 2 \end{array}$$

which is equivalent to $5932 - 4451 = 1481$, which is correct.

Finally, $K + L + M + N = 9 + 2 + 1 + 8 = 20$.

ANSWER: (A)

14. The difference between $\frac{1}{6}$ and $\frac{1}{12}$ is $\frac{1}{6} - \frac{1}{12} = \frac{2}{12} - \frac{1}{12} = \frac{1}{12}$, so $LP = \frac{1}{12}$.

Since LP is divided into three equal parts, then this distance is divided into three equal parts, each equal to $\frac{1}{12} \div 3 = \frac{1}{12} \times \frac{1}{3} = \frac{1}{36}$.

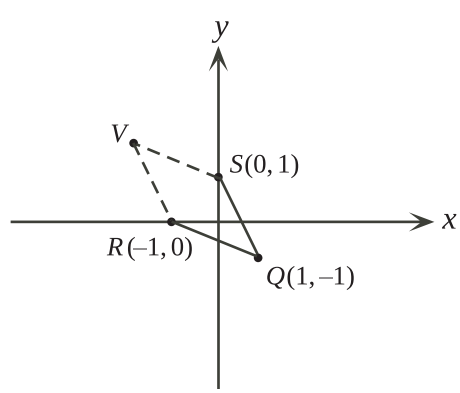
Therefore, M is located $\frac{1}{36}$ to the right of L.

Thus, the value at M is $\frac{1}{12} + \frac{1}{36} = \frac{3}{36} + \frac{1}{36} = \frac{4}{36} = \frac{1}{9}$.

ANSWER: (C)

15. We plot the first three vertices on a graph.

We see that one possible location for the fourth vertex, V, is in the second quadrant:



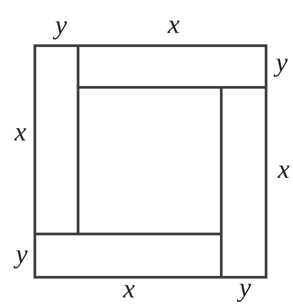
If $VSQR$ is a parallelogram, then SV is parallel and equal to QR.
To get from Q to R, we go left 2 units and up 1 unit.
Therefore, to get from S to V, we also go left 2 units and up 1 unit.
Since the coordinates of S are $(0, 1)$, then the coordinates of V are $(0 - 2, 1 + 1) = (-2, 2)$.
This is choice (A).
There are two other possible locations for the fourth vertex, which we can find in a similar way.
These are $U(0, -2)$ and $W(2, 0)$.
Using these points, we can see that $SQUR$ and $SWQR$ are parallelograms. But $(0, -2)$ and $(2, 0)$ are not among the possible answers.
Therefore, of the given choices, the only one that completes a parallelogram is $(-2, 2)$.

Answer: (A)

16. It is possible that after buying 7 gumballs, Wally has received 2 red, 2 blue, 1 white, and 2 green gumballs.
This is the largest number of each colour that he could receive without having three gumballs of any one colour.
If Wally buys another gumball, he will receive a blue or a green or a red gumball.
In each of these cases, he will have at least 3 gumballs of one colour.
In summary, if Wally buys 7 gumballs, he is not guaranteed to have 3 of any one colour; if Wally buys 8 gumballs, he is guaranteed to have 3 of at least one colour.
Therefore, the least number that he must buy to guarantee receiving 3 of the same colour is 8.

Answer: (E)

17. Suppose that each of the smaller rectangles has a longer side of length x cm and a shorter side of length y cm.



Since the perimeter of each of the rectangles is 40 cm, then $2x + 2y = 40$ or $x + y = 20$.
But the side length of the large square is $x + y$ cm.
Therefore, the area of the large square is $(x + y)^2 = 20^2 = 400$ cm^2.

Answer: (C)

18. *Solution 1*
When a number n is divided by x, the remainder is the difference between n and the largest multiple of x less than n. When 100 is divided by a positive integer x, the remainder is 10.
This means that $100 - 10 = 90$ is exactly divisible by x. It also means that x is larger than 10, otherwise the remainder would be smaller than 10.
Since 90 is exactly divisible by x, then $11 \times 90 = 990$ is also exactly divisible by x.
Since $x > 10$, then the next multiple of x is $990 + x$, which is larger than 1000.
Thus, 990 is the largest multiple of x less than 1000, and so the remainder when 1000 is divided by x is $1000 - 990 = 10$.

Solution 2
When 100 is divided by a positive integer x, the remainder is 10. The remainder is the difference between 100 and the largest multiple of x less than 100.
Therefore, the largest multiple of x less than 100 is $100 - 10 = 90$.
It also means that x is larger than 10, otherwise the remainder would be smaller than 10.
We can choose $x = 15$, since $6 \times 15 = 90$ and 15 is larger than 10.
What is the remainder when 1000 is divided by 15? Using a calculator, $1000 \div 15 = 66.666\ldots$ and $66 \times 15 = 990$.
Thus, the difference between 1000 and the largest multiple of 15 less than 1000 (that is, 990) equals 10 and so the remainder is equal to 10.

ANSWER: (A)

19. Let $\angle XYW = \theta$.
Since $\triangle XYW$ is isosceles with $WX = WY$, then $\angle YXW = \angle XYW = \theta$.
Since the sum of the angles in $\triangle XYW$ is 180°, then $\angle XWY = 180^\circ - 2\theta$.
Since $\angle XWY + \angle ZWY = 180^\circ$, then $\angle ZWY = 180^\circ - (180^\circ - 2\theta) = 2\theta$.
Since $\triangle YWZ$ is isosceles with $YW = YZ$, then $\angle YZW = \angle ZWY = 2\theta$.
Since $\triangle XYZ$ is isosceles, with $XY = XZ$, then $\angle XYZ = \angle XZY = 2\theta$.
Since the sum of the angles in $\triangle XYZ$ is 180°, then $\angle XYZ + \angle XZY + \angle YXZ = 180^\circ$ or $2\theta + 2\theta + \theta = 180^\circ$, or $5\theta = 180^\circ$, or $\theta = 36^\circ$.

ANSWER: (D)

20. For $n^3 + 5n^2$ to be the square of an integer, $\sqrt{n^3 + 5n^2}$ must be an integer.
We know that $\sqrt{n^3 + 5n^2} = \sqrt{n^2(n+5)} = \sqrt{n^2}\sqrt{n+5} = n\sqrt{n+5}$.
For $n\sqrt{n+5}$ to be an integer, $\sqrt{n+5}$ must be an integer. In other words, $n+5$ must be a perfect square.
Since n is between 1 and 100, then $n+5$ is between 6 and 105.
The perfect squares in this range are $3^2 = 9$, $4^2 = 16$, $\ldots$, $10^2 = 100$.
Thus, there are 8 perfect squares in this range.
Therefore, there are 8 values of n for which $\sqrt{n+5}$ is an integer, and thus for which $n^3 + 5n^2$ is the square of an integer.

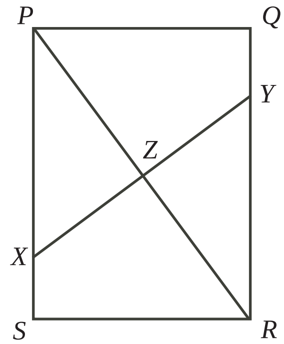
ANSWER: (B)

21. *Solution 1*
If we multiply the second and third equations together, we obtain $x(y+1)y(y+1) = \frac{7}{9} \cdot \frac{5}{18}$ or $xy(x+1)(x+1) = \frac{35}{162}$.
From the first equation, $xy = \frac{1}{9}$.
Therefore, $\frac{1}{9}(x+1)(y+1) = \frac{35}{162}$ or $(x+1)(y+1) = 9(\frac{35}{162}) = \frac{35}{18}$.

Solution 2
If we expand the left side of the second equation, we obtain $xy + x = \frac{7}{9}$.
Since $xy = \frac{1}{9}$ (from the first equation), then $x = \frac{7}{9} - xy = \frac{7}{9} - \frac{1}{9} = \frac{2}{3}$.
If we expand the left side of the third equation, we obtain $xy + y = \frac{5}{18}$.
Since $xy = \frac{1}{9}$ (from the first equation), then $y = \frac{5}{18} - xy = \frac{5}{18} - \frac{1}{9} = \frac{3}{18} = \frac{1}{6}$.
Therefore, $(x+1)(y+1) = (\frac{2}{3} + 1)(\frac{1}{6} + 1) = \frac{5}{3} \cdot \frac{7}{6} = \frac{35}{18}$.

ANSWER: (E)

22. Suppose that the crease intersects PS at X, QR at Y, and the line PR at Z. We want to determine the length of XY.



Since P folds on top of R, then line segment PZ folds on top of line segment RZ, since after the fold Z corresponds with itself and P corresponds with R. This means that $PZ = RZ$ and PR must be perpendicular to XY at point Z.
Since $PS = RQ$ and $SR = QP$, then right-angled triangles $\triangle PSR$ and $\triangle RQP$ are congruent (side-angle-side).
Therefore, $\angle XPZ = \angle YRZ$.
Since $PZ = RZ$, then right-angled triangles $\triangle PZX$ and $\triangle RZY$ are congruent too (angle-side-angle).
Thus, $XZ = ZY$ and so $XY = 2XZ$.
Since $\triangle PSR$ is right-angled at S, then by the Pythagorean Theorem,

$$PR = \sqrt{PS^2 + SR^2} = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$$

since $PR > 0$.
Since $PZ = RZ$, then $PZ = \frac{1}{2}PR = 5$.
Now $\triangle PZX$ is similar to $\triangle PSR$ (common angle at P and right angle), so $\dfrac{XZ}{PZ} = \dfrac{RS}{PS}$ or $XZ = \frac{5\cdot 6}{8} = \frac{30}{8} = \frac{15}{4}$.
Therefore, $XY = 2XZ = \frac{15}{2}$, so the length of the fold is $\frac{15}{2}$ or 7.5.

Answer: (C)

23. First, we calculate the number of pairs that can be formed from the integers from 1 to n.
One way to form a pair is to choose one number to be the first item of the pair (n choices) and then a different number to be the second item of the pair ($n - 1$ choices).
There are $n(n - 1)$ ways to choose these two items in this way.
But this counts each pair twice; for example, we could choose 1 then 3 and we could also choose 3 then 1.
So we have double-counted the pairs, meaning that there are $\frac{1}{2}n(n-1)$ pairs that can be formed.

Next, we examine the number of rows in the table.
Since each row has three entries, then each row includes three pairs (first and second numbers, first and third numbers, second and third numbers).
Suppose that the completed table has r rows.
Then the total number of pairs in the table is $3r$.
Since each pair of the numbers from 1 to n appears exactly once in the table and the total number of pairs from these numbers is $\frac{1}{2}n(n - 1)$, then $3r = \frac{1}{2}n(n - 1)$, which tells us that $\frac{1}{2}n(n - 1)$ must be divisible by 3, since $3r$ is divisible by 3.
We make a table listing the possible values of n and the corresponding values of $\frac{1}{2}n(n - 1)$:

n	3	4	5	6	7	8	9	10	11	12
$\frac{1}{2}n(n-1)$	3	6	10	15	21	28	36	45	55	66

Since $\frac{1}{2}n(n-1)$ must be divisible by 3, then the possible values of n are 3, 4, 6, 7, 9, 10, and 12.
Next, consider a fixed number m from the list 1 to n.
In each row that m appears, it will belong to 2 pairs (one with each of the other two numbers in its row).
If the number m appears in s rows, then it will belong to $2s$ pairs.
Therefore, each number m must belong to an even number of pairs.
But each number m from the list of integers from 1 to n must appear in $n-1$ pairs (one with each other number in the list), so $n-1$ must be even, and so n is odd.
Therefore, the possible values of n are 3, 7, 9.
Finally, we must verify that we can create a Fano table for each of these values of n. We are given the Fano table for $n=7$.
Since the total number of pairs when $n=3$ is 3 and when $n=9$ is 36, then a Fano table for $n=3$ will have $3 \div 3 = 1$ row and a Fano table for $n=9$ will have $36 \div 3 = 12$ rows.
For $n=3$ and $n=9$, possible tables are shown below:

1	2	3

1	2	3
1	4	5
1	6	7
1	8	9
2	4	7
2	5	8
2	6	9
3	4	9
3	5	6
3	7	8
4	6	8
5	7	9

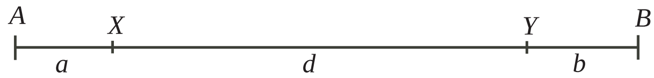
In total, there are 3 values of n in this range for which a Fano table can be created.

Answer: (B)

24. First, we note that the three people are interchangeable in this problem, so it does not matter who rides and who walks at any given moment. We abbreviate the three people as D, M and P.
We call their starting point A and their ending point B.
Here is a strategy where all three people are moving at all times and all three arrive at B at the same time:

D and M get on the motorcycle while P walks.
D and M ride the motorcycle to a point Y before B.
D drops off M and rides back while P and M walk toward B.
D meets P at point X.
D picks up P and they drive back to B meeting M at B.
Point Y is chosen so that D, M and P arrive at B at the same time.

Suppose that the distance from A to X is a km, from X to Y is d km, and the distance from Y to B is b km.



In the time that it takes P to walk from A to X at 6 km/h, D rides from A to Y and back to X at 90 km/h.
The distance from A to X is a km.
The distance from A to Y and back to X is $a + d + d = a + 2d$ km.
Since the time taken by P and by D is equal, then $\frac{a}{6} = \frac{a+2d}{90}$ or $15a = a + 2d$ or $7a = d$.

In the time that it takes M to walk from Y to B at 6 km/h, D rides from Y to X and back to B at 90 km/h.
The distance from Y to B is b km, and the distance from Y to X and back to B is $d+d+b = b+2d$ km.
Since the time taken by M and by D is equal, then $\frac{b}{6} = \frac{b+2d}{90}$ or $15b = b + 2d$ or $7b = d$.

Therefore, $d = 7a = 7b$, and so we can write $d = 7a$ and $b = a$.
Thus, the total distance from A to B is $a + d + b = a + 7a + a = 9a$ km.
However, we know that this total distance is 135 km, so $9a = 135$ or $a = 15$.
Finally, D rides from A to Y to X to B, a total distance of $(a + 7a) + 7a + (7a + a) = 23a$ km.
Since $a = 15$ km and D rides at 90 km/h, then the total time taken for this strategy is $\frac{23\times 15}{90} = \frac{23}{6} \approx 3.83$ h.
Since we have a strategy that takes 3.83 h, then the smallest possible time is no more than 3.83 h. Can you explain why this is actually the smallest possible time?

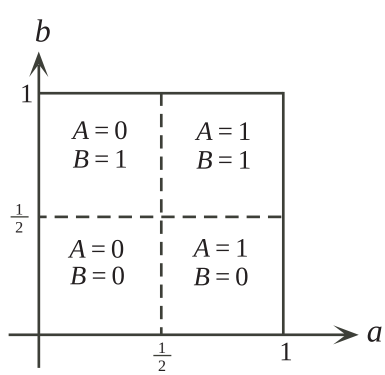
If we didn't think of this strategy, another strategy that we might try would be:

D and M get on the motorcycle while P walks.
D and M ride the motorcycle to B.
D drops off M at B and rides back to meet P, who is still walking.
D picks up P and they drive back to B. (M rests at B.)

This strategy actually takes 4.125 h, which is longer than the strategy shown above, since M is actually sitting still for some of the time.

Answer: (A)

25. By definition, if $0 \le a < \frac{1}{2}$, then $A = 0$ and if $\frac{1}{2} \le a \le 1$, then $A = 1$.
Similarly, if $0 \le b < \frac{1}{2}$, then $B = 0$ and if $\frac{1}{2} \le b \le 1$, then $B = 1$.
We keep track of our information on a set of axes, labelled a and b.



The area of the region of possible pairs (a, b) is 1, since the region is a square with side length 1. Next, we determine the sets of points where $C = 2A + 2B$ and calculate the combined area of these regions.
We consider the four sub-regions case by case. In each case, we will encounter lines of the form $a + b = Z$ for some number Z. We can rewrite these equations as $b = -a + Z$ which shows that this equation is the equation of the line with slope -1 and b-intercept Z. Since the slope is -1, the a-intercept is also Z.

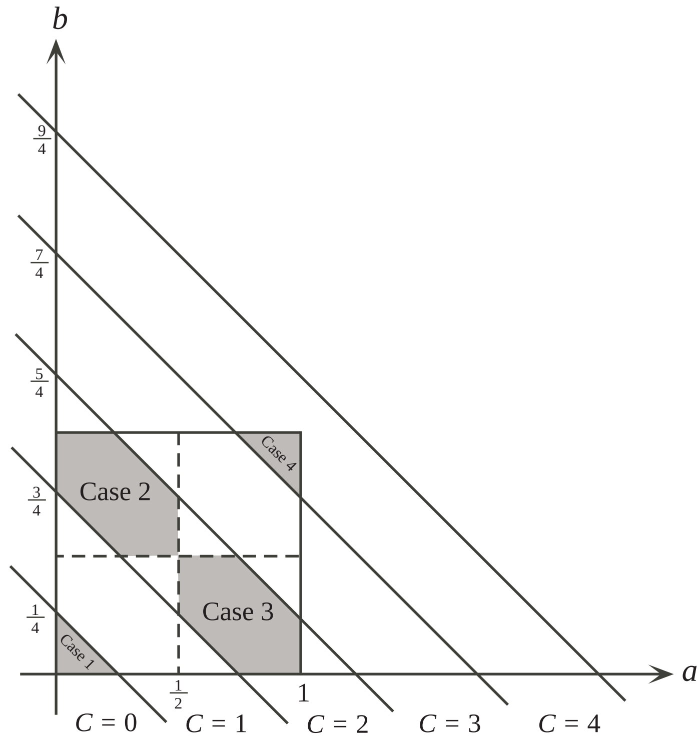
<u>Case 1: $A = 0$ and $B = 0$</u>
For C to equal $2A + 2B$, we need $C = 0$.
Since C is obtained by rounding c, then we need $0 \leq c < \frac{1}{2}$.
Since $c = 2a + 2b$ by definition, this is true when $0 \leq 2a + 2b < \frac{1}{2}$ or $0 \leq a + b < \frac{1}{4}$.
This is the set of points in this subregion above the line $a + b = 0$ and below the line $a + b = \frac{1}{4}$.

<u>Case 2: $A = 0$ and $B = 1$</u>
For C to equal $2A + 2B$, we need $C = 2$.
Since C is obtained by rounding c, then we need $\frac{3}{2} \leq c < \frac{5}{2}$.
Since $c = 2a + 2b$ by definition, this is true when $\frac{3}{2} \leq 2a + 2b < \frac{5}{2}$ or $\frac{3}{4} \leq a + b < \frac{5}{4}$.
This is the set of points in this subregion above the line $a + b = \frac{3}{4}$ and below the line $a + b = \frac{5}{4}$.

<u>Case 3: $A = 1$ and $B = 0$</u>
For C to equal $2A + 2B$, we need $C = 2$.
Since C is obtained by rounding c, then we need $\frac{3}{2} \leq c < \frac{5}{2}$.
Since $c = 2a + 2b$ by definition, this is true when $\frac{3}{2} \leq 2a + 2b < \frac{5}{2}$ or $\frac{3}{4} \leq a + b < \frac{5}{4}$.
This is the set of points in this subregion above the line $a + b = \frac{3}{4}$ and below the line $a + b = \frac{5}{4}$.

<u>Case 4: $A = 1$ and $B = 1$</u>
For C to equal $2A + 2B$, we need $C = 4$.
Since C is obtained by rounding c, then we need $\frac{7}{2} \leq c < \frac{9}{2}$.
Since $c = 2a + 2b$ by definition, this is true when $\frac{7}{2} \leq 2a + 2b < \frac{9}{2}$ or $\frac{7}{4} \leq a + b < \frac{9}{4}$.
This is the set of points in this subregion above the line $a + b = \frac{7}{4}$ and below the line $a + b = \frac{9}{4}$.

We shade the appropriate set of points in each of the subregions:



The shaded regions are the regions of points (a, b) where $2A + 2B = C$. To determine the required probability, we calculate the combined area of these regions and divide by the total area of the set of all possible points (a, b). This total area is 1, so the probability will actually be equal to the combined area of the shaded regions.

The region from Case 1 is a triangle with height $\frac{1}{4}$ and base $\frac{1}{4}$, so has area $\frac{1}{2} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{32}$.
The region from Case 4 is also a triangle with height $\frac{1}{4}$ and base $\frac{1}{4}$. This is because the line $a + b = \frac{7}{4}$ intersects the top side of the square (the line $b = 1$) when $a = \frac{3}{4}$ and the right side of the square (the line $a = 1$) when $b = \frac{3}{4}$.
The regions from Case 2 and Case 3 have identical shapes and so have the same area. We calculate the area of the region from Case 2 by subtracting the unshaded area from the area of the entire subregion (which is $\frac{1}{4}$).
Each unshaded portion of this subregion is a triangle with height $\frac{1}{4}$ and base $\frac{1}{4}$. We can confirm this by calculating points of intersection as in Case 4.
Therefore, the area of the shaded region in Case 2 is $\frac{1}{4} - 2 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$.

Therefore, the combined area of the shaded regions is $\frac{1}{32} + \frac{1}{32} + \frac{3}{16} + \frac{3}{16} = \frac{14}{32} = \frac{7}{16}$.
Thus, the required probability is $\frac{7}{16}$.

(Note that because calculating the probability is equivalent to calculating areas in this problem, we do not have to pay special attention as to whether points on the boundaries of the various regions are included or excluded.)

Answer: (D)

Canadian
Mathematics
Competition
An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2010 Cayley Contest

(Grade 10)

Thursday, February 25, 2010

Solutions

1. Calculating, $6 + 4 \div 2 = 6 + 2 = 8$.

   Answer: (D)

2. Since there are 12 equally spaced numbers and the total angle in a complete circle is 360°, then the angle between two consecutive numbers is $360^\circ \div 12 = 30^\circ$.
   To rotate 120°, the minute hand must move by $120^\circ \div 30^\circ = 4$ numbers clockwise from the 12.
   Therefore, the hand will be pointing at the 4.

   Answer: (C)

3. Since $x + \sqrt{25} = \sqrt{36}$, then $x + 5 = 6$ or $x = 1$.

   Answer: (A)

4. Evaluating, $\dfrac{1}{2+\frac{2}{3}} = \dfrac{1}{\frac{8}{3}} = \dfrac{3}{8}$.

   Answer: (E)

5. In a rectangle, length times width equals area, so width equals area divided by length.
   Therefore, the width is $\frac{1}{3} \div \frac{3}{5} = \frac{1}{3} \times \frac{5}{3} = \frac{5}{9}$.

   Answer: (B)

6. Since the sum of the angles in a triangle is 180°, then $3x^\circ + x^\circ + 6x^\circ = 180^\circ$ or $10x = 180$ or $x = 18$.
   The largest angle in the triangle is $6x^\circ = 6(18^\circ) = 108^\circ$.

   Answer: (E)

7. *Solution 1*
   Since the mean of five consecutive integers is 9, then the middle of these five integers is 9.
   Therefore, the integers are $7, 8, 9, 10, 11$, and so the smallest of the five integers is 7.

   Solution 2
   Suppose that x is the smallest of the five consecutive integers.
   Then the integers are x, $x+1$, $x+2$, $x+3$, and $x+4$.
   The mean of these integers is $\dfrac{x+(x+1)+(x+2)+(x+3)+(x+4)}{5} = \dfrac{5x+10}{5} = x+2$.
   Since the mean is 9, then $x + 2 = 9$ or $x = 7$.
   Thus, the smallest of the five integers is 7.

   Answer: (D)

8. Since square $PQRS$ has an area of 900, then its side length is $\sqrt{900} = 30$.
   Thus, $PQ = PS = 30$.
   Since M and N are the midpoints of PQ and PS, respectively, then $PN = PM = \frac{1}{2}(30) = 15$.
   Since $PQRS$ is a square, then the angle at P is 90°, so $\triangle PMN$ is right-angled.
   Therefore, the area of $\triangle PMN$ is $\frac{1}{2}(PM)(PN) = \frac{1}{2}(15)(15) = \frac{225}{2} = 112.5$.

   An alternate approach would be to divide the square into 8 triangles, each congruent to $\triangle PMN$. (Can you see how to do this?) This would tell us that the area of $\triangle PMN$ is $\frac{1}{8}$ of the area of the square, or $\frac{1}{8}(30^2) = 112.5$.

   Answer: (B)

9. *Solution 1*

Since the triangle will include the two axes, then the triangle will have a right angle.
For the triangle to be isosceles, the other two angles must be 45°.
For a line to make an angle of 45° with both axes, it must have slope 1 or -1.
Of the given possibilities, the only such line is $y = -x + 4$.

Solution 2

The vertices of a triangle formed in this way will be the points of intersection of the pairs of lines forming the triangle.
The point of intersection of the x- and y-axes is $(0, 0)$.
The line $y = -x + 4$ crosses the y-axis at $(0, 4)$.
The line $y = -x + 4$ crosses the x-axis where $y = 0$, or $-x + 4 = 0$ and so at the point $(4, 0)$.
Therefore, if the third line is $y = -x+4$, the vertices of the triangle are $(0, 0)$, $(0, 4)$ and $(4, 0)$.
This triangle is isosceles since two of its sides have length 4.
Therefore, the third side is $y = -x + 4$, since we know that only one possibility can be correct.

Answer: (C)

10. Since the ratio of boys to girls at Pascal H.S. is $3 : 2$, then $\frac{3}{3+2} = \frac{3}{5}$ of the students at Pascal H.S. are boys.
Thus, there are $\frac{3}{5}(400) = \frac{1200}{5} = 240$ boys at Pascal H.S.
Since the ratio of boys to girls at Fermat C.I. is $2 : 3$, then $\frac{2}{2+3} = \frac{2}{5}$ of the students at Fermat C.I. are boys.
Thus, there are $\frac{2}{5}(600) = \frac{1200}{5} = 240$ boys at Fermat C.I.
There are $400 + 600 = 1000$ students in total at the two schools.
Of these, $240 + 240 = 480$ are boys, and so the remaining $1000 - 480 = 520$ students are girls.
Therefore, the overall ratio of boys to girls is $480 : 520 = 48 : 52 = 12 : 13$.

Answer: (B)

11. First, we note that the values of x and y cannot be equal since they are integers and $x + y$ is odd.
Next, we look at the case when $x > y$.
We list the fifteen possible pairs of values for x and y and the corresponding values of xy:

x	y	xy	x	y	xy	x	y	xy
30	1	30	25	6	150	20	11	220
29	2	58	24	7	168	19	12	228
28	3	84	23	8	184	18	13	234
27	4	108	22	9	198	17	14	238
26	5	130	21	10	210	16	15	240

Therefore, the largest possible value for xy is 240.
Note that the largest value occurs when x and y are as close together as possible.
The case of $x < y$ will give us exactly the same result. We can see this by switching the headings of x and y in the table above.

Answer: (A)

12. *Solution 1*

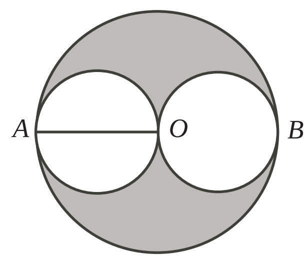
Since the sale price has been reduced by 20%, then the sale price of \$112 is 80% or $\frac{4}{5}$ of the regular price.

Therefore, $\frac{1}{5}$ of the regular price is $\$112 \div 4 = \$28$.
Thus, the regular price is $\$28 \times 5 = \$140$.
If the regular price is reduced by 30%, the new sale price would be 70% of the regular price, or $\frac{7}{10}(\$140) = \$98$.

Solution 2
Suppose that the original price was $\$x$.
Since the price has been reduced by 20%, then the sale price is 80% of the original price, or $\frac{80}{100}x = \frac{8}{10}x$.
Therefore, $\frac{8}{10}x = 112$.
If it were on sale for 30% off, then the price would be 70% of the original price, or $\frac{7}{10}x$.
Now, $\frac{7}{10}x = \frac{7}{8}\left(\frac{8}{10}x\right) = \frac{7}{8}(112) = 98$.
Thus, the price would be \$98.

Answer: (E)

13. Label the centre of the larger circle O and the points of contact between the larger circle and the smaller circles A and B. Draw the radius OA of the larger circle.



Since the smaller circle and the larger circle touch at A, then the diameter through A of the smaller circle lies along the diameter through A of the larger circle. (This is because each diameter is perpendicular to the common tangent at the point of contact.)
Since AO is a radius of the larger circle, then it is a diameter of the smaller circle.
Since the radius of the larger circle is 6, then the diameter of the smaller circle is 6, so the radius of the smaller circle on the left is 3.
Similarly, we can draw a radius through O and B and deduce that the radius of the smaller circle on the right is also 3.
The area of the shaded region equals the area of the larger circle minus the combined area of the two smaller circles.
Thus, the area of the shaded region is $6^2\pi - 3^2\pi - 3^2\pi = 36\pi - 9\pi - 9\pi = 18\pi$.

Answer: (D)

14. Since b is a positive integer, then $b^2 \geq 1$, and so $a^2 \leq 49$, which gives $1 \leq a \leq 7$, since a is a positive integer.
If $a = 7$, then $b^2 = 50 - 7^2 = 1$, so $b = 1$.
If $a = 6$, then $b^2 = 50 - 6^2 = 14$, which is not possible since b is an integer.
If $a = 5$, then $b^2 = 50 - 5^2 = 25$, so $b = 5$.
If $a = 4$, then $b^2 = 50 - 4^2 = 34$, which is not possible.
If $a = 3$, then $b^2 = 50 - 3^2 = 41$, which is not possible.
If $a = 2$, then $b^2 = 50 - 2^2 = 46$, which is not possible.
If $a = 1$, then $b^2 = 50 - 1^2 = 49$, so $b = 7$.
Therefore, there are 3 pairs (a, b) that satisfy the equation, namely $(7, 1), (5, 5), (1, 7)$.

Answer: (C)

15. Since the coins in the bag of loonies are worth \$400, then there are 400 coins in the bag.
Since 1 loonie has the same mass as 4 dimes, then 400 loonies have the same mass as 4(400) or 1600 dimes.
Therefore, the bag of dimes contains 1600 dimes, and so the coins in this bag are worth \$160.

ANSWER: (C)

16. The sum of the odd numbers from 5 to 21 is

$$5+7+9+11+13+15+17+19+21=117$$

Therefore, the sum of the numbers in any row is one-third of this total, or 39.
This means as well that the sum of the numbers in any column or diagonal is also 39.
Since the numbers in the middle row add to 39, then the number in the centre square is $39-9-17=13$.
Since the numbers in the middle column add to 39, then the number in the middle square in the bottom row is $39-5-13=21$.

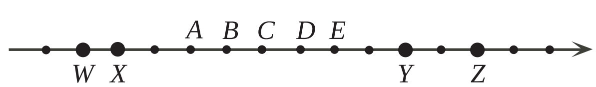
	5	
9	13	17
x	21	

Since the numbers in the bottom row add to 39, then the number in the bottom right square is $39-21-x=18-x$.
Since the numbers in the bottom left to top right diagonal add to 39, then the number in the top right square is $39-13-x=26-x$.
Since the numbers in the rightmost column add to 39, then $(26-x)+17+(18-x)=39$ or $61-2x=39$ or $2x=22$, and so $x=11$.
We can complete the magic square as follows:

19	5	15
9	13	17
11	21	7

ANSWER: (B)

17. We label the points corresponding to the four larger dots as W, X, Y, and Z.



Two of these four points must be multiples of 5.
These multiples of 5 must differ from each other by a multiple of 5.
We look at the possible differences between the four numbers W, X, Y, and Z:

$$X-W=1,\ Y-W=9,\ Y-X=8,\ Z-W=11,\ Z-X=10,\ Z-Y=2$$

Only X and Z differ by a multiple of 5. Thus, X and Z must be the two multiples of 5.
We know that one of A, B, C, D, E is a multiple of 5, and so it must differ from X by a multiple of 5.
Since $D - X = 5$, then D is the only multiple of 5 among A, B, C, D, E (the others differ from X by 2, 3, 4, and 6).
Since D is the only possibility that is a multiple of 5, then D must be the multiple of 15.
(We can check that X and Z are multiples of 5 and W and Y are multiples of 3, which means that D is also a multiple of 3.)

Answer: (D)

18. Since $\triangle PQR$ is isosceles, then $\angle PRQ = \angle PQR = 2x^\circ$.
Since $\angle PRQ$ and $\angle SRT$ are opposite angles, then $\angle SRT = \angle PRQ = 2x^\circ$.
Since $\triangle RST$ is isosceles with $RS = RT$, then

$$\angle RST = \tfrac{1}{2}(180^\circ - \angle SRT) = \tfrac{1}{2}(180^\circ - 2x^\circ) = 90^\circ - x^\circ = (90 - x)^\circ$$

Answer: (E)

19. We write a general three-digit positive integer in terms of its digits as ABC.
There are 9 possible values for the digit A (the digits 1 to 9) and 10 possible values for each of B and C (the digits 0 to 9).
We want to count the number of such integers with exactly one even digit. We consider the three cases separately.
Suppose that A is even. In this case, B and C are odd.
There are 4 possible values of A (2, 4, 6, 8) and 5 possible values for each of B and C (1, 3, 5, 7, 9). This means that there are $4 \times 5 \times 5 = 100$ integers in this case.
Suppose that B is even. In this case, A and C are odd.
There are 5 possible values of B (0, 2, 4, 6, 8) and 5 possible values for each of A and C (1, 3, 5, 7, 9). This means that there are $5 \times 5 \times 5 = 125$ integers in this case.
Suppose that C is even. In this case, A and B are odd.
There are 5 possible values of C (0, 2, 4, 6, 8) and 5 possible values for each of A and B (1, 3, 5, 7, 9). This means that there are $5 \times 5 \times 5 = 125$ integers in this case. (This case is very similar to the second case.)
Therefore, there are $100 + 125 + 125 = 350$ such integers in total.

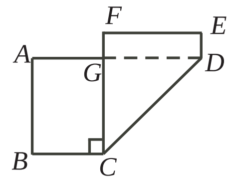
Answer: (A)

20. *Solution 1*
Note that $n^{200} = (n^2)^{100}$ and $3^{500} = (3^5)^{100}$.
Since n is a positive integer, then $n^{200} < 3^{500}$ is equivalent to $n^2 < 3^5 = 243$.
Note that $15^2 = 225$, $16^2 = 256$ and if $n \geq 16$, then $n^2 \geq 256$.
Therefore, the largest possible value of n is 15.

Solution 2
Since n is a positive integer and $500 = 200(2.5)$, then $n^{200} < 3^{500}$ is equivalent to $n^{200} < (3^{2.5})^{200}$, which is equivalent to $n < 3^{2.5} = 3^2 3^{0.5} = 9\sqrt{3}$.
Since $9\sqrt{3} \approx 15.59$ and n is an integer, the largest possible value of n is 15.

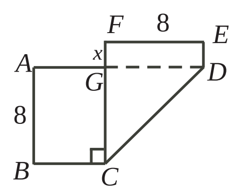
Answer: (C)

21. *Solution 1*
The area of the original piece of paper is $17(8) = 136\text{ cm}^2$.



When the paper is folded in this way, the portion of the original bottom face of the paper that is visible has the same area as the original portion of the top side of the paper to the right of the fold. (This is quadrilateral $CDEF$.)
Of the portion of the original sheet to the left of the fold, the part that is hidden (and thus not included in the area of the new figure) is the triangular portion under the folded part. (This is the section under $\triangle CDG$.) The hidden triangle is congruent to $\triangle CDG$.
Thus, the area of the portion of the original top face of the paper that is visible is the area to the left of the fold, minus the area of the hidden triangle.
Therefore, the area of the new figure equals the area of the original rectangle minus the area of $\triangle CDG$.
$\triangle CDG$ has height $GC = 8$ cm (the height of the rectangle), is right-angled (since the folded portion of the original bottom edge is perpendicular to the top and bottom edges), and has base $GD = 8$ cm.
Therefore, $\triangle CDG$ has area $\frac{1}{2}(8)(8) = 32$ cm^2.
In summary, the area of the new figure is $136 - 32 = 104$ cm^2.

Solution 2
We show the hidden portion of the original top edge and label the vertices of the new figure:



We want to determine the area of figure $ABCDEFG$.
Suppose that $FG = x$.
Since quadrilateral $AGCB$ has three right angles (at A, B and C), then it is a rectangle, and FC is perpendicular to AD.
Since quadrilateral $DEFG$ has three right angles (at E, F and G), then it is a rectangle.
We know that $FE = AB = 8$ since both are heights of the original rectangle.
Since $AGCB$ and $DEFG$ are rectangles, then $GC = AB = 8$ and $GD = FE = 8$.
Note that BC, CG and GF made up the original bottom edge of the rectangle.
Thus, $BC + CG + GF = 17$, and so $BC = 17 - 8 - x = 9 - x$.
The area of rectangle $AGCB$ is $8(9 - x) = 72 - 8x$.
The area of rectangle $DEFG$ is $8x$.
The area of $\triangle CGD$ (which is right-angled at G) is $\frac{1}{2}(8)(8) = 32$.
Therefore, the total area of figure $ABCDEFG$ is $(72 - 8x) + 8x + 32 = 104$ cm^2.

Answer: (A)

22. *Solution 1*
We label the terms $x_1, x_2, x_3, \ldots, x_{2009}, x_{2010}$.
Suppose that S is the sum of the odd-numbered terms in the sequence; that is,

$$S = x_1 + x_3 + x_5 + \cdots + x_{2007} + x_{2009}$$

We know that the sum of all of the terms is 5307; that is,

$$x_1 + x_2 + x_3 + \cdots + x_{2009} + x_{2010} = 5307$$

Next, we pair up the terms: each odd-numbered term with the following even-numbered term. That is, we pair the first term with the second, the third term with the fourth, and so on, until we pair the 2009th term with the 2010th term. There are 1005 such pairs.
In each pair, the even-numbered term is one bigger than the odd-numbered term. That is, $x_2 - x_1 = 1$, $x_4 - x_3 = 1$, and so on.
Therefore, the sum of the even-numbered terms is 1005 greater than the sum of the odd-numbered terms. Thus, the sum of the even-numbered terms is $S + 1005$.
Since the sum of all of the terms equals the sum of the odd-numbered terms plus the sum of the even-numbered terms, then $S + (S + 1005) = 5307$ or $2S = 4302$ or $S = 2151$.
Thus, the required sum is 2151.

Solution 2
Suppose that the first term is x.
Since each term after the first is 1 larger than the previous term, we can label the terms in the sequence as $x, x+1, x+2, \ldots, x+2008, x+2009$.
Since the sum of all 2010 terms is 5307, then

$$\begin{aligned} x + (x+1) + (x+2) + \cdots + (x+2008) + (x+2009) &= 5307 \\ 2010x + (1 + 2 + \cdots + 2008 + 2009) &= 5307 \\ 2010x + \tfrac{1}{2}(2009)(2010) &= 5307 \\ 2010x &= -2013738 \end{aligned}$$

When we add up every second term beginning with the first term, we obtain

$$\begin{aligned} & x + (x+2) + (x+4) + \cdots + (x+2006) + (x+2008) \\ &= 1005x + (2 + 4 + \cdots + 2006 + 2008) \\ &= 1005x + 2(1 + 2 + \cdots + 1003 + 1004) \\ &= 1005x + 2\left(\tfrac{1}{2}(1004)(1005)\right) \\ &= \tfrac{1}{2}(2010x) + (1004)(1005) \\ &= \tfrac{1}{2}(-2013738) + (1004)(1005) \\ &= 2151 \end{aligned}$$

Therefore, the required sum is 2151.

ANSWER: (C)

23. *Solution 1*
Connie gives 24 bars that account for 45% of the total weight to Brennan. Thus, each of these 24 bars accounts for an average of $\frac{45}{24}\% = \frac{15}{8}\% = 1.875\%$ of the total weight.
Connie gives 13 bars that account for 26% of the total weight to Maya. Thus, each of these 13 bars accounts for an average of $\frac{26}{13}\% = 2\%$ of the total weight.
Since each of the bars that she gives to Blair is heavier than each of the bars given to Brennan (which were the 24 lightest bars) and is lighter than each of the bars given to Maya (which were the 13 heaviest bars), then the average weight of the bars given to Blair must be larger than 1.875% and smaller than 2%.

Note that the bars given to Blair account for $100\% - 45\% - 26\% = 29\%$ of the total weight.
If there were 14 bars accounting for 29% of the total weight, the average weight would be $\frac{29}{14}\% \approx 2.07\%$, which is too large. Thus, there must be more than 14 bars accounting for 29% of the total weight.
If there were 15 bars accounting for 29% of the total weight, the average weight would be $\frac{29}{15}\% \approx 1.93\%$, which is in the correct range.
If there were 16 bars accounting for 29% of the total weight, the average weight would be $\frac{29}{16}\% \approx 1.81\%$, which is too small. The same would be true if there were 17 or 18 bars.
Therefore, Blair must have received 15 bars.

Solution 2
We may assume without loss of generality that the total weight of all of the bars is 100.
Therefore, the bars given to Brennan weigh 45 and the bars given to Maya weigh 26.
Suppose that Connie gives n bars to Blair.
These bars weigh $100 - 45 - 26 = 29$.
Let $b_1, b_2, \ldots, b_{24}$ be the weights of the 24 bars given to Brennan.
We may assume that $b_1 < b_2 < \cdots < b_{23} < b_{24}$, since the weights are all different.
Let $m_1, m_2, \ldots, m_{13}$ be the weights of the 13 bars given to Maya.
We may assume that $m_1 < m_2 < \cdots < m_{12} < m_{13}$, since the weights are all different.
Let $x_1, x_2, \ldots, x_n$ be the weights of the n bars given to Blair.
We may assume that $x_1 < x_2 < \cdots < x_{n-1} < x_n$, since the weights are all different.
Note that

$$b_1 < b_2 < \cdots < b_{23} < b_{24} < x_1 < x_2 < \cdots < x_{n-1} < x_n < m_1 < m_2 < \cdots < m_{12} < m_{13}$$

since the lightest bars are given to Brennan and the heaviest to Maya.
Also,

$$\begin{aligned} b_1 + b_2 + \cdots + b_{23} + b_{24} &= 45 \\ x_1 + x_2 + \cdots + x_{n-1} + x_n &= 29 \\ m_1 + m_2 + \cdots + m_{12} + m_{13} &= 26 \end{aligned}$$

Now b_{24} is the heaviest of the bars given to Brennan, so

$$45 = b_1 + b_2 + \cdots + b_{23} + b_{24} < b_{24} + b_{24} + \cdots + b_{24} + b_{24} = 24b_{24}$$

and so $b_{24} > \frac{45}{24} = \frac{15}{8}$.
Also, m_1 is the lightest of the bars given to Maya, so

$$26 = m_1 + m_2 + \cdots + m_{12} + m_{13} > m_1 + m_1 + \cdots + m_1 + m_1 = 13m_1$$

and so $m_1 < \frac{26}{13} = 2$.
But each of the n bars given to Blair is heavier than b_{24} and each is lighter than m_1.
Thus, $nb_{24} < x_1 + x_2 + \cdots + x_{n-1} + x_n < nm_1$ or $nb_{24} < 29 < nm_1$.
Thus, $\frac{15}{8}n < nb_{24} < 29 < nm_1 < 2n$ and so $n < 29\left(\frac{8}{15}\right) = \frac{232}{15} = 15\frac{7}{15}$ and $n > \frac{29}{2} = 14\frac{1}{2}$.
Since n is an integer, then $n = 15$, so Blair receives 15 bars.

ANSWER: (B)

24. Since the grid is a 5 by 5 grid of squares and each square has side length 10 cm, then the whole grid is 50 cm by 50 cm.

Since the diameter of the coin is 8 cm, then the radius of the coin is 4 cm.
We consider where the centre of the coin lands when the coin is tossed, since the location of the centre determines the position of the coin.
Since the coin lands so that no part of it is off of the grid, then the centre of the coin must land at least 4 cm (1 radius) away from each of the outer edges of the grid.
This means that the centre of the coin lands anywhere in the region extending from 4 cm from the left edge to 4 cm from the right edge (a width of $50 - 4 - 4 = 42$ cm) and from 4 cm from the top edge to 4 cm to the bottom edge (a height of $50 - 4 - 4 = 42$ cm).
Thus, the centre of the coin must land in a square that is 42 cm by 42 cm in order to land so that no part of the coin is off the grid.
Therefore, the total admissible area in which the centre can land is $42 \times 42 = 1764$ cm^2.
Consider one of the 25 squares. For the coin to lie completely inside the square, its centre must land at least 4 cm from each edge of the square.
As above, it must land in a region of width $10 - 4 - 4 = 2$ cm and of height $10 - 4 - 4 = 2$ cm.
There are 25 possible such regions (one for each square) so the area in which the centre of the coin can land to create a winning position is $25 \times 2 \times 2 = 100$ cm^2.
Thus, the probability that the coin lands in a winning position is equal to the area of the region in which the centre lands giving a winning position, divided by the area of the region in which the coin may land, or $\frac{100}{1764} = \frac{25}{441}$.

Answer: (A)

25. First, we define $u(n)$ to be the units digit of the positive integer n (for example, $u(25) = 5$).
Next, we make three important notes:

- It is the final position on the circle that we are seeking, not the total number of times travelled around the circle. Therefore, moving 25 steps, for example, around the circle is equivalent to moving 5 steps around the circle because in both cases the counter ends up in the same position. Since 10 steps gives one complete trip around the circle, then we only care about the units digit of the number of steps (that is, $u(n^n)$).
- To determine the final position, we want to determine the sum of the number of steps for each of the 1234 moves; that is, we want to determine

$$S = 1^1 + 2^2 + \cdots + 1233^{1233} + 1234^{1234}$$

since to calculate the position after a move we add the number of steps to the previous position. In fact, as above in the first note, we only care about the units digit of the sum of the numbers of steps $u(S)$, which is equal to

$$u\Big(u(1^1) + u(2^2) + \cdots + u(1233^{1233}) + u(1234^{1234})\Big)$$

- To calculate $u(n^n)$, we only need to worry about the units digit of the base n, $u(n)$, and not the other digits of the base. In other words, we need to determine the units digit of the product of n factors $u(n) \times u(n) \times \cdots \times u(n)$ – namely, $u\Big((u(n))^n\Big)$. (As we will see below, we cannot necessarily truncate the exponent n to its units digit.) To actually perform this calculation, we can always truncate to the units digit at each step because only the units digits affect the units digits.
For example, to calculate $u(13^{13})$, we can calculate $u\Big((u(13))^{13}\Big)$ which equals $u(3^{13})$. In

other words, we need to calculate $3 \times 3 \times \cdots \times 3$, which we can do by multiplying and truncating at each step if necessary:

$$3, 9, 27 \rightarrow 7, 21 \rightarrow 1, 3, 9, 27 \rightarrow 7, 21 \rightarrow 1, 3, 9, 27 \rightarrow 7, 21 \rightarrow 1, 3$$

Thus, $u(13^{13}) = 3$.

Next, we consider the different possible values of $u(n)$ and determine a pattern of the units digits of powers of n:

- If $u(n)$ is 0, 1, 5, or 6, then $u(n^k)$ is 0, 1, 5, or 6, respectively, for any positive integer k, because $u(n^2) = u(n)$ in each case, and so raising to higher powers does not affect the units digit.
- If $u(n) = 4$, then the units digits of powers of n alternate $4, 6, 4, 6$, and so on. (We can see by multiplying and truncating as above.)
- If $u(n) = 9$, then the units digits of powers of n alternate $9, 1, 9, 1$, and so on.
- If $u(n) = 2$, then the units digits of powers of n cycle as $2, 4, 8, 6, 2, 4, 8, 6$ and so on.
- If $u(n) = 8$, then the units digits of powers of n cycle as $8, 4, 2, 6, 8, 4, 2, 6$ and so on.
- If $u(n) = 3$, then the units digits of powers of n cycle as $3, 9, 7, 1, 3, 9, 7, 1$ and so on.
- If $u(n) = 7$, then the units digits of powers of n cycle as $7, 9, 3, 1, 7, 9, 3, 1$ and so on.

Next, we determine $u(n^n)$, based on $u(n)$:

- If $u(n)$ is 0, 1, 5, or 6, then $u(n^n)$ is 0, 1, 5, or 6, respectively.
- If $u(n) = 4$, then $u(n^n) = 6$, since the exponent is even so the units digit will be that occurring in even positions in the pattern.
- If $u(n) = 9$, then $u(n^n) = 9$, since the exponent is odd so the units digit will be that occurring in odd positions in the pattern.
- If $u(n) = 2$, then $u(n^n)$ will be either 4 or 6, depending on the exponent n, because the exponent is certainly even, but the pattern of units digits cycles with length 4. In particular, $u(2^2) = 4$ and $u(12^{12}) = 6$ (the exponent 12 is a multiple of 4, so the units digit appears at the end of a cycle).
- If $u(n) = 8$, then $u(n^n)$ will be either 4 or 6, depending on the exponent n, because the exponent is certainly even, but the pattern of units digits cycles with length 4. In particular, $u(8^8) = 6$ and $u(18^{18}) = 4$, using a similar argument to above.
- Similarly, $u(3^3) = 7$, $u(13^{13}) = 3$, $u(7^7) = 3$, and $u(17^{17}) = 7$.
- Since the units digits of the base n repeat in a cycle of length 10 and the units digits of the powers of n for a fixed n repeat every 1, 2 or 4 powers, then $u(n^n)$ repeats in a cycle of length 20 (the least common multiple of 10, 1, 2, and 4).

We can now determine $u(1) + u(2) + \cdots + u(19) + u(20)$ to be

$$u(1+4+7+6+5+6+3+6+9+0+1+6+3+6+5+6+7+4+9+0) = u(94) = 4$$

To calculate the total for 1234 steps, we note that 61 cycles of 20 bring us to 1220 steps. After 1220 steps, the units digit of the sum is $u(61 \cdot 4) = u(244) = 4$.

We then add the units digit of the sum of 14 more steps, starting at the beginning of the cycle, to obtain a final position equal to the units digit of

$$u(4 + (1 + 4 + 7 + 6 + 5 + 6 + 3 + 6 + 9 + 0 + 1 + 6 + 3 + 6)) = u(67) = 7$$

Therefore, the final position is 7.

Answer: (D)

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2009 Cayley Contest

(Grade 10)

Wednesday, February 18, 2009

Solutions

1. Calculating, $\dfrac{10^2-10}{9}=\dfrac{100-10}{9}=\dfrac{90}{9}=10$.

   Answer: (A)

2. On Saturday, Deepit worked 6 hours. On Sunday, he worked 4 hours.
   Therefore, he worked $6+4=10$ hours in total on Saturday and Sunday.

   Answer: (E)

3. Since $3(-2)=\nabla+2$, then $-6=\nabla+2$ so $\nabla=-6-2=-8$.

   Answer: (C)

4. Since $\sqrt{5+n}=7$ and $7=\sqrt{49}$, then $5+n=49$, so $n=44$.

   Answer: (D)

5. *Solution 1*
   Calculating, $3^2+4^2+12^2=9+16+144=25+144=169=13^2$.

   Solution 2
   From our work with the Pythagorean Theorem, we might remember that $3^2+4^2=5^2$.
   (This comes from the "3-4-5" right-angled triangle.)
   Thus, $3^2+4^2+12^2=5^2+12^2$.
   We might also remember, from the "5-12-13" triangle that $5^2+12^2=13^2$.
   Therefore, $3^2+4^2+12^2=5^2+12^2=13^2$.

   Answer: (A)

6. Since the shaded area is 20% of the area of the circle, then the central angle should be 20% of the total possible central angle.
   Thus, $x^\circ=\frac{20}{100}(360^\circ)$ or $x=\frac{1}{5}(360)=72$.

   Answer: (D)

7. Since the sum of the angles in any triangle is 180°, then looking at $\triangle QSR$, we have

   $$\angle SQR=180^\circ-\angle QSR-\angle SRQ=180^\circ-90^\circ-65^\circ=25^\circ$$

   Since $PQ=PR$, then $\angle PQR=\angle PRQ$.
   Thus, $x^\circ+25^\circ=65^\circ$ or $x+25=65$, and so $x=40$.

   Answer: (E)

8. According to the problem, we want to find the choice that works no matter what three consecutive positive integers we choose.
   Thus, we try $1(2)(3)$ to see if we can eliminate any of the given choices.
   Now, $1(2)(3)=6$. 6 is not a multiple of 4, 5 or 12, and is not odd.
   Therefore, only "a multiple of 6" is possible.
   (In fact, the product of three consecutive positive integers will always be a multiple of 6, because at least one of the three integers is even and one of the three must be a multiple of 3.)

   Answer: (B)

9. *Solution 1*
   Since there are 24 hours in a day, Francis spends $\frac{1}{3}\times 24=8$ hours sleeping.
   Also, he spends $\frac{1}{4}\times 24=6$ hours studying, and $\frac{1}{8}\times 24=3$ hours eating.
   The number of hours that he has left is $24-8-6-3=7$ hours.

   Solution 2
   Francis spends $\frac{1}{3}+\frac{1}{4}+\frac{1}{8}=\frac{8+6+3}{24}=\frac{17}{24}$ either sleeping, studying or eating.
   This leaves him $1-\frac{17}{24}=\frac{7}{24}$ of his day.
   Since there are 24 hours in a full day, then he has 7 hours left.

   Answer: (D)

10. *Solution 1*

Suppose that the height is h cm, the width is w cm, and the depth is d cm.
Using the given areas of the faces, $wh = 12$, $dw = 8$ and $dh = 6$.
Multiplying these together, we get $whdwdh = 12(8)(6)$ or $d^2h^2w^2 = 576$.
Thus, $(dhw)^2 = 576$, so $dhw = \sqrt{576} = 24$, since $dhw > 0$.
The volume is dhw cm^3, so the volume is 24 cm^3.

Solution 2

We try to find dimensions that work to give the three face areas that we know.
After some trial and error, we see that if the width is 4 cm, the height is 3 cm, and the depth is 2 cm, then the given face areas are correct.
Thus, the volume is 4 cm $\times$ 3 cm $\times$ 2 cm = 24 cm^3.

Solution 3

Suppose that the height is h cm, the width is w cm, and the depth is d cm.
Using the given areas of the faces, $wh = 12$, $dw = 8$ and $dh = 6$.
From the first equation, $w = \frac{12}{h}$.
Substituting into the second equation, we obtain $\frac{12d}{h} = 8$.
Multiplying this equation by the equation $dh = 6$, we obtain $12d^2 = 48$ which gives $d^2 = 4$ and so $d = 2$, since $d > 0$.
Since $dh = 6$, then $2h = 6$ so $h = 3$.
Since $dw = 8$, then $2w = 8$ so $w = 4$.
Thus, the volume is 4 cm $\times$ 3 cm $\times$ 2 cm = 24 cm^3.

ANSWER: (A)

11. To maximize the number of songs used, Gillian should use as many of the shortest length songs as possible. (This is because she can always trade a longer song for a shorter song and shorten the total time used.)
If Gillian uses all 50 songs of 3 minutes in length, this takes 150 minutes.
There are $180 - 150 = 30$ minutes left, so she can play an additional $30 \div 5 = 6$ songs that are 5 minutes in length.
In total, she plays $50 + 6 = 56$ songs.

ANSWER: (C)

12. Since there are 6 columns and each term in the sequence is 3 greater than the previous term, then the number in each box in the final column is 18 more than the number above it. (Each number in the final column is 6 terms further along in the sequence than the number above it.)
Thus, the number in the bottom right box should be $17 + 5(18) = 17 + 90 = 107$, since it is 5 rows below the 17.

ANSWER: (C)

13. The coin starts on square number 8, counting from left to right.
After the first roll, it moves 1 square to the left, since 1 is odd.
After the second roll, it moves 2 squares to the right, since 2 is even.
The coin continues to move, ending on square $8-1+2-3+4-5+6 = 11$, and so is 3 squares to the right of where it started.

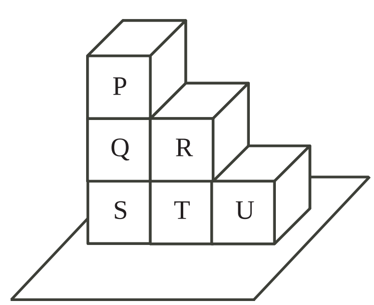
ANSWER: (E)

14. Of the integers from 3 to 20, the numbers 3, 5, 7, 11, 13, 17, and 19 are prime.
Thus, the integers 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, and 20 are composite.
The sum of three different composite numbers has to be at least $4 + 6 + 8 = 18$.
Thus, the smallest prime number that could be the sum of three different composite numbers is 19.
Can we write 19 as the sum of three different composite numbers? Yes, because $19 = 4+6+9$.
Answer: (D)

15. We write the list of five numbers in increasing order.
We know that the number 8 occurs at least twice in the list.
Since the median of the list is 9, then the middle number (that is, the third number) in the list is 9.
Thus, the list can be written as a, b, 9, d, e.
Since 8 occurs more than once and the middle number is 9, then 8 must occur twice only with $a = b = 8$.
Thus, the list can be written as 8, 8, 9, d, e.
Since the average is 10 and there are 5 numbers in the list, then the sum of the numbers in the list is $5(10) = 50$.
Therefore, $8 + 8 + 9 + d + e = 50$ or $25 + d + e = 50$ or $d + e = 25$.
Since 8 is the only integer that occurs more than once in the list, then $d > 9$.
Thus, $10 \leq d < e$ and $d + e = 25$.
To make e as large as possible, we make d as small as possible, so we make $d = 10$, and so $e = 15$.
The list 8, 8, 9, 10, 15 has the desired properties, so the largest possible integer that could appear in the list is 15.
Answer: (A)

16. Suppose that the side length of each of the small squares is x. (Since the 4 small squares have the same height, they must have the same side length.)
Then the side length of the largest square is $4x$.
Since the side length of each of the shaded squares is 10, then $QR = 3(10) = 30 = PS$.
Thus, $PS = 30 = 4x + x$ or $5x = 30$ or $x = 6$.
Therefore, the side length of the largest square is $4x = 24$.
Answer: (B)

17. Label the six dice as shown:



The maximum overall exposed sum occurs when the sum of the exposed faces on each die is maximized.
Die P has 5 exposed faces. The sum of these faces is a maximum when the 1 is hidden, so the

maximum exposed sum on die P is $2 + 3 + 4 + 5 + 6 = 20$.
Dice Q and S each have 3 exposed faces. Two of these are opposite to each other, so have a sum of 7. Thus, to maximize the exposed sum on these dice, we position them with the 6 as the unpaired exposed face. (This is on the left face of the stack.) Each of these dice has a maximum exposed sum of $6 + 7 = 13$.
Dice R and U each have 4 exposed faces. Two of these are opposite to each other, so have a sum of 7. Thus, to maximize the exposed sum on these dice, we position them with the 6 and the 5 as the unpaired exposed faces (on the top and right of the stack). Each of these dice have a maximum exposed sum of $5 + 6 + 7 = 18$.
Die T has 2 exposed faces, which are opposite each other, so have a sum of 7.
Therefore, the maximum possible sum of the exposed faces is $20 + 13 + 13 + 18 + 18 + 7 = 89$.

ANSWER: (C)

18. The slope of line segment QP is 1. Since the "rise" of QP is 6 units, then the "run" of QP should also be 6 units. Therefore, Q is 6 units horizontally to the left of P, and so has coordinates $(-5, 0)$.
The slope of line segment RP is 2. Since the rise of RP is 6 units, then the run of RP is $\frac{1}{2}(6) = 3$ units. Therefore, R is 3 units horizontally to the left of P, and so has coordinates $(-2, 0)$.
(We could have used the coordinates of P and the slopes of the lines to find that the equations of the lines are $y = x + 5$ and $y = 2x + 4$ and used them to find the coordinates of Q and R.)
Therefore, $QR = -2 - (-5) = 3$ and P is 6 units above the x-axis.
Thus, treating QR as the base of $\triangle PQR$, we find that its area is $\frac{1}{2}(3)(6) = 9$.

ANSWER: (B)

19. *Solution 1*
The product of the digits of n is 0 if at least one of the digits of n equals 0.
Consider the integers from 5000 to 5999, inclusive, which are each of the form $5xyz$.
How many of these *do not* include a 0?
If $5xyz$ does not include a 0, there are 9 possibilities for each of x, y and z (namely 1 to 9), and so there are $9^3 = 729$ such integers.
Therefore, $1000 - 729 = 271$ of the integers from 5000 to 5999 actually do include a 0.
We must also consider 6000, which includes a 0.
Therefore, there are $271 + 1 = 272$ integers n with $5000 \leq n \leq 6000$ with the property that the product of the digits of n is 0.

Solution 2
The product of the digits of n is 0 if at least one of the digits of n equals 0.
We carefully count the number of integers with 0 in each position, being careful not to "double count" any integers.
The integer $n = 6000$ includes a 0, so contributes 1 integer to the total.
There are 100 integers of the form $50xy$ (namely, 5000 to 5099).
There are 10 integers of the form $510y$ (namely, 5100 to 5109). Similarly, there are 10 integers of each of the forms $520y$, $530y$, and so on to $590y$. This gives 9 sets of 10 integers, or 90 integers more.
There are 9 integers of the form $51x0$ where x is not 0 (namely, 5110, 5120 and so on up to 5190). Similarly, there are 9 integers of each of the forms $52x0$, $53x0$, and so on. This gives 9 sets of 9 integers, or 81 integers more.
In total, there are thus $1 + 100 + 90 + 81 = 272$ such integers.

ANSWER: (D)

20. Let the length of his route be d km.
Since he arrives 1 minute early when travelling at 75 km/h and 1 minute late when travelling at 70 km/h, then the difference between these times is 2 minutes, or $\frac{1}{30}$ of an hour.
The time that his trip takes while travelling at 75 km/h is $\frac{d}{75}$ hours, and at 70 km/h is $\frac{d}{70}$ hours.
Therefore,

$$\begin{aligned}\frac{d}{70}-\frac{d}{75} &= \frac{1}{30}\\ 75d-70d &= \frac{75(70)}{30}\\ 5d &= 25(7)\\ d &= 35\end{aligned}$$

Therefore, the route is 35 km long.

Answer: (B)

21. We count the lattice points by starting with the point closest to PS and proceeding to just below QR.
For a lattice point (a,b) on the line $y=3x-5$ to be above PS, we need $b\geq 0$ or $3a-5\geq 0$ or $3a\geq 5$ or $a\geq \frac{5}{3}$.
Since a is an integer, then $a\geq 2$. ($a=2$ gives the point $(2,1)$.)
For a lattice point (a,b) on the line $y=3x-5$ to be below QR, we need $b\leq 2009$ or $3a-5\leq 2009$ or $3a\leq 2014$ or $a\leq\frac{2014}{3}$.
Since a is an integer, then $a\leq 671$. ($a=671$ gives the point $(671,2008)$.)
Thus, for a lattice point (a,b) to be on the line and inside the square, it must have $2\leq a\leq 671$.
In fact, every such integer a gives a lattice point, because it gives an integer value of $b=2a-5$.
The number of such integers a is $671-1=670$, so there are 670 such lattice points.

Answer: (E)

22. *Solution 1*
If we add the second and third equations, we obtain

$$\begin{aligned}ac+b+bc+a &= 18+6\\ c(a+b)+(a+b) &= 24\\ (c+1)(a+b) &= 24\end{aligned}$$

From the first given equation, $a+b=3$ and so we get $(c+1)(3)=24$ or $c+1=8$.
Thus, $c=7$.

Solution 2
From the first equation, $b=3-a$. (We could solve for a in terms of b instead.)
The second equation becomes $ac+(3-a)=18$ or $ac-a=15$.
The third equation becomes $(3-a)c+a=6$ or $-ac+3c+a=6$.
Adding these two equations, we obtain $3c=15+6=21$ and so $c=7$.

Answer: (E)

23. *Solution 1*
Suppose that Angela and Barry share 100 hectares of land. (We may assume any convenient total area, since this is a multiple choice problem.)

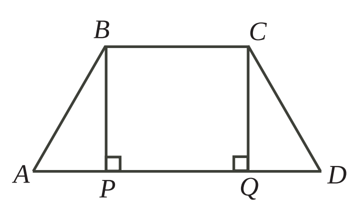
Since the ratio of the area of Angela's land to the area of Barry's land is $3:2$, then Angela has $\frac{3}{5}$ of the 100 hectares, or 60 hectares. Barry has the remaining 40 hectares.
Since the land is covered by corn and peas in the ratio $7:3$, then $\frac{7}{10}$ of the 100 hectares (that is, 70 hectares) is covered with corn and the remaining 30 hectares with peas.
On Angela's land, the ratio is $4:1$ so $\frac{4}{5}$ of her 60 hectares, or 48 hectares, is covered with corn and the remaining 12 hectares with peas.
Since there are 70 hectares of corn in total, then Barry has $70 - 48 = 22$ hectares of corn.
Since there are 30 hectares of peas in total, then Barry has $30 - 12 = 18$ hectares of peas.
Therefore, the ratio of corn to peas on Barry's land is $22:18 = 11:9$.

Solution 2
Suppose that the total combined area of land is x.
Since the ratio of the area of Angela's land to the area of Barry's land is $3:2$, then Angela has $\frac{3}{5}$ of the land, or $\frac{3}{5}x$, while Barry has the remaining $\frac{2}{5}x$.
Since the land is covered by corn and peas in the ratio $7:3$, then $\frac{7}{10}x$ is covered with corn and the remaining $\frac{3}{10}x$ with peas.
On Angela's land, the ratio is $4:1$ so $\frac{4}{5}$ of her $\frac{3}{5}x$, or $\frac{4}{5}(\frac{3}{5}x) = \frac{12}{25}x$, is covered with corn and the remaining $\frac{3}{5}x - \frac{12}{25}x = \frac{3}{25}x$ with peas.
Since the area of corn is $\frac{7}{10}x$ in total, then Barry's area of corn is $\frac{7}{10}x - \frac{12}{25}x = \frac{35}{50}x - \frac{24}{50}x = \frac{11}{50}x$.
Since the area of peas is $\frac{3}{10}x$ in total, then Barry's area of peas is $\frac{3}{10}x - \frac{3}{25}x = \frac{15}{50}x - \frac{6}{50}x = \frac{9}{50}x$.
Therefore, the ratio of corn to peas on Barry's land is $\frac{11}{50}x : \frac{9}{50}x = 11:9$.

ANSWER: (A)

24. We first note that the given quadrilateral is a trapezoid, because $60° + 120° = 180°$, and so the top and bottom sides are parallel.
We need to determine the total area of the trapezoid and then what fraction of that area is closest to the longest side.

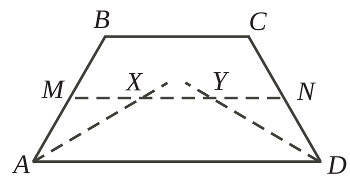
<u>Area of trapezoid</u>
Label the trapezoid as $ABCD$ and drop perpendiculars from B and C to P and Q on AD.



Since $\triangle ABP$ is right-angled at P and $\angle BAP = 60°$, then $AP = 100\cos(60°) = 100(\frac{1}{2}) = 50$ m and $BP = 100\sin(60°) = 100(\frac{\sqrt{3}}{2}) = 50\sqrt{3}$ m. (We could have used the ratios in a 30°-60°-90° triangle to do these calculations.)
By symmetry, $QD = 50$ m as well.
Also, since BC is parallel to PQ, and BP and CQ are perpendicular to PQ, then $BPQC$ is a rectangle, so $PQ = BC = 100$ m.
Thus, the area of trapezoid $ABCD$ is $\frac{1}{2}(BC + AD)(BP) = \frac{1}{2}(100 + (50 + 100 + 50))(50\sqrt{3})$ or $7500\sqrt{3}$ square metres.

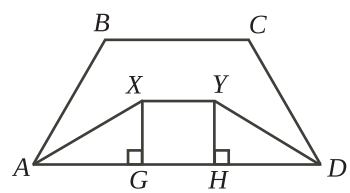
<u>Determination of region closest to AD</u>
Next, we need to determine what region of the trapezoid is closest to side AD.

To be closest to side AD, a point inside the trapezoid must be closer to AD than to each of BC, AB, and DC.
For a point in the trapezoid to be closer to AD than to BC, it must be below the "half-way mark", which is the midsegment MN.
Thus, such a point must be below the parallel line that is $\frac{1}{2}(50\sqrt{3}) = 25\sqrt{3}$ m above AD.
For a point in the trapezoid to be closer to AD than to AB, it must be below the angle bisector of $\angle BAD$. (See the end of this solution for a justification of this.)
Similarly, for a point in the trapezoid to be closer to AD than to DC, it must be below the angle bisector of $\angle CDA$.
Define points X and Y to be the points of intersection between the angle bisectors of $\angle BAD$ and $\angle CDA$, respectively, with the midsegment MN. We will confirm later in the solution that the angle bisectors intersect above MN, not below MN.



Area of trapezoid $AXYD$

Lastly, we need to determine the area of trapezoid $AXYD$.
Note that $\angle XAD = \angle YDA = \frac{1}{2}(60^\circ) = 30^\circ$.
Drop perpendiculars from X and Y to G and H, respectively, on AD.



We know that $AD = 200$ m and $XG = YH = 25\sqrt{3}$ m.
Since each of $\triangle AXG$ and $\triangle DYH$ is a 30°-60°-90° triangle, then

$$AG = DH = \sqrt{3}XG = \sqrt{3}(25\sqrt{3}) = 75$$

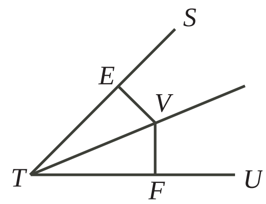
This tells us that the angle bisectors must intersect above MN, since $AG + HD = 150$ and $AD = 200$, so $AG + HD < AD$.
Since $XGHY$ is a rectangle (by similar reasoning as for $BPQC$), then

$$XY = GH = AD - AG - DH = 200 - 75 - 75 = 50$$

Therefore, the area of trapezoid $AXYD$ is $\frac{1}{2}(AD + XY)(XG) = \frac{1}{2}(50 + 200)(25\sqrt{3})$ or $3125\sqrt{3}$ square metres.
This tells us that the fraction of the crop that is brought to AD is $\frac{3125\sqrt{3}}{7500\sqrt{3}} = \frac{25}{60} = \frac{5}{12}$.

Property of angle bisectors

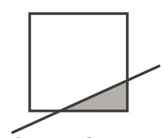
We must still justify the fact about angle bisectors above.
Consider $\angle STU$ and point V on its angle bisector.
Drop perpendiculars from V to E and F on ST and UT, respectively.



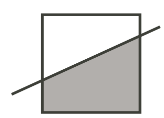
Now $\triangle VET$ is congruent to $\triangle VFT$ since each is right-angled, the two have equal angles at T, and the two share a common hypotenuse TV.
This tells us that $EV = FV$; that is, V is the same distance from ST and from UT.
This also tells us that any point on the angle bisector will be equidistant from ST and UT.
We can also deduce from this that any point below the angle bisector will be closer to UT, as moving from a point on the angle bisector to such a point moves closer to UT and further from ST.

Answer: (B)

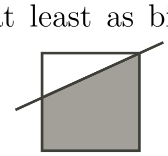
25. We add coordinates to the diagram, with the bottom left corner at $(0,0)$, the bottom right at $(m,0)$, the top right at (m,n), and the top left at $(0,n)$.
Thus, the slope of the diagonal is $\dfrac{n}{m}$.
This tells us that the equation of the diagonal is $y = \dfrac{n}{m}x$.
Since $2n < m < 3n$, then $\dfrac{1}{3} < \dfrac{n}{m} < \dfrac{1}{2}$; that is, the slope is between $\frac{1}{3}$ and $\frac{1}{2}$.
There are three possible configurations of shading in these partially shaded squares:

- A small triangle is shaded, while the rest is unshaded: 

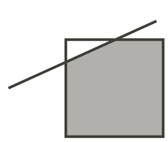
  Here, the maximum possible length of the base is 1 and so the maximum possible height is when the slope is as large as possible, so is $\frac{1}{2}$.
  Thus, in this case, the maximum shaded area is $\frac{1}{2}(1)(\frac{1}{2}) = \frac{1}{4}$.
  Since we want a shaded area of more than 0.999, then this is not the case we need.

- A trapezoid is shaded and a trapezoid is unshaded: 

  (Note that since the slope is less than 1, then the case  is not possible.)

  Consider the unshaded area. For the shaded area to be more than 0.999, the unshaded area is less than 0.001.
  But the unshaded trapezoid is at least as big as the triangle that is cut off when the diagonal passes through a vertex: 

  Such a triangle has base 1 and height at least $\frac{1}{3}$ (since the slope is at least $\frac{1}{3}$).
  Thus, the area of such a trapezoid is at least $\frac{1}{2}(1)(\frac{1}{3}) = \frac{1}{6}$ and so cannot be less than 0.001.

- A triangle is unshaded: 

It is this last case upon which we need to focus.
Suppose that the coordinates of the top left corner of such a unit square are (p,q).
The point where the diagonal ($y = \frac{n}{m}x$) crosses the top edge of the square ($y = q$) has coordinates $\left(\frac{m}{n}q, q\right)$, since if $y = q$, then $q = \frac{n}{m}x$ gives $x = \frac{m}{n}q$.
Similarly, the point where the diagonal crosses the left edge ($x = p$) of the square has coordinates $\left(p, \frac{n}{m}p\right)$.
Thus, the triangle has (horizontal) base of length $\frac{m}{n}q - p$ and (vertical) height of length $q - \frac{n}{m}p$.
We also know that neither the base nor the height is 0, since there is some unshaded area.
Since the area of the unshaded triangle is less than 0.001, then

$$\begin{array}{rclcll} 0 & < & \frac{1}{2}\left(\frac{m}{n}q - p\right)\left(q - \frac{n}{m}p\right) & < & 0.001 & \\ 0 & < & \left(\frac{m}{n}q - p\right)\left(q - \frac{n}{m}p\right) & < & 0.002 & \\ 0 & < & (mq - pn)(mq - pn) & < & 0.002mn & \quad \text{(multiplying by } mn\text{)} \\ 0 & < & 500(mq - pn)^2 & < & mn & \end{array}$$

Now m, n, p and q are integers and $mq - pn$ is not zero. In fact, $mq - pn = n\left(\frac{m}{n}q - p\right) > 0$.
Thus, $(mq - pn)^2 \geq 1$ because m, q, p, n are all integers and $(mq - pn)^2 > 0$.
Thus, $mn > 500(1) = 500$.
Note that if $(mq - pn)^2 > 1$, then mn would be much bigger.
So, since we want the smallest value of mn, we try to see if we can find a solution with $(mq - pn)^2 = 1$.

So we need to try to find m and n with $2n < m < 3n$, with the product mn as close to 500 as possible, and so that we can also find p and q with $mq - pn = 1$.
We consider the restriction that $2n < m < 3n$ first and look at the integers from 501 to 510 to see if we can find m and n with a product equal to one of these numbers and with $2n < m < 3n$.

- $501 = 3(167)$ and 167 is prime, so this is not possible
- $502 = 2(251)$ and 251 is prime, so this is not possible
- 503 is prime, so this is not possible
- $504 = 8(7)(9)$ so we can choose $n = 14$ and $m = 36$ (this is the only such way)
- $505 = 5(101)$ and 101 is prime, so this is not possible
- $506 = 11(2)(23)$ which cannot be written in this way
- $507 = 3(13)(13)$ which cannot be written in this way
- $508 = 4(127)$ and 127 is prime, so this is not possible
- 509 which is prime, so this is not possible
- $510 = 2(3)(5)(17)$, so we can choose $n = 15$ and $m = 34$ (this is only such way)

This gives two possible pairs m and n to consider so far. If one of them works, then this pair will give the smallest possible value of mn.
In order to verify if one of these works, we do need to determine if we can find an appropriate p and q.
Consider $n = 14$ and $m = 36$. In this case, we want to find integers p and q with $36q - 14p = 1$.

This is not possible since the left side is even and the right side is odd.
Consider $n = 15$ and $m = 34$. In this case, we want to find integers p and q with $34q - 15p = 1$.
The integers $q = 4$ and $p = 9$ satisfy this equation.
Therefore, $(m, n) = (34, 15)$ is a pair with the smallest possible value of mn that satisfies the given conditions, and so $mn = 510$.

ANSWER: (C)

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2008 Cayley Contest

(Grade 10)

Tuesday, February 19, 2008

Solutions

1. Calculating, $3^2 - 2^2 + 1^2 = 9 - 4 + 1 = 6$.

   Answer: (E)

2. Calculating, $\dfrac{\sqrt{25-16}}{\sqrt{25}-\sqrt{16}} = \dfrac{\sqrt{9}}{5-4} = \dfrac{3}{1} = 3$.

   Answer: (B)

3. In decimal form, the five possible answers are

   $$0.75, 1.2, 0.81, 1.333\ldots, 0.7$$

   The differences between 1 and these possiblities are

   $1 - 0.75 = 0.25 \qquad 1.2 - 1 = 0.2 \qquad 1 - 0.81 = 0.19 \qquad 1.333\ldots - 1 = 0.333\ldots \qquad 1 - 0.7 = 0.3$

   The possibility with the smallest difference with 1 is 0.81, so 0.81 is closest to 1.

   Answer: (C)

4. In total, there are $5 + 6 + 7 + 8 = 26$ jelly beans in the bag.
   Since there are 8 blue jelly beans, the probability of selecting a blue jelly bean is $\frac{8}{26} = \frac{4}{13}$.

   Answer: (D)

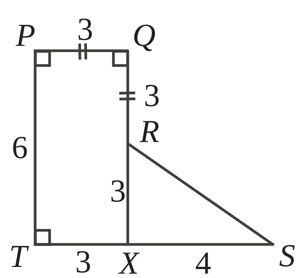
5. Since $52\,28\square$ is a multiple of 6, then it must be a multiple of 2 and a multiple of 3.
   Since it is a multiple of 2, the digit represented by $\square$ must be even.
   Since it is a multiple of 3, the sum of its digits is divisible by 3.
   The sum of its digits is $5 + 2 + 2 + 8 + \square = 17 + \square$.
   Since $\square$ is even, the possible sums of digits are 17, 19, 21, 23, 25 (for the possible values 0, 2, 4, 6, 8 for $\square$).
   Of these possibilities, only 21 is divisible by 3, so $\square$ must equal 4.
   We can check that 52 284 is divisible by 6.
   (An alternate approach would have been to use a calculator and test each of the five possible values for $\square$ by dividing the resulting values of $52\,28\square$ by 6.)

   Answer: (C)

6. Since $\dfrac{40}{x} - 1 = 19$, then $\dfrac{40}{x} = 20$.
   Thus, $x = 2$, since the number that we must divide 40 by to get 20 is 2.

   Answer: (D)

7. We extend QR to meet TS at X.
   Since $PQ = QR$, then $QR = 3$.
   Since $PQXT$ has three right angles, it must be a rectangle, so $TX = PQ = 3$.
   Also, $QX = PT = 6$.
   Since $TS = 7$ and $TX = 3$, then $XS = TS - TX = 7 - 3 = 4$.
   Since $QX = 6$ and $QR = 3$, then $RX = QX - QR = 6 - 3 = 3$.



Since $PQXT$ is a rectangle, then $\angle RXS = 90^\circ$.
By the Pythagorean Theorem in $\triangle RXS$,

$$RS^2 = RX^2 + XS^2 = 3^2 + 4^2 = 9 + 16 = 25$$

so $RS = 5$, since $RS > 0$.
Therefore, the perimeter is $PQ + QR + RS + ST + TP = 3 + 3 + 5 + 7 + 6 = 24$.

Answer: (A)

8. Since $PQ = QR$, then $\angle QPR = \angle QRP$.
Since $\angle PQR + \angle QPR + \angle QRP = 180^\circ$, then $40^\circ + 2(\angle QRP) = 180^\circ$, so $2(\angle QRP) = 140^\circ$ or $\angle QRP = 70^\circ$.
Since $\angle PRQ$ and $\angle SRT$ are opposite angles, then $\angle SRT = \angle PRQ = 70^\circ$.
Since $RS = RT$, then $\angle RST = \angle RTS = x^\circ$.
Since $\angle SRT + \angle RST + \angle RTS = 180^\circ$, then $70^\circ + 2x^\circ = 180^\circ$ or $2x = 110$ or $x = 55$.

Answer: (C)

9. *Solution 1*
Since a and b are both odd, then ab is odd.
Therefore, the largest even integer less than ab is $ab - 1$.
Since every other positive integer less than or equal to $ab - 1$ is even, then the number of even positive integers less than or equal to $ab - 1$ (thus, less than ab) is $\dfrac{ab-1}{2}$.

Solution 2
Since $a = 7$ and $b = 13$, then $ab = 91$.
The even positive integers less than $ab = 91$ are $2, 4, 6, \ldots, 90$.
There are $90 \div 2 = 45$ such integers.
Using $a = 7$ and $b = 13$, the five possible answers are

$$\frac{ab-1}{2} = 45 \qquad \frac{ab}{2} = \frac{91}{2} \qquad ab - 1 = 90 \qquad \frac{a+b}{4} = 5 \qquad (a-1)(b-1) = 72$$

Therefore, the answer must be $\dfrac{ab-1}{2}$.

Answer: (A)

10. For her 200 daytime minutes, Vivian is charged $200 \times \$0.10 = \$20$.
Since Vivian used 300 evening minutes and has 200 free evening minutes, then she is charged for $300 - 200 = 100$ of these minutes, and so is charged $100 \times \$0.05 = \$5$.
Her total bill is thus $\$20 + \$20 + \$5 = \$45$.

Answer: (C)

11. Lex has 265 cents in total.
Since a quarter is worth 25 cents, the total value in cents of the quarters that Lex has is a multiple of 25, and so must end in 00, 25, 50 or 75.
Since the remaining part of the 265 cents is made up of dimes only, the remaining part is a multiple of 10, so ends in a 0.
Thus, the value of the quarters must end with a 5, so ends with 25 or 75.
Since Lex has more quarters than dimes, we start by trying to determine the largest possible number of quarters that he could have.

The largest possible value of his quarters is thus 225 cents, which would be $225 \div 25 = 9$ quarters, leaving $265 - 225 = 40$ cents in dimes, or 4 dimes.
Thus, Lex has $9 + 4 = 13$ coins in total.
(The next possible largest value of his quarters is 175 cents, which would come from 7 quarters and so 90 cents in dimes or 9 dimes. This does not satisfy the condition that the number of quarters is larger than the number of dimes.)

Answer: (B)

12. *Solution 1*
Since $\angle OMP$ and $\angle GMH$ are opposite angles, then $\angle OMP = \angle GMH$.
The x-axis and the y-axis are perpendicular, so $\angle POM = 90^\circ$, so $\angle POM = \angle GHM$.
Since M is the midpoint of OH, then $OM = HM$.
Therefore, $\triangle POM$ and $\triangle GHM$ are congruent by Angle-Side-Angle.
Thus, $GH = OP = 4$.
Since GH is perpendicular to the x-axis, then the x-coordinate of G is 12, so G has coordinates $(12, 4)$.

Solution 2
Since GH is perpendicular to the x-axis, then the x-coordinate of G is 12, so G has coordinates $(12, g)$, for some value of g.
Since $OH = 12$ and M is the midpoint of OH, then $OM = \frac{1}{2}(12) = 6$, so M has coordinates $(6, 0)$.
To get from P to M, we move 6 units right and 4 units up.
To get from M to G, we also move 6 units right. Since P, M and G lie on a line, then we must also move 4 units up to get from M to G.
Therefore, the coordinates of G are $(12, 4)$.

Answer: (E)

13. In the given layout, the white face and the face containing the "U" are joined so that the "U" opens towards the edge joining these faces. Therefore, (A) cannot be correct as the "U" does not open towards the white face across the common edge.
The given layout shows that the white face and the grey face cannot be joined along along an edge when folded. Therefore, (C) cannot be correct.
The given layout also shows that the face containing the "U" and the face containing the "V" cannot be joined along along an edge when folded. Therefore, (D) cannot be correct.
In the given layout, the grey face and the face containing the "U" are joined so that the "U" opens away from the edge joining these faces. Therefore, (E) cannot be correct as the "U" does not open away from the grey face across the common edge.
Having eliminated the other possibilities, the correct answer must be (B). (We can check by visualizing the folding process that this cube can indeed be made.)

Answer: (B)

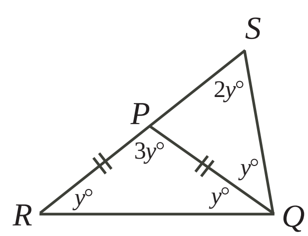
14. The third term is odd ($t = 5$), so the fourth term is $3(5) + 1 = 16$, which is even.
Thus, the fifth term is $\frac{1}{2}(16) = 8$, which is even.
Thus, the sixth term is $\frac{1}{2}(8) = 4$, which is even.
Thus, the seventh term is $\frac{1}{2}(4) = 2$, which is even.
Thus, the eighth term is $\frac{1}{2}(2) = 1$, which is odd.
Thus, the ninth term is $3(1) + 1 = 4$, which is even.
Thus, the tenth term is $\frac{1}{2}(4) = 2$.

Answer: (A)

15. First, we find the prime factors of 555.
Since 555 ends with a 5, it is divisible by 5, with $555 = 5 \times 111$.
Since the sum of the digits of 111 is 3, then 111 is divisible by 3, with $111 = 3 \times 37$.
Therefore, $555 = 3 \times 5 \times 37$, and each of 3, 5 and 37 is a prime number.
The possible ways to write 555 as the product of two integers are 1×555, 3×185, 5×111, and 15×37. (In each of these products, two or more of the prime factors have been combined to give a composite divisor.)
The only pair where both members are two-digit positive integers is 37 and 15, so $x + y$ is $37 + 15 = 52$.

ANSWER: (A)

16. *Solution 1*
Since RPS is a straight line, then $\angle SPQ = 180^\circ - \angle RPQ = 180^\circ - 3y^\circ$.
Using the angles in $\triangle PQS$, we have $\angle PQS + \angle QSP + \angle SPQ = 180^\circ$.
Thus, $x^\circ + 2y^\circ + (180^\circ - 3y^\circ) = 180^\circ$ or $x - y + 180 = 180$ or $x = y$.
(We could have instead looked at $\angle RPQ$ as being an external angle to $\triangle SPQ$.)
Since $x = y$, then $\angle RQS = 2y^\circ$.
Since $RP = PQ$, then $\angle PRQ = \angle PQR = x^\circ = y^\circ$.



Therefore, the angles of $\triangle RQS$ are y°, $2y^\circ$ and $2y^\circ$.
Thus, $y^\circ + 2y^\circ + 2y^\circ = 180^\circ$ or $5y = 180$ or $y = 36$.
Therefore, $\angle RPQ = 3y^\circ = 3(36)^\circ = 108^\circ$.

Solution 2
Since $RP = PQ$, then $\angle PRQ = \angle PQR = x^\circ$.
Looking at the sum of the angles in $\triangle RPQ$ and $\triangle RSQ$, we have $x^\circ + 3y^\circ + x^\circ = 180^\circ$ (or $2x + 3y = 180$) and $x^\circ + 2y^\circ + 2x^\circ = 180$ (or $3x + 2y = 180$).
Adding these two equations gives $5x + 5y = 360$ or $x + y = \frac{1}{5}(360) = 72$.
Thus, $2x + 2y = 2(72) = 144$, so $2x + 3y = 180$, gives $y = 180 - (2x + 2y) = 180 - 144 = 36$.
Therfore, $\angle RPQ = 3y^\circ = 3(36)^\circ = 108^\circ$.

ANSWER: (B)

17. The maximum possible value of $\dfrac{p}{q}$ is when p is as large as possible (that is, 10) and q is as small as possible (that is, 12). Thus, the maximum possible value of $\dfrac{p}{q}$ is $\dfrac{10}{12} = \dfrac{5}{6}$.
The minimum possible value of $\dfrac{p}{q}$ is when p is as small as possible (that is, 3) and q is as large as possible (that is, 21). Thus, the maximum possible value of $\dfrac{p}{q}$ is $\dfrac{3}{21} = \dfrac{1}{7}$.
The difference between these two values is $\dfrac{5}{6} - \dfrac{1}{7} = \dfrac{35}{42} - \dfrac{6}{42} = \dfrac{29}{42}$.

ANSWER: (A)

18. Suppose that there are x \$1 bills.
Thus, there are $(x+11)$ \$2 bills and $(x-18)$ \$3 bills.
Since the total value of the money is \$100, then

$$\begin{aligned} 1(x)+2(x+11)+3(x-18) &= 100 \\ x+2x+22+3x-54 &= 100 \\ 6x-32 &= 100 \\ 6x &= 132 \\ x &= 22 \end{aligned}$$

Therefore, there are 22 \$1 bills.

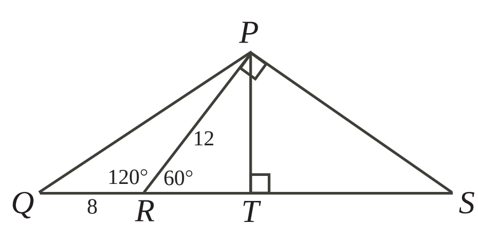
Answer: (C)

19. *Solution 1*
Since $\frac{2}{3}$ of the apples are rotten, $\frac{3}{4}$ of the pears are rotten, and the number of rotten apples and pears are equal, then we could try 6 rotten pieces of each type of fruit. (We choose 6 as it is a multiple of the numerator of each fraction.)
If there are 6 rotten apples, then the total number of apples is $\frac{3}{2}(6)=9$.
If there are 6 rotten pears, then the total number of pears is $\frac{4}{3}(6)=8$.
Therefore, there are $9+8=17$ pieces of fruit in total, of which $6+6=12$ are rotten.
Thus, $\frac{12}{17}$ of the fruit are rotten.

Solution 2
Suppose there are a apples and p pears in total.
Since the number of rotten apples and rotten pears are equal, then $\frac{2}{3}a=\frac{3}{4}p$, so $p=\frac{4}{3}(\frac{2}{3}a)=\frac{8}{9}a$.
Therefore, the total number of pieces of fruit is $a+p=a+\frac{8}{9}a=\frac{17}{9}a$.
Also, the total number of rotten fruit is $2(\frac{2}{3}a)=\frac{4}{3}a$, so the fraction of the total amount of fruit that is rotten is $\frac{\frac{4}{3}a}{\frac{17}{9}a}=\frac{4}{3}\cdot\frac{9}{17}=\frac{12}{17}$.

Answer: (D)

20. Since $\angle QRP=120^\circ$ and QRS is a straight line, then $\angle PRS=180^\circ-120^\circ=60^\circ$.
Since $\angle RPS=90^\circ$, then $\triangle SRP$ is a 30°-60°-90° triangle.
Therefore, $RS=2PR=2(12)=24$.
Drop a perpendicular from P to T on RS.



Since $\angle PRT=60^\circ$ and $\angle PTR=90^\circ$, then $\triangle PRT$ is also a 30°-60°-90° triangle.
Therefore, $PT=\frac{\sqrt{3}}{2}PR=6\sqrt{3}$.
Consider $\triangle QPS$. We may consider QS as its base with height PT.
Thus, its area is $\frac{1}{2}(6\sqrt{3})(8+24)=96\sqrt{3}$.

Answer: (E)

21. Since the radius of the inner circular pane is 20 cm, then its area is $\pi 20^2 = 400\pi$ cm^2.
Therefore, the area of each of the outer panes is also 400π cm^2, so the total area of the circular window is $9(400) = 3600\pi$ cm^2.
If the radius of the larger circle is R, then $\pi R^2 = 3600\pi$, so $R^2 = 3600$ or $R = 60$, since $R > 0$.
Since each of the outer lines can be extended to form a radius by joining its inner end to the centre using a radius of the inner circle, then the radius of the larger circle is $x + 20 = 60$, so $x = 40 = 40.0$, to the nearest tenth.
(It was not actually necessary to calculate the area of the circles. Since the large circle is formed from 9 pieces of equal area, its area is 9 times that of the inner circle. Thus, its radius is $\sqrt{9} = 3$ times that of the inner circle.)

ANSWER: (A)

22. There are 52 terms in the sum: the number 1, the number 11, and the 50 numbers starting with a 1, ending with a 1 and with 1 to 50 zeroes in between. The longest of these terms thus has 52 digits (50 zeroes and 2 ones).
When the units digits of all 52 terms are added up, their sum is 52, so the units digit of N is 2, and a 5 carried to the tens digit.
In the tens digit, there is only 1 non-zero digit: the 1 in the number 11. Therefore, using the carry, the tens digit of N is $1 + 5 = 6$.
In each of positions 3 to 52 from the right-hand end, there is only one non-zero digit, which is a 1.
Therefore, the digit in each of these positions in N is also a 1. (There is no carrying to worry about.)
Therefore, $N = 11 \cdots 1162$, where N has $52 - 2 = 50$ digits equal to 1.
This tells us that the sum of the digits of N is $50(1) + 6 + 2 = 58$.

ANSWER: (D)

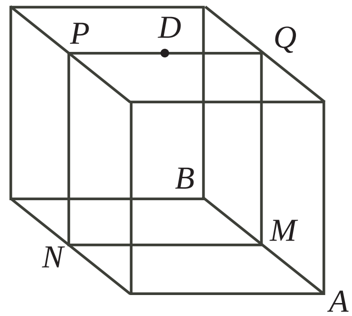
23. The number 4^3 equals 64.
To express 64 as a^b where a and b are integers, we can use 64^1, 8^2, 4^3, 2^6, $(-2)^6$, and $(-8)^2$.
We make a table to evaluate x and y:

$y-1$	$x+y$	y	x
64	1	65	-64
8	2	9	-7
4	3	5	-2
2	6	3	3
-2	6	-1	7
-8	2	-7	9

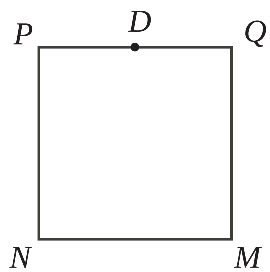
Therefore, there are 6 possible values for x.

ANSWER: (E)

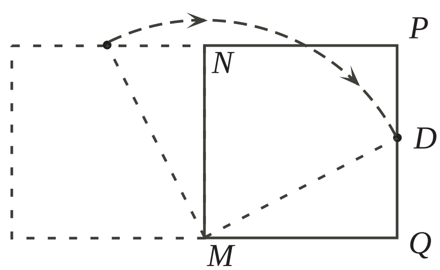
24. Suppose the cube rolls first over edge AB.
Consider the cube as being made up of two half-cubes (each of dimensions $1 \times 1 \times \frac{1}{2}$) glued together at square $PQMN$. (Note that $PQMN$ lies on a vertical plane.)
Since dot D is in the centre of the top face, then D lies on square $PQMN$.



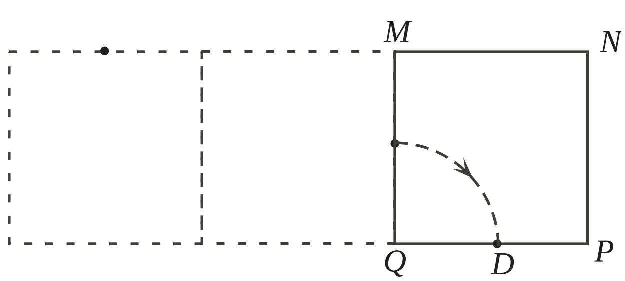
Since the cube always rolls in a direction perpendicular to AB, then the dot will always roll in the plane of square $PQMN$.



So we can convert the original three-dimensional problem to a two-dimensional problem of this square slice rolling.
Square $MNPQ$ has side length 1 and $DQ = \frac{1}{2}$, since D was in the centre of the top face.
By the Pythagorean Theorem, $MD^2 = DQ^2 + QM^2 = \frac{1}{4} + 1 = \frac{5}{4}$, so $MD = \frac{\sqrt{5}}{2}$ since $MD > 0$.
In the first segment of the roll, we start with NM on the table and roll, keeping M stationary, until Q lands on the table.



This is a rotation of 90° around M. Since D is at a constant distance of $\frac{\sqrt{5}}{2}$ from M, then D rotates along one-quarter (since 90° is $\frac{1}{4}$ of 360°) of a circle of radius $\frac{\sqrt{5}}{2}$, for a distance of $\frac{1}{4}(2\pi\frac{\sqrt{5}}{2}) = \frac{\sqrt{5}}{4}\pi$.
In the next segment of the roll, Q stays stationary and the square rolls until P touches the table.



Again, the roll is one of 90°. Note that $QD = \frac{1}{2}$. Thus, again D moves through one-quarter of a circle this time of radius $\frac{1}{2}$, for a distance of $\frac{1}{4}(2\pi\frac{1}{2}) = \frac{1}{4}\pi$.
Through the next segment of the roll, P stays stationary and the square rolls until N touches the table. This is similar to the second segment, so D rolls through a distance of $\frac{1}{4}\pi$.
Through the next segment of the roll, N stays stationary and the square rolls until M touches the table. This will be the end of the process as the square will end up in its initial position. This segment is similar to the first segment so D rolls through a distance of $\frac{\sqrt{5}}{4}\pi$.
Therefore, the total distance through which the dot travels is $\frac{\sqrt{5}}{4}\pi + \frac{1}{4}\pi + \frac{1}{4}\pi + \frac{\sqrt{5}}{4}\pi$ or $\left(\frac{1+\sqrt{5}}{2}\right)\pi$.

Answer: (E)

25. There are $7! = 7(6)(5)(4)(3)(2)(1)$ possible arrangements of the 7 numbers $\{1, 2, 3, 11, 12, 13, 14\}$.
To determine the average value of

$$(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-e)^2 + (e-f)^2 + (f-g)^2 \qquad (*)$$

we determine the sum of the values of this expression over all possible arrangements, and then divide by the number of arrangements.
Let x and y be two of these seven numbers.
In how many of these arrangements are x and y adjacent?
Treat x and y as a single unit $(x\ y)$ with 5 other numbers to be placed on either side of, but not between, $x\ y$.
This gives 6 "numbers" ($x\ y$ and 5 others) to arrange, which can be done in $6(5)(4)(3)(2)(1)$ or $6!$ ways.
But y could be followed by x, so there are $2(6!)$ arrangements with x and y adjacent, since there are the same number of arrangements with x followed by y as there are with y followed by x.
Therefore, when we add up the values of (*) over all possible arrangements, the term $(x-y)^2$ (which is equal to $(y-x)^2$) will occur $2(6!)$ times.
This is true for any pair x and y.
Therefore, the average value is $2(6!)$ times the sum of $(x-y)^2$ over all choices of x and y with $x < y$, divided by $7!$.
The sum of all possible values of $(x-y)^2$ is

$$\begin{array}{cccccccccccccc}
1^2 & + & 2^2 & + & 10^2 & + & 11^2 & + & 12^2 & + & 13^2 & & \\
 & + & 1^2 & + & 9^2 & + & 10^2 & + & 11^2 & + & 12^2 & & \\
 & & & + & 8^2 & + & 9^2 & + & 10^2 & + & 11^2 & & \\
 & & & & & + & 1^2 & + & 2^2 & + & 3^2 & & \\
 & & & & & & & + & 1^2 & + & 2^2 & & \\
 & & & & & & & & & + & 1^2 & = & 1372
\end{array}$$

(Here, we have paired 1 with each of the 6 larger numbers, then 2 with each of the 5 larger numbers, and so on. We only need to pair each number with all of the larger numbers because we have accounted for the reversed pairs in our method above.)
Therefore, the average value is $\dfrac{2(6!)(1372)}{7!} = \dfrac{2(1372)}{7} = 392$.

Answer: (D)

Canadian
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Competition
An activity of the Centre for Education
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2007 Cayley Contest

(Grade 10)

Tuesday, February 20, 2007

Solutions

1. Calculating, $8 + 2(3^2) = 8 + 2(9) = 8 + 18 = 26$.

   Answer: (A)

2. Calculating, $\dfrac{7+21}{14+42} = \dfrac{28}{56} = \dfrac{1}{2}$.

   Answer: (C)

3. If $3x - 2x + x = 3 - 2 + 1$, then $2x = 2$ or $x = 1$.

   Answer: (B)

4. In 3 hours, Leona earns \$24.75, so she makes $\$24.75 \div 3 = \$8.25$ per hour.
   Therefore, in a 5 hour shift, Leona earns $5 \times \$8.25 = \$41.25$.

   Answer: (E)

5. The value of $\frac{1}{4}$ of 100 is $\frac{1}{4} \times 100 = 25$.
   We evaluate each of the five given possibilities:

   (A): 20% of 200 is $0.2(200) = 40$
   (B): 10% of 250 is $0.1(250) = 25$
   (C): 15% of 100 is 15
   (D): 25% of 50 is $0.25(50) = 12.5$
   (E): 5% of 300 is $0.05(300) = 15$

   Therefore, the correct answer is (B).
   (Note that we did not have to evaluate (C), (D) and (E) after seeing that (B) was correct.)

   Answer: (B)

6. Evaluating each of the five given possibilities with $a = 2$ and $b = 5$,
   (A) $\dfrac{a}{b} = \dfrac{2}{5}$ (B) $\dfrac{b}{a} = \dfrac{5}{2}$ (C) $a - b = 2 - 5 = -3$ (D) $b - a = 5 - 2 = 3$ (E) $\frac{1}{2}a = \frac{1}{2}(2) = 1$
   The largest of these five is 3, or (D).

   Answer: (D)

7. The mean of 6, 9 and 18 is $\dfrac{6+9+18}{3} = \dfrac{33}{3} = 11$.
   Since the mean of 12 and y is thus 11, then the sum of 12 and y is $2(11) = 22$, so $y = 10$.

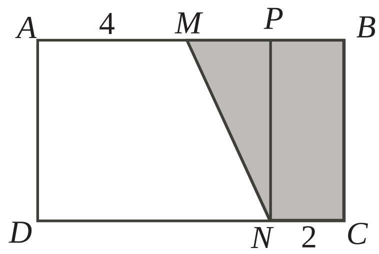
   Answer: (C)

8. In $\triangle ABC$, $\angle ABC = \angle BAC$, so $AC = BC$.
   In $\triangle BCD$, $\angle CBD = \angle CDB$, so $CD = BC$.
   Since the perimeter of $\triangle CBD$ is 19 and $BD = 7$, then $7 + BC + CD = 19$ or $2(BC) = 12$ or $BC = 6$.
   Since the perimeter of $\triangle ABC$ is 20, $BC = 6$, and $AC = BC$, then $AB + 6 + 6 = 20$ or $AB = 8$.

   Answer: (D)

9. *Solution 1*
   Since the area of rectangle $ABCD$ is 40 and $AB = 8$, then $BC = 5$.
   Therefore, $MBCN$ is a trapezoid with height 5 and parallel bases of lengths 4 and 2, so has area $\frac{1}{2}(5)(4 + 2) = 15$.

   Solution 2
   Since the area of rectangle $ABCD$ is 40 and $AB = 8$, then $BC = 5$.
   Draw a line from N to AB parallel to BC (and so perpendicular to AB) meeting AB at P.



This line divides $MBCN$ into a rectangle $PBCN$ of width 2 and height 5, and a triangle MPN with base MP of length 2 and height PN of height 5.
The area of $MBCN$ is sum of the areas of these two parts, or $2(5) + \frac{1}{2}(2)(5) = 10 + 5 = 15$.

Answer: (A)

10. *Solution 1*
To get from one term to the next, we double and add 4.
Since the first term is x, the second term is $2x + 4$.
Since the second term is $2x + 4$, the third term is $2(2x + 4) + 4 = 4x + 12$.
Since the third term equals 52, then $4x + 12 = 52$ so $4x = 40$ or $x = 10$.

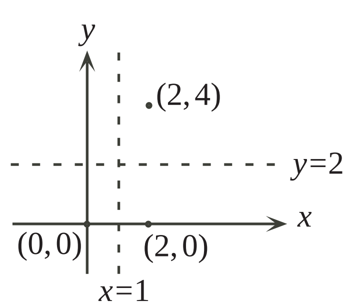
Solution 2
To get from one term to the next, we double and add 4.
Therefore, to get from one term to the previous term, we subtract 4 and divide by 2.
Since the third term is 52, then subtracting 4 gives 48 and dividing by 2 gives 24, which is the second term.
Since the second term is 24, then subtracting 4 gives 20 and dividing by 2 gives 10, which is the first term.
Thus, $x = 10$.

Answer: (D)

11. Suppose that Ivan ran a distance of x km on Monday.
Then on Tuesday, he ran $2x$ km, on Wednesday, he ran x km, on Thursday, he ran $\frac{1}{2}x$ km, and on Friday he ran x km.
The shortest of any of his runs was on Thursday, so $\frac{1}{2}x = 5$ or $x = 10$.
Therefore, his runs were 10 km, 20 km, 10 km, 5 km, and 10 km, for a total of 55 km.

Answer: (A)

12. When $(0, 0)$ is reflected in the line $x = 1$, the image is $(2, 0)$.



When $(2, 0)$ is reflected in the line $y = 2$, the image is $(2, 4)$.

Answer: (E)

13. *Solution 1*
Since the ratio $AN : AC$ equals the ratio $AP : PB$ (each is $1 : 2$) and $\angle A$ is common in $\triangle APN$ and $\triangle ABC$, then $\triangle APN$ is similar to $\triangle ABC$.
Since the ratio of side lengths between these two triangles is $1 : 2$, then the ratio of areas is $1 : 2^2 = 1 : 4$.
Thus, the area of $\triangle ABC$ is $4 \times 2 = 8$ cm^2.

Solution 2
Since the ratio $AN : AC$ equals the ratio $AP : PB$ (each is $1 : 2$) and $\angle A$ is common in $\triangle APN$ and $\triangle ABC$, then $\triangle APN$ is similar to $\triangle ABC$.
Therefore, $\angle ANP = \angle ACB = 90^\circ$.
Similarly, $\triangle PMB$ is similar to $\triangle ACB$, and $\angle PMB = \angle ACB = 90^\circ$.
Therefore, $AN = NC = PM = \frac{1}{2}AC$ and $NP = CM = MB = \frac{1}{2}CB$.
Thus, $\triangle PMB$ is congruent to $\triangle ANP$, so has area 2 cm^2.
Rectangle $NPMC$ has the same width and height as $\triangle ANP$, so has double the area of $\triangle APN$ or 4 cm^2.
Thus, the area of $\triangle ACB$ is $2 + 4 + 2 = 8$ cm^2.

Solution 3
Join C to P.
Since CP is the diagonal of the rectangle, then $\triangle CPN$ and $\triangle PCM$ are congruent, so have equal areas.
Since $AN = NC$ and $\triangle PNA$ and $\triangle PNC$ have equal height (PN), then the areas of $\triangle PNA$ and $\triangle PNC$ are equal.
Similarly, the areas of $\triangle PMC$ and $\triangle PMB$ are equal.
In summary, the areas of the four small triangles are equal.
Since the area of $\triangle APN$ is 2 cm^2, then the total area of $\triangle ABC$ is 8 cm^2.

Answer: (A)

14. Using a common denominator, $\dfrac{3}{x-3} + \dfrac{5}{2x-6} = \dfrac{6}{2x-6} + \dfrac{5}{2x-6} = \dfrac{11}{2x-6}$.
Therefore, $\dfrac{11}{2x-6} = \dfrac{11}{2}$, so $2x - 6 = 2$.

Answer: (A)

15. Since $\triangle ABC$ and $\triangle PQR$ are equilateral, then $\angle ABC = \angle ACB = \angle RPQ = 60^\circ$.
Therefore, $\angle YBP = 180^\circ - 65^\circ - 60^\circ = 55^\circ$ and $\angle YPB = 180^\circ - 75^\circ - 60^\circ = 45^\circ$.
In $\triangle BYP$, we have $\angle BYP = 180^\circ - \angle YBP - \angle YPB = 180^\circ - 55^\circ - 45^\circ = 80^\circ$.
Since $\angle XYC = \angle YBP$, then $\angle XYC = 80^\circ$.
In $\triangle CXY$, we have $\angle CXY = 180^\circ - 60^\circ - 80^\circ = 40^\circ$.

Answer: (C)

16. Since 60% of the 10 000 students are in Arts, 6000 students are in Arts, and so 4000 students are in Science.
Since 40% of the Science students are male, then $0.4(4000) = 1600$ of the Science students are male.
Since half of the total of 10 000 students are male, then $5000 - 1600 = 3400$ of the male students are in Arts.
Since there are 6000 students in Arts, then $6000 - 3400 = 2600$ of the Arts students are female, or $\dfrac{2600}{6000} \times 100\% \approx 43.33\%$ of the Arts students are female.

Answer: (E)

17. First, we note that no matter how many Heroes are present, all four would always reply "Hero" when asked "Are you a Hero or a Villain?". (This is because Heroes will tell the truth and answer "Hero" and Villains will lie and answer "Hero".)
When each is asked "Is the person on your right a Hero or a Villain?", all four reply "Villain", so any Hero that is at the table must have a Villain on his right (or he would have answered "Hero") and any Villain at the table must have a Hero on his right (or he would have had a Villain on his right and answered "Hero").
In other words, Heroes and Villains must alternate around the table, so there are 2 Heroes and 2 Villains.
(It is worth checking that when 2 Heroes and 2 Villains sit in alternate seats, the answers are indeed as claimed.)

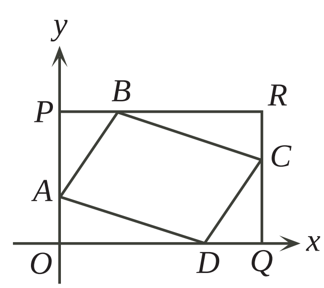
Answer: (C)

18. *Solution 1*
Suppose there are x balls in total in the bag.
Then there are $\frac{1}{3}x$ red balls and $\frac{2}{7}x$ blue balls.
This tells us that the number of green balls is $x - \frac{1}{3}x - \frac{2}{7}x = \frac{21}{21}x - \frac{7}{21}x - \frac{6}{21}x = \frac{8}{21}x$.
But we know that the number of green balls is $2 \times \frac{2}{7}x - 8$.
Thus, $\frac{8}{21}x = 2 \times \left(\frac{2}{7}x\right) - 8$ or $\frac{8}{21}x = \frac{12}{21}x - 8$ or $\frac{4}{21}x = 8$ or $x = 42$.
Since $x = 42$, the number of green balls is $\frac{8}{21}x = \frac{8}{21}(42) = 16$.

Solution 2
Suppose that there were 21 balls in the bag. (We choose 21 since there are fractions with denominator 3 and fractions with denominator 7 in the problem.)
Since $\frac{1}{3}$ of the balls are red, then 7 balls are red.
Since $\frac{2}{7}$ of the balls are blue, then 6 balls are red.
Thus, there are $21 - 7 - 6 = 8$ green balls in the bag. However, this is only 4 less than twice the number of blue balls, so there cannot be 21 balls in the bag.
To get from "4 less" to "8 less", we try doubling the number of balls in the bag.
If there are 42 balls in the bag, then 14 are red and 12 are blue, so 16 are green, which is 8 less than twice the number of blue balls.
Therefore, the number of green balls is 16.

Answer: (B)

19. We draw a horizontal line through B (meeting the y-axis at P) and a vertical line through C (meeting the x-axis at Q). Suppose the point of intersection of these two lines is R.



We know that P has coordinates $(0, 3)$ (since B has y-coordinate 3) and Q has coordinates $(5, 0)$ (since C has x-coordinate 5), so R has coordinates $(5, 3)$.
Using the given coordinates, $OA = 1$, $AP = 2$, $PB = 1$, $BR = 4$, $RC = 1$, $CQ = 2$, $QD = 1$,

and $DO = 4$.
The area of $ABCD$ equals the area of $PRQO$ minus the areas of triangles APB, BRC, CQD, and DOA.
$PRQO$ is a rectangle, so has area $3 \times 5 = 15$.
Triangles APB and CQD have bases PB and QD of length 1 and heights AP and CQ of length 2, so each has area $\frac{1}{2}(1)(2) = 1$.
Triangles BRC and DOA have bases BR and DO of length 4 and heights CR and AO of length 1, so each has area $\frac{1}{2}(4)(1) = 2$.
Thus, the area of $ABCD$ is $15 - 1 - 1 - 2 - 2 = 9$.
(Alternatively, we could notice that $ABCD$ is a parallelogram. Therefore, if we draw the diagonal AC, the area is split into two equal pieces. Dropping a perpendicular from C to Q on the x-axis produces a trapezoid $ACQO$ from which only two triangles need to be removed to determine half of the area of $ABCD$.)

ANSWER: (A)

20. Since $3(n^{2007}) < 3^{4015}$, then $n^{2007} < \dfrac{3^{4015}}{3} = 3^{4014}$.
But $3^{4014} = (3^2)^{2007} = 9^{2007}$ so we have $n^{2007} < 9^{2007}$.
Therefore, $n < 9$ and so the largest integer n that works is $n = 8$.

ANSWER: (D)

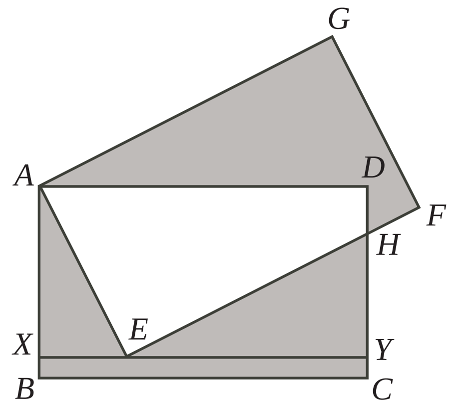
21. Since T has played 5 matches so far, then T has played P, Q, R, S, and W (ie. each of the other teams).
Since P has played only 1 match and has played T, then P has played no more matches.
Since S has played 4 matches and has not played P, then S must have played each of the remaining 4 teams (namely, Q, R, T, and W).
Since Q has played only 2 matches and has played T and S, then Q has played no more matches.
Since R has played 3 matches, has played T and S but has not played P or Q, then R must have played W as well.
Therefore, W has played T, S and R, or 3 matches in total. (Since we have considered all possible opponents for W, then W has played no more matches.)

ANSWER: (C)

22. Suppose the first integer in the list is n.
Then the remaining four integers are $n + 3$, $n + 6$, $n + 9$, and $n + 12$.
Since the fifth number is a multiple of the first, then $\dfrac{n+12}{n} = \dfrac{n}{n} + \dfrac{12}{n} = 1 + \dfrac{12}{n}$ is an integer.
Since $1 + \dfrac{12}{n}$ is an integer, then $\dfrac{12}{n}$ is an integer, or n is a positive divisor of 12.
The positive divisors of 12 are 1, 2, 3, 4, 6, and 12, so there are 6 possible values of n and so 6 different lists.
(We can check that each of 6 values of n produces a different list, each of which has the required property.)

ANSWER: (D)

23. Since the same region ($AEHD$) is unshaded inside each rectangle, then the two shaded regions have equal area, since the rectangles have equal area.
Thus, the total shaded area is twice the area of $AEHCB$.
Draw a horizontal line through E, meeting AB at X and HC at Y.



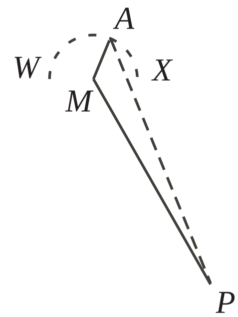
We know $\angle BAE = 30^\circ$ and $AE = 12$, so $EX = \frac{1}{2}AE = 6$ and $AX = \sqrt{3}EX = 6\sqrt{3}$, by the ratios in a 30°-60°-90° triangle.
Thus, the area of $\triangle EXA$ is $\frac{1}{2}(6)(6\sqrt{3}) = 18\sqrt{3}$.
Since $AB = 12$ and $AX = 6\sqrt{3}$, then $XB = 12 - 6\sqrt{3}$.
Therefore, rectangle $BXYC$ has height $12 - 6\sqrt{3}$ and width 18, so has area $18(12 - 6\sqrt{3})$ or $216 - 108\sqrt{3}$.
Since $XY = BC = 18$ and $EX = 6$, then $EY = 12$.
Since $\angle AEB = 60^\circ$ and $\angle AEH = 90^\circ$, then $\angle HEY = 30^\circ$.
Since $EY = 12$ and $\angle HEY = 30^\circ$, then $HY = \dfrac{12}{\sqrt{3}} = 4\sqrt{3}$.
Thus, $\triangle EYH$ has area $\frac{1}{2}(12)(4\sqrt{3}) = 24\sqrt{3}$.
So the total area of $AEHCB$ is $18\sqrt{3} + (216 - 108\sqrt{3}) + 24\sqrt{3} = 216 - 66\sqrt{3}$.
Therefore, the total shaded area is $2(216 - 66\sqrt{3}) = 432 - 132\sqrt{3} \approx 203.369$.

Answer: (C)

24. To form such a collection of integers, our strategy is to include some integers larger than 1 whose product is 2007 and then add enough 1s to the collection to make the sum 2007.
In order to make n as small as possible, we would like to include as few 1s as possible, and so make the sum of the initial integers (whose product is 2007) as large as possible.
Since we would like to consider integers whose product is 2007, we should find the divisors of 2007.
We see that $2007 = 3 \times 669 = 3 \times 3 \times 223$, and 223 is a prime number.
So the collections of positive integers larger than 1 whose product is 2007 are $\{3, 669\}$, $\{3, 3, 223\}$, and $\{9, 223\}$.
The collection with the largest sum is $\{3, 669\}$ whose sum is 672. To get a sum of 2007, we must add $2007 - 672 = 1335$ copies of 1, which means that we are using $1335 + 2 = 1337$ integers.
Therefore, the smallest value of n is 1337.

Answer: (B)

25. Since $\angle WAX = 90^\circ$ regardless of the position of square $ABCD$, then A always lies on the semi-circle with diameter WX.
The centre of this semi-circle is the midpoint, M, of WX.
To get from P to M, we must go up 4 units and to the left 3 units (since $WX = 2$), so $PM^2 = 3^2 + 4^2 = 25$ or $PM = 5$.
Since the semi-circle with diameter WX has diameter 2, it has radius 1, so $AM = 1$.
So we have $AM = 1$ and $MP = 5$.



Therefore, the maximum possible length of AP is $5 + 1 = 6$, when A, M, P lie on a straight line.

ANSWER: (E)

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2006 Cayley Contest

(Grade 10)

Wednesday, February 22, 2006

Solutions

1. Calculating, $\frac{1}{2} + \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$.

   Answer: (E)

2. Calculating, $(\sqrt{100} - \sqrt{36})^2 = (10 - 6)^2 = 4^2 = 16$.

   Answer: (A)

3. We determine the value of this expression, by calculating each difference first, so

$$43 - 41 + 39 - 37 + 35 - 33 + 31 - 29 = 2 + 2 + 2 + 2 = 8$$

   Answer: (A)

4. When $a = -3$ and $b = 2$, we have $a(b - 3) = (-3)(2 - 3) = (-3)(-1) = 3$.

   Answer: (C)

5. To determine the number by which we multiply the third term to obtain the fourth term, we can divide the second term by the first term. Doing this, we get $\dfrac{0.02}{0.001} = 20$.
   (We can check that this is also equal to the third term divided by the second term.)
   Therefore, the fourth term is $20(0.4) = 8$.

   Answer: (B)

6. The base of $\triangle ABC$ (that is, BC) has length 20.
   Since the area of $\triangle ABC$ is 240, then $240 = \frac{1}{2}bh = \frac{1}{2}(20)h = 10h$, so $h = 24$.
   Since the height of $\triangle ABC$ (from base BC) is 24, then the y-coordinate of A is 24.

   Answer: (D)

7. *Solution 1*
   Rewriting $\frac{3}{2}$ as $\frac{6}{4}$, we see that $\dfrac{6}{x+1} = \dfrac{6}{4}$ and so comparing denominators, $x + 1 = 4$ or $x = 3$.

   Solution 2
   Since $\dfrac{6}{x+1} = \dfrac{3}{2}$, then cross-multiplying, we obtain $2(6) = 3(x + 1)$ or $12 = 3x + 3$ or $3x = 9$ or $x = 3$.

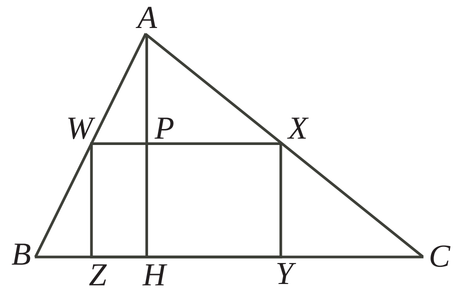
   Solution 3
   Inverting both sides of the equation, we get $\dfrac{x+1}{6} = \dfrac{2}{3}$ or $x + 1 = 6 \times \frac{2}{3} = 4$.
   Thus, $x = 3$.

   Answer: (C)

8. *Solution 1*
   Since $WXYZ$ is a rectangle, then $\angle ZWX = \angle WXY = 90^\circ$.
   Therefore, $\angle AWX = 180^\circ - \angle ZWX - \angle BWZ = 180^\circ - 90^\circ - 22^\circ = 68^\circ$.
   Also, $\angle AXW = 180^\circ - \angle WXY - \angle CXY = 180^\circ - 90^\circ - 65^\circ = 25^\circ$.
   So, looking at $\triangle AWX$, $\angle BAC = \angle WAX = 180^\circ - \angle AWX - \angle AXW = 180^\circ - 68^\circ - 25^\circ = 87^\circ$.

   Solution 2
   Since $WXYZ$ is a rectangle, then $\angle WZY = \angle XYZ = 90^\circ$, so $\triangle WZB$ and $\triangle XYC$ are both right-angled.
   Thus, $\angle WBZ = 90^\circ - \angle BWZ = 90^\circ - 22^\circ = 68^\circ$ and $\angle XCY = 90^\circ - \angle CXY = 90^\circ - 65^\circ = 25^\circ$.
   Looking at $\triangle ABC$, $\angle BAC = 180^\circ - \angle ABC - \angle ACB = 180^\circ - 68^\circ - 25^\circ = 87^\circ$.

Solution 3
Draw a perpendicular from A to H on BC.



Then WZ, AH and XY are all parallel since each is perpendicular to BC, so $\angle BAH = \angle BWZ = 22^\circ$ and $\angle CAH = \angle CXY = \angle 65^\circ$.
Therefore, $\angle BAC = 22^\circ + 65^\circ = 87^\circ$.

ANSWER: (A)

9. Since the perimeter of the triangle is 36, then $7 + (x + 4) + (2x + 1) = 36$ or $3x + 12 = 36$ or $3x = 24$ or $x = 8$.
Thus, the lengths of the three sides of the triangle are 7, $8+4 = 12$ and $2(8)+1 = 17$, of which the longest is 17.

ANSWER: (C)

10. From the given information, the total amount of marks obtained by the class is $20(80) + 8(90) + 2(100) = 2520$.
Therefore, the class average is $\dfrac{2520}{30} = 84$.

ANSWER: (B)

11. *Solution 1*
Since the side lengths of $\triangle DEF$ are 50% larger than the side lengths of $\triangle ABC$, then these new side lengths are $\frac{3}{2}(6) = 9$, $\frac{3}{2}(8) = 12$, and $\frac{3}{2}(10) = 15$.
We know that $\triangle DEF$ is right-angled, and this right angle must occur between the sides of length 9 and 12 (since it is opposite the longest side).
Therefore, the area of $\triangle DEF$ is $\frac{1}{2}(9)(12) = 54$.

Solution 2
The area of $\triangle ABC$ is $\frac{1}{2}(6)(8) = 24$.
Since the side lengths of $\triangle ABC$ are each multiplied by $\frac{3}{2}$ to get the side lengths in $\triangle DEF$, then the area of $\triangle DEF$ is $\left(\frac{3}{2}\right)^2$ times that of $\triangle ABC$.
Therefore, the area of $\triangle DEF$ is $\left(\frac{3}{2}\right)^2(24) = \frac{9}{4}(24) = 54$.

ANSWER: (E)

12. *Solution 1*
Since Jim drives from 7:45 p.m. to 9:30 p.m., then Jim drives for 1 hour and 45 minutes or $1\frac{3}{4}$ hours or $\frac{7}{4}$ hours.
Since Jim drives 84 km in $\frac{7}{4}$ hours at a constant speed, then this speed is $\dfrac{84}{\frac{7}{4}} = 84 \times \dfrac{4}{7} = 48$ km/h.

Solution 2
Since Jim drives from 7:45 p.m. to 9:30 p.m., then Jim drives for 1 hour and 45 minutes, which is the same as 7 quarters of an hour.
Since he drives 84 km in 7 quarters of an hour, he drives 12 km in 1 quarter of an hour, or 48 km in one hour, so his speed is 48 km/h.

ANSWER: (E)

13. Since $x + 1 = y - 8$ and $x = 2y$, then $2y + 1 = y - 8$ or $y = -9$.
Thus, $x = 2(-9) = -18$, and so $x + y = (-9) + (-18) = -27$.

ANSWER: (D)

14. We check the value of each of the five possibilities when $x = -3$:

$$(-3)^2 - 3 = 6 \qquad (-3-3)^2 = 36 \qquad (-3)^2 = 9 \qquad (-3+3)^2 = 0 \qquad (-3)^2 + 3 = 12$$

So the smallest value is that of $(x + 3)^2$.

ANSWER: (D)

15. *Solution 1*
Since the units digit of the product $39P \times Q3$ comes from multiplying $P \times 3$, and this units digit is a 1, then P must be the digit 7.
Therefore, $397 \times Q3 = 32\,951$ so $Q3 = \dfrac{32\,951}{397} = 83$, so $Q = 8$.
Thus, $P + Q = 15$.

Solution 2
Since the units digit of the product $39P \times Q3$ comes from multiplying $P \times 3$, and this units digit is a 1, then P must be the digit 7.
Therefore, the product becomes

$$\begin{array}{r} 3\,9\,7 \\ \times \quad Q\,3 \\ \hline 1\,1\,9\,1 \\ \square \\ \hline 3\,2\,9\,5\,1 \end{array}$$

Therefore, the product $Q \times 7$ must have a units digit of 6 (where the $\square$ is) in order to get the tens digit of 5 in the product. Since $Q \times 7$ has a units digit of 6, then $Q = 8$.
Thus, $P + Q = 15$.

ANSWER: (C)

16. Let c cents be the cost of downloading 1 song in 2005.
Then the cost of downloading 1 song in 2004 was $c + 32$ cents.
The total cost in 2005 was $360c$ and the total cost in 2004 was $200(c + 32)$.
Thus, $360c = 200(c + 32)$ or $160c = 6400$ or $c = 40$ cents, and so the total cost in 2005 was $360(40) = 14400$ cents, or \$144.00.

ANSWER: (A)

17. Since w is a positive integer, then $w \neq 0$, so $w^3 = 9w$ implies $w^2 = 9$ (we can divide by w since it is non-zero).
Since $w^2 = 9$, then $w = 3$ (because we know $w > 0$).
Thus, $w^5 = 3^5 = 243$.

ANSWER: (B)

18. Suppose the side lengths of the triangle are a, b, and c, with c the hypotenuse.
Then $c^2 = a^2 + b^2$ by the Pythagorean Theorem.
We are told that $a^2 + b^2 + c^2 = 1800$.
Since $a^2 + b^2 = c^2$, then $c^2 + c^2 = 1800$ or $2c^2 = 1800$ or $c^2 = 900$ or $c = 30$ (since the side lengths are positive).
So the hypotenuse has length 30.

ANSWER: (D)

19. *Solution 1*
Of the original 200 candies, since 90% (or 180 candies) are black, then 10% (or 20 candies) are gold.
Since Yehudi eats only black candies, the number of gold candies does not change.
After Yehudi has eaten some of the black candies, the 20 gold candies then represent 20% of the total number of candies, so there must be 100 candies in total remaining.
Thus, Yehudi ate $200 - 100 = 100$ black candies.

Solution 2
Of the original 200 candies, 90% or 180 candies are black.
Suppose that Yehudi eats b black candies.
This will leave $200 - b$ candies in total, of which $180 - b$ are black.
Since 80% of the remaining candies are black, then $\frac{180-b}{200-b} = \frac{4}{5}$ or $5(180 - b) = 4(200 - b)$ or $900 - 5b = 800 - 4b$ or $b = 100$.
Thus, Yehudi ate 100 black candies.

ANSWER: (D)

20. *Solution 1*
The y-intercept of the line $y = -\frac{3}{4}x + 9$ is $y = 9$, so Q has coordinates $(0, 9)$.
To determine the x-intercept, we set $y = 0$, and so obtain $0 = -\frac{3}{4}x + 9$ or $\frac{3}{4}x = 9$ or $x = 12$.
Thus, P has coordinates $(12, 0)$.
Therefore, the area of $\triangle POQ$ is $\frac{1}{2}(12)(9) = 54$, since $\triangle POQ$ is right-angled at O.
Since we would like the area of $\triangle TOP$ to be $\frac{1}{3}$ that of $\triangle POQ$, then the area of $\triangle TOP$ should be 18.
If T has coordinates (r, s), then $\triangle TOP$ has base TO of length 12 and height s, so $\frac{1}{2}(12)(s) = 18$ or $6s = 18$ or $s = 3$.
Since T lies on the line, then $s = -\frac{3}{4}r + 9$ or $3 = -\frac{3}{4}r + 9$ or $\frac{3}{4}r = 6$ or $r = 8$.
Thus, $r + s = 8 + 3 = 11$.

Solution 2
The y-intercept of the line $y = -\frac{3}{4}x + 9$ is $y = 9$, so Q has coordinates $(0, 9)$.
To determine the x-intercept, we set $y = 0$, and so obtain $0 = -\frac{3}{4}x + 9$ or $\frac{3}{4}x = 9$ or $x = 12$.
Thus, P has coordinates $(12, 0)$.
The areas of triangles are proportional to their heights if the bases are equal.
Since the area of $\triangle POQ$ is 3 times the area of $\triangle TOP$, then the height of $\triangle TOP$ is $\frac{1}{3}$ that of $\triangle POQ$.
Thus, T is $\frac{1}{3}$ along PQ from P towards Q.
Since P has coordinates $(12, 0)$ and Q has coordinates $(0, 9)$, then T has coordinates $\left(\frac{2}{3}(12), \frac{1}{3}(9)\right) = (8, 3)$.
Thus, $r + s = 8 + 3 = 11$.

ANSWER: (C)

21. *Solution 1*

We know $p + \dfrac{1}{q + \dfrac{1}{r}} = \dfrac{25}{19} = 1 + \dfrac{6}{19} = 1 + \dfrac{1}{\frac{19}{6}} = 1 + \dfrac{1}{3 + \dfrac{1}{6}}$.

Therefore, comparing the two fractions, $p = 1$, $q = 3$ and $r = 6$.

Solution 2

Since p, q and r are positive integers, then $q + \dfrac{1}{r}$ is at least 1, so $\dfrac{1}{q + \dfrac{1}{r}}$ is between 0 and 1.

Since $p + \dfrac{1}{q + \dfrac{1}{r}}$ is equal to $\dfrac{25}{19}$ which is between 1 and 2, then p must be equal to 1.

Therefore, $\dfrac{1}{q + \dfrac{1}{r}} = \dfrac{25}{19} - 1 = \dfrac{6}{19}$ or $q + \dfrac{1}{r} = \dfrac{19}{6}$.

Since r is a positive integer, then $\dfrac{1}{r}$ is between 0 and 1, so since $\dfrac{19}{6}$ is between 3 and 4, then $q = 3$.

(We are not asked to determine what the value of r, but we can check that $r = 6$.)

Answer: (C)

22. Suppose that n is a multiplicatively perfect positive integer.

If n can be written as $n = a \times b$ with a and b different positive integers, neither equal to 1, then n cannot have any other proper divisors other than 1, since $n = 1 \times a \times b$, so any other proper divisors would make this product too large.

So n can only have 2 proper divisors other than 1.

This tells us that n cannot have more than two distinct prime divisors (otherwise, it would certainly have three proper divisors larger than one).

If n has 2 distinct prime divisors, p and q, then neither can occur in the prime factorization of n more than once, otherwise p^2 or q^2 would be another proper divisor of n. So if n has 2 distinct prime divisors, then $n = pq$ (which indeed has only three proper divisors: 1, p and q).

The integers between 2 and 30 which are of the form $n = pq$ are 6, 10, 14, 15, 21, 22, and 26, or a total of 7 integers.

If n has only 1 prime factor, p, then $n = p^3$ in order to have exactly 3 proper divisors (namely, 1, p and p^2). Higher powers of p will have more than 3 proper divisors and lower powers of p will have fewer than 3 proper divisors. (For example, p^2 has 2 proper divisors and p^5 has 5 proper divisors.)

The integers between 2 and 30 which are of the form $n = p^3$ are 8 and 27, or 2 integers in total.

Therefore, there are 9 multiplicatively perfect numbers between 2 and 30.

Answer: (A)

23. First, we do some experimentation.

Since Celine moves small boxes faster and Quincy moves large boxes faster, then suppose Celine moves all 16 small boxes (taking 32 minutes) and Quincy moves all 10 large boxes (taking 50 minutes). Thus, they would finish in 50 minutes.

We could transfer two large boxes from Quincy to Celine, who now moves 16 small and 2 large boxes, taking 44 minutes. Quincy would then move 8 large boxes, taking 40 minutes. So they would finish in 44 minutes. (So (E) is not the answer.)

If we transfer one small box from Celine to Quincy, then Quincy moves 8 large boxes and 1 small box, taking 43 minutes, and Celine moves 15 small and 2 large boxes, taking 42 minutes.

So they would finish in 43 minutes. (So (D) is not the answer.)

Why is 43 minutes the smallest possible total time?
Suppose that it took them at most 42 minutes to finish the job. Then the total amount of working time would be at most 84 minutes.
Suppose that Celine moves x small boxes and y large boxes, which would take $2x+6y$ minutes.
Then Quincy moves $16-x$ small boxes and $10-y$ large boxes, which would take
$3(16-x)+5(10-y)=98-3x-5y$ minutes.
Since the total working time is at most 84 minutes, then $(2x+6y)+(98-3x-5y) \leq 84$ or $14 \leq x-y$.
Since $0 \leq x \leq 16$ and $0 \leq y \leq 10$, then the possible pairs of x and y which satisfy $14 \leq x-y$ are $(16,0)$, $(16,1)$, $(16,2)$, $(15,0)$, $(15,1)$, $(14,0)$, which produce working times as follows:

x	y	Celine Small	Celine Large	Celine Time	Quincy Small	Quincy Large	Quincy Time
16	0	16	0	32	0	10	50
16	1	16	1	38	0	9	45
16	2	16	2	44	0	8	40
15	0	15	0	30	1	10	53
15	1	15	1	36	1	9	48
14	0	14	0	28	2	10	56

In each of these cases, while the total working time is no more than 84 minutes, it takes longer than 43 minutes for the two of them to finish.
Therefore, it is impossible for them to finish in 42 minutes or less, so the earliest possible finishing time is 9:43 a.m.

Answer: (C)

24. Let us consider some small values for n.
If $n=1$, Anne wins by taking the first toothpick.
If $n=2$, Anne must take 1 toothpick (since she cannot take 3 or 4), so Brenda takes 1 toothpick and wins.
If $n=3$ or $n=4$, Anne can win by taking all of the toothpicks.
When $n=5$, if Anne takes 3 toothpicks to start, then Brenda is left with 2 toothpicks, and so cannot win, since Anne could not win when starting with 2 toothpicks. So Anne wins.
When $n=6$, if Anne takes 4 toothpicks to start, then Anne will win in the same way as in the $n=5$ case. So Anne wins.
When $n=7$, if Anne takes 1, 3 or 4 toothpicks, then 6, 4 or 3 toothpicks will be left for Brenda. But choosing from 6, 4 or 3 toothpicks is a winning position for the first person choosing (Brenda in this case). So $n=7$ is a losing position for Anne, since Brenda can use Anne's winning strategy in each case.
In general, Brenda will have a winning strategy for n if $n-1$, $n-3$ and $n-4$ all give Anne a winning strategy (since if Anne starts with n toothpicks an chooses 1, 3 or 4, then Brenda will start choosing from $n-1$, $n-3$ or $n-4$ toothpicks).
Anne will have a winning strategy otherwise (since if Anne would lose starting with an initial position of $n-1$, $n-3$ or $n-4$, then Anne should win starting with an initial position of n because she can leave Brenda with whichever of $n-1$, $n-3$ or $n-4$ is a losing position for the first person choosing).
We construct lists of the winning positions for each, adding a new n to Brenda's list if $n-1$, $n-3$ and $n-4$ are all in Anne's list, and to Anne's list otherwise:

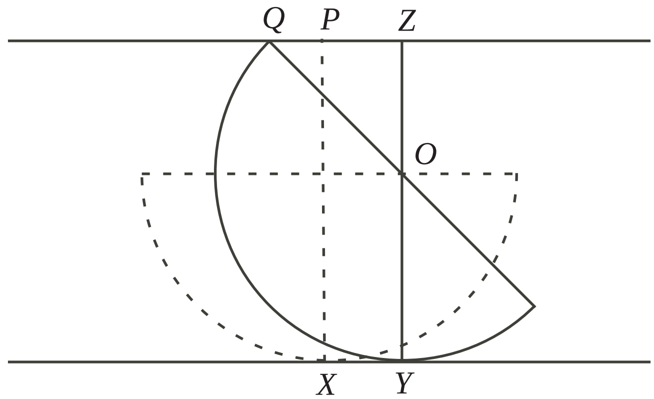
A's winning positions: 1, 3, 4, 5, 6, 8, 10, 11, 12, 13, 15, 17, 18, 19, 20, 22, 24, 25, 26, 27, 29, 31, 32, 33, 34

B's winning positions: 2, 7, 9, 14, 16, 21, 23, 28, 30, 35

So of the given possibilities, Brenda has a winning strategy when $n = 35$.

ANSWER: (E)

25. In its initial position, suppose the semi-circle touches the bottom line at X, with point P directly above X on the top line.
Consider when the semi-circle rocks to the right.



Suppose now the semi-circle touches the bottom line at Y (with O the point on the top of the semi-circle directly above Y, and Z the point on the top line directly above Y) and touches the top line at Q. Note that $XY = PZ$.
Q is one of the desired points where the semi-circle touches the line above. Because the diagram is symmetrical, the other point will be the mirror image of Q in line XP. Thus, the required distance is 2 times the length of PQ.
Now $PQ = QZ - PZ = QZ - XY$.
Since the semi-circle is tangent to the bottom line, YO is perpendicular to the bottom line, and O lies on a diameter, then O is the centre of the circle, so $OY = OQ = 8$ cm, since both are radii (or since the centre always lies on a line parallel to the bottom line and a distance of the radius away).
Also, $OZ = 4$ cm, since the distance between the two lines is 12 cm
By the Pythagorean Theorem (since $\angle QZO = 90°$), then

$$QZ^2 = QO^2 - ZO^2 = 8^2 - 4^2 = 64 - 16 = 48$$

so $QZ = 4\sqrt{3}$ cm.
Also, since $QZ : ZO = \sqrt{3} : 1$, then $\angle QOZ = 60°$.
Thus, the angle from QO to the horizontal is $30°$, so the semi-circle has rocked through an angle of $30°$, ie. has rocked through $\frac{1}{12}$ of a full revolution (if it was a full circle).
Thus, the distance of Y from X is $\frac{1}{12}$ of the circumference of what would be the full circle of radius 8, or $XY = \frac{1}{12}(2\pi(8)) = \frac{4}{3}\pi$ cm. (We can think of a wheel turning through $30°$ and the related horizontal distance through which it travels.)
Thus, $PQ = QZ - XY = 4\sqrt{3} - \frac{4}{3}\pi$ cm.
Therefore, the required distance double this, or $8\sqrt{3} - \frac{8}{3}\pi$ cm or about 5.4788 cm, which is closest to 55 mm.

ANSWER: (A)

Canadian Mathematics Competition
An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2005 Cayley Contest

(Grade 10)

Wednesday, February 23, 2005

Solutions

1. Simplifying, $a + 1 + a - 2 + a + 3 + a - 4 = a + a + a + a + 1 - 2 + 3 - 4 = 4a - 2$.

   Answer: (C)

2. Cancelling common factors in the numerators and denominators,

$$\left(\frac{4}{5}\right)\left(\frac{5}{6}\right)\left(\frac{6}{7}\right)\left(\frac{7}{8}\right)\left(\frac{8}{9}\right) = \left(\frac{4}{\cancel{5}}\right)\left(\frac{\cancel{5}}{\cancel{6}}\right)\left(\frac{\cancel{6}}{\cancel{7}}\right)\left(\frac{\cancel{7}}{\cancel{8}}\right)\left(\frac{\cancel{8}}{9}\right) = \frac{4}{9}$$

   Answer: (A)

3. The largest multiple of 17 less than 70 is 68. Therefore, $70 = 4(17) + 2$, so the remainder is 2.

   Answer: (D)

4. Since $\dfrac{3}{x+10} = \dfrac{1}{2x}$, then cross-multiplying, we get $6x = x + 10$ or $5x = 10$ or $x = 2$.

   Answer: (D)

5. Calculating, $(5^2 - 4^2)^3 = (25 - 16)^3 = 9^3 = 729$.

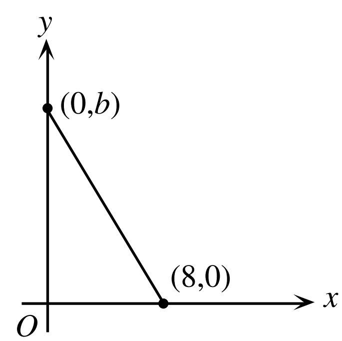
   Answer: (E)

6. 8 volunteers who each work 40 hours and each raise \$18 per hour will raise $8 \times 40 \times 18 = \$5760$. If 12 volunteers each work 32 hours and raise a total of \$5760, then they each raise $\dfrac{5760}{12 \times 32}$ or \$15 per hour.

   Answer: (C)

7. *Solution 1*

   Since the slope is $-\frac{3}{2}$, then for every 2 units we move to the right along the line, we must move 3 units down.



   To get from $(0, b)$ to $(8, 0)$, we move 8 units right, or 2 units right four times.
   Therefore, we must move 3 units down four times, or 12 units down in total.
   Therefore, $b = 12$.

   Solution 2

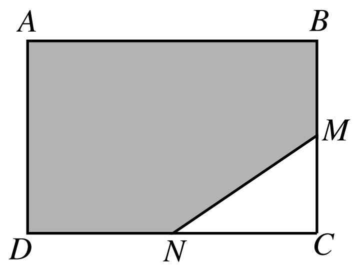
   Since the slope of the line is $-\frac{3}{2}$, then $\frac{b-0}{0-8} = -\frac{3}{2}$ or $-\frac{b}{8} = -\frac{3}{2}$ or $b = 8 \times \frac{3}{2} = 12$.

   Answer: (B)

8. Jack ran a total of 24 km.
Since he ran the first 12 km at 12 km/h, then it took him 1 hour to run the first 12 km.
Since he ran the second 12 km at 6 km/h, then it took him 2 hours to run the second 12 km.
Therefore, his run took a total of 3 hours.
Thus, Jill ran 24 km in 3 hours, so her speed was 8 km/h.

Answer: (A)

9. *Solution 1*
Since M is the midpoint of BC and $CM = 4$, then $BC = 8$.

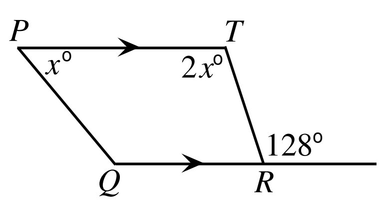


Since N is the midpoint of CD and $NC = 5$, then $CD = 10$.
Since $ABCD$ is a rectangle, its area is $10 \times 8 = 80$.
Also, the area of $\triangle NCM$ is $\frac{1}{2}(4)(5) = 10$, so the shaded area of the rectangle is the area of the whole rectangle minus the area of $\triangle NCM$, or 70.
Thus, the fraction of the rectangle that is shaded is $\dfrac{70}{80} = 0.875$, so 87.5% of the area is shaded.

Solution 2
The calculation from Solution 1 can also be done more generally.
Suppose $BC = 2x$ and $CD = 2y$.
Since M is the midpoint of BC, then $CM = x$.
Since N is the midpoint of CD, then $NC = y$.
Since $ABCD$ is a rectangle, its area is $(2x)(2y) = 4xy$.
Also, the area of $\triangle NCM$ is $\frac{1}{2}(NC)(MC) = \frac{1}{2}xy$, so the shaded area of the rectangle is the area of the whole rectangle minus the area of $\triangle NCM$, or $4xy - \frac{1}{2}xy = \frac{7}{2}xy$.
Thus, the fraction of the rectangle that is shaded is $\dfrac{\frac{7}{2}xy}{4xy} = \dfrac{7}{8} = 0.875$, so 87.5% of the area is shaded.

Answer: (D)

10. *Solution 1*
Since PT and RQ are parallel, then $2x^\circ = 128^\circ$, so $x = 64$, so $\angle TPQ = 64^\circ$.



Since PT and QR are parallel, then $\angle TPQ$ and $\angle PQR$ are supplementary.
Thus, $\angle PQR + 64^\circ = 180^\circ$, so $\angle PQR = 116^\circ$.

Solution 2
Since the two angles at R add to 180°, then $\angle QRT + 128^\circ = 180^\circ$, so $\angle QRT = 52^\circ$.
Since PT and QR are parallel, then $\angle PTR$ and $\angle QRT$ are supplementary, so $2x^\circ + 52^\circ = 180^\circ$ or $2x^\circ = 128^\circ$ or $x = 64$.
Therefore, three of the angles of quadrilateral $PQRT$ are 64°, 128° and 52°.
Since the angles in a quadrilateral add to 360°, then $\angle PQR = 360^\circ - 64^\circ - 128^\circ - 52^\circ = 116^\circ$.

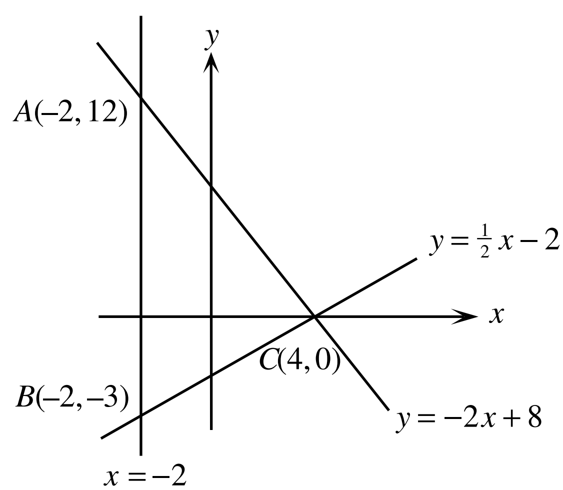
ANSWER: (A)

11. *Solution 1*
Matt's longest kick was 6 metres more than the average.
Thus, the other two kicks must be six metres less than the average when combined (that is, when we add up the difference between each of these kicks and the average, we get 6).
Since the other two kicks were the same length, then they each must have been 3 metres less than the average, or 34 metres each.

Solution 2
Since Matt's three kicks averaged 37 metres, then the sum of the lengths of the three kicks was $3 \times 37 = 111$ metres.
Let x be the length of each of the two kicks of unknown length.
Then $43 + 2x = 111$ or $x = 34$.

ANSWER: (D)

12. We first determine where the lines $y = -2x + 8$ and $y = \frac{1}{2}x - 2$ cross the line $x = -2$.
For the line $y = -2x + 8$, when $x = -2$, $y = -2(-2) + 8 = 12$, so the point of intersection is $(-2, 12)$.
For the line $y = \frac{1}{2}x - 2$, when $x = -2$, $y = \frac{1}{2}(-2) - 2 = -3$, so the point of intersection is $(-2, -3)$.

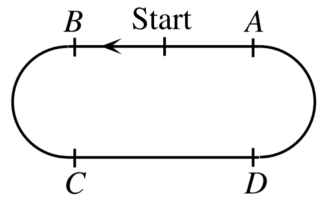


Therefore, we can think of $\triangle ABC$ as having base AB of length $12 - (-3) = 15$ and height being the distance from C to the line segment AB, or $4 - (-2) = 6$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(15)(6) = 45$.

ANSWER: (E)

13. If Andrew walks 1.4 metres per second, then he walks $60 \times 1.4 = 84$ metres per minute.
Since Andrew is walking for 30 minutes, then he walks a total of $30 \times 84 = 2520$ m.

Now the total length of track is 400 m, so after walking 2400 m, Andrew is back at the Start line.
Since the points A, B, C, and D are equally spaced, then consecutive points are 100 m apart.
Since the Start is half-way between A and B, then the Start is 50 m from B.



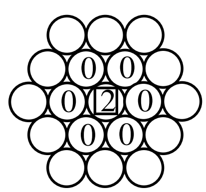
Therefore, after walking 2450 m, Andrew is at B.
After walking 70 m more to get to his total of 2520 m, Andrew will be 70 m beyond B and 30 m from C, so he will be closest to C.

Answer: (C)

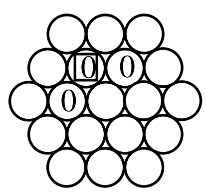
14. To make $\sqrt{1+2+3+4+x}$ an integer, we need $1+2+3+4+x=10+x$ to be a perfect square.
So we can rephrase the question as "For how many positive integers x less than 100 is $10+x$ a perfect square?".
Since x is between 1 and 99, then $10+x$ is between 11 and 109.
There are 7 perfect squares in this interval: 16, 25, 36, 49, 64, 81, and 100, so there are 7 possible values of x: 6, 15, 26, 39, 54, 71, and 90.

Answer: (B)

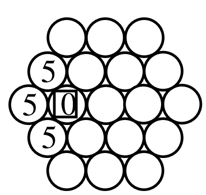
15. From the 2 in the centre, there are 6 possible 0s to which we can move.



From any 0, there are 2 possible 0s to which we can move.



From any 0, there are 3 possible 5s to which we can move.



For each of the 6 choices of the first 0, we can choose either of the 2 choices for the second 0, and from whichever second 0 is chosen we can choose any of the 3 possible 5s.
Therefore, there are $6 \times 2 \times 3 = 36$ possible paths that can be followed.

Answer: (A)

16. A good first step is to write out more terms in the sequence to see if we see a pattern:

$$88, 24, 64, 40, 24, 16, 8, 8, 0, 8, 8, 0, 8, 8, 0, 8, \ldots$$

So after some beginning terms, the sequence starts to repeat blocks of "8, 8, 0". (We can see that this will always happen: after "8, 0", the next term is $8-0=8$, so we get "8, 0, 8"; after "0, 8", the next term is $8-0=8$, so we get "8, 0, 8, 8"; after "8, 8", the next term is $8-8=0$, so we get "8, 0, 8, 8, 0", so the pattern continues.)
So in the first 100 numbers we have the first 6 terms (88, 24, 64, 40, 24, 16), and then 31 blocks of "8, 8, 0" (93 terms in total), and then the 100th term will be the beginning of a new block "8, 8, 0" (ie. the number 8).
Therefore, the sum of the first 100 terms is

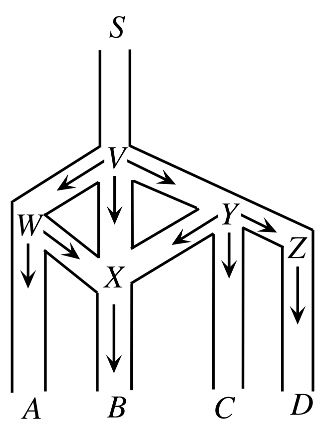
$$88+24+64+40+24+16+31(8+8+0)+8 = 256+31(16)+8 = 760$$

Answer: (B)

17. Using exponent laws, $1000^{100} = (10^3)^{100} = 10^{300} = (10^{100})^3 = \text{googol}^3$.

Answer: (E)

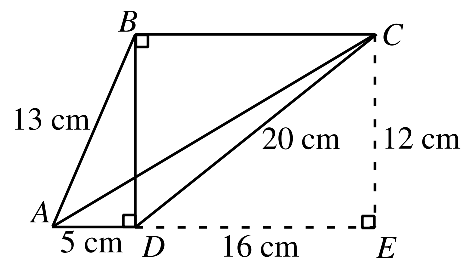
18. We label the five junctions as V, W, X, Y, and Z.



From the arrows which Harry can follow, we see that in order to get to B, he must get to X. So we calculate the probability that he gets to X.
To get to X, Harry can go S to V to W to X, or S to V to Y to X, or S to V to X directly.
At V, the probability that Harry goes down any of the three paths (that is, towards W, X or Y) is $\frac{1}{3}$.
So the probability that Harry goes directly from V to X to $\frac{1}{3}$.
At W, the probability that Harry turns to X is $\frac{1}{2}$, so the probability that he goes from V to W to X is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.
At Y, the probability that Harry turns to X is $\frac{1}{3}$, so the probability that he goes from V to Y to X is $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.
Therefore, the probability that Harry gets to X (and thus to B) is $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{6+3+2}{18} = \frac{11}{18}$.

Answer: (C)

19. We extend AD to the point E where it intersects the perpendicular to BC from C.



By the Pythagorean Theorem in $\triangle ADB$, $BD^2 = BA^2 - AD^2 = 13^2 - 5^2 = 144$, so $BD = 12$ cm.
By the Pythagorean Theorem in $\triangle DBC$, $BC^2 = DC^2 - BD^2 = 20^2 - 12^2 = 256$, so $BC = 16$ cm.
Since $BCED$ has three right angles (and in fact, a fourth right angle at E), then it is a rectangle, so $DE = BC = 16$ cm and $CE = BD = 12$ cm.
Therefore, if we look at $\triangle AEC$, we see that $AE = 16 + 5 = 21$ cm, so by the Pythagorean Theorem, $AC^2 = 21^2 + 12^2 = 585$, so $AC \approx 24.2$ cm, to the nearest tenth of a centimetre.

ANSWER: (A)

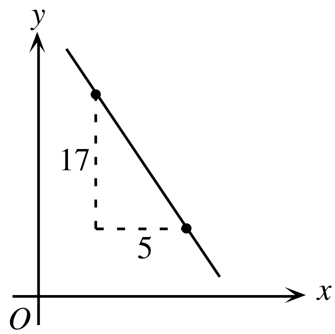
20. Let B be the total number of Beetles in the parking lot.
Then the number of Acuras is $\frac{1}{2}B$.
Also, the number of Camrys is 80% of $\frac{1}{2}B + B$, so the number of Camrys is $\frac{4}{5} \times \frac{3}{2}B = \frac{6}{5}B$.
Therefore, since the total number of cars in the parking lot is 81, then $B + \frac{1}{2}B + \frac{6}{5}B = 81$ or $\frac{27}{10}B = 81$, or $B = \frac{10}{27} \times 81 = 30$.
Therefore, the number of Beetles is 30.

ANSWER: (B)

21. We start by determining the combination of these bills totalling 453 Yacleys which uses the fewest 17 Yacley bills.
To do this, we notice that since we can use as many 5 Yacley bills as we'd like, then any multiple of 17 less than 453 which ends in a 5 or an 8 can be "topped up" to 453 Yacleys using 5 Yacley bills.
The first few multiples of 17 are 17, 34, 51, 68.
So if we use four 17 Yacley bills, we have 68 Yacleys, leaving $453 - 68 = 385$ Yacleys to get to 453. For 385 Yacleys, we need 77 of the 5 Yacley bills.
So 4 of the 17 Yacley bills and 77 of the 5 Yacley bills works.
To get other combinations, we use the fact that 5 of the 17 Yacley bills are worth the same as 17 of the 5 Yacley bills, so we can subtract 17 of the 5 Yacley bills and add 5 of the 17 Yacley bills and keep the total the same.
Thus, $77 - 17 = 60$ of the 5 Yacley bills and $4 + 5 = 9$ of the 17 Yacley bills make 453 Yacleys. (Check this!)
Also, 43 and 14 of the two types of bills, and 26 and 19 of the two types of bills, and 9 and 24 of the two types of bills.
Since we are down to 9 of the 5 Yacley bills, we can no longer use this exchanging process (since we need at least 17 of the 5 Yacley bills to be able to do this).
Therefore, there are 5 combinations that work.

(An alternate approach would be to let x be the number of 17 Yacley bills and y the number of 5 Yacley bills used.

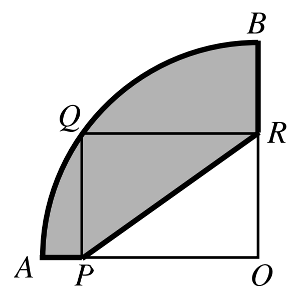
We then would like to consider the equation $17x + 5y = 453$, and find the number of pairs (x, y) which satisfy this equation and where both x and y are positive integers.
Geometrically, we are trying to find the number of points (x, y), with x and y both positive integers, lying on the line $17x + 5y = 453$.
In a similar way to above, we can find that $(x, y) = (4, 77)$ is such a point.
Since the slope of the line is $-\frac{17}{5}$, then to get a another point with integer coordinates on the line, we can move 5 units right and 17 units down.



We can repeat this process as above, and get 5 combinations that work.)

ANSWER: (C)

22. In order to determine the perimeter of the shaded region, we need to determine the total combined length of arc AQB and segments AP, PR and RB.



Since AOB is a quarter circle of radius 10, then arc AQB has length $\frac{1}{4}(2\pi(10)) = 5\pi$.
Since $PQRO$ is a rectangle, then $PR = QO$ and QO is a radius of the quarter circle, so $PR = QO = 10$. So we now need to calculate $AP + RB$.
But $AP + RB = (AO - PO) + (BO - RO) = AO + BO - (PO + RO)$. We know that $AO = BO = 10$ since each is a radius of the quarter circle.
Also, $PO + RO$ is half of the perimeter of the rectangle (which has total perimeter 26), so $PO + RO = 13$.
Therefore, $AP + RB = 10 + 10 - 13 = 7$.
Thus, the perimeter of the shaded region is $5\pi + 10 + 7 = 17 + 5\pi$.

ANSWER: (C)

23. We solve this problem by systematically keeping track of the distance from home of each of Anna, Bill and Dexter. At 12:00 noon, each is 0 km from home.
At 12:15:

Dexter is 0 km from home (since he hasn't started running)
Anna is $\frac{1}{4} \times 4 = 1$ km from home (since she has walked at 4 km/h for $\frac{1}{4}$ of an hour)
Bill is $\frac{1}{4} \times 3 = \frac{3}{4}$ km from home (since he has walked at 3 km/h for $\frac{1}{4}$ of an hour)

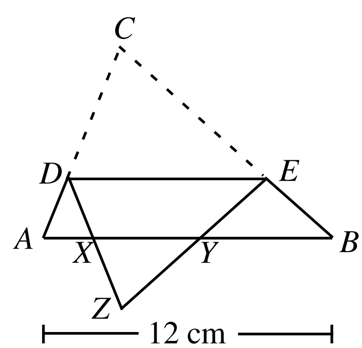
At 12:15, Dexter leaves and runs until he catches up to Anna.
How long does this take? Since Anna walks at 4 km/h and Dexter runs at 6 km/h in the same direction, then Dexter gains 2 km on Anna every hour. Since Anna starts 1 km ahead of Dexter, then it takes Dexter $\frac{1}{2}$ hour to catch Anna, so he catches her at 12:45.
At 12:45:

Dexter is 3 km from home (since he has run at 6 km/h for $\frac{1}{2}$ hour)
Anna is $\frac{3}{4} \times 4 = 3$ km from home (since she has walked at 4 km/h for $\frac{3}{4}$ of an hour)
Bill is $\frac{3}{4} \times 3 = \frac{9}{4}$ km from home (since he has walked at 3 km/h for $\frac{3}{4}$ of an hour)

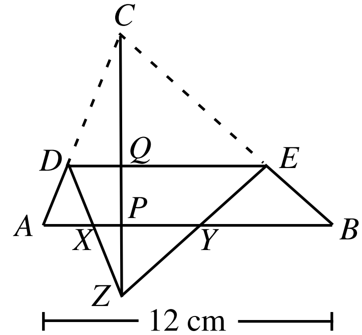
At 12:45, Dexter turns around instantaneously and runs back to Bill from Anna.
How long does this take? Since Bill walks at 3 km/h and Dexter runs at 6 km/h in the opposite direction, then Dexter and Bill are getting closer at a rate of 9 km/h.
Since Dexter and Bill start $3 - \frac{9}{4} = \frac{3}{4}$ km apart, then it takes Dexter $\frac{1}{9} \times \frac{3}{4} = \frac{1}{12}$ hour (or 5 minutes) to meet Bill.
Therefore, Bill and Dexter meet at 12:50 p.m.

ANSWER: (E)

24. *Solution 1*
Let X and Y be the points where the folded portion of the triangle crosses AB, and Z be the location of the original vertex C after folding.

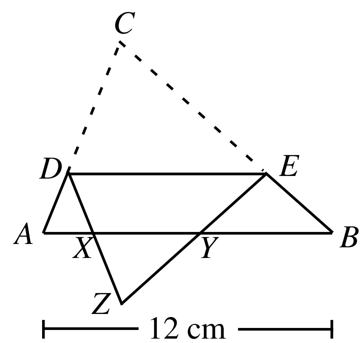


We are told that the area of $\triangle XYZ$ is 16% that of the area of $\triangle ABC$.
Now $\triangle ACB$ is similar to $\triangle XZY$, since $\angle XZY$ is the folded over version of $\angle ACB$ and since $\angle XYZ = \angle EYB = \angle DEY = \angle CED = \angle CBA$ by parallel lines and folds.
Since $\triangle XZY$ is similar to $\triangle ACB$ and its area is $0.16 = (0.4)^2$ that of $\triangle ACB$, then the sides of $\triangle XZY$ are 0.4 times as long as the sides of $\triangle ACB$.
Draw the altitude of $\triangle ACB$ from C down to P on AB (crossing DE at Q) and extend it through to Z.



Now $CP = CQ + QP = ZQ + QP = ZP + 2PQ$.
Since the sides of $\triangle XZY$ are 0.4 times as long as the sides of $\triangle ACB$, then $ZP = 0.4CP$.
Since $CP = ZP + 2PQ$, then $PQ = 0.3CP$, and so $CQ = CP - PQ = 0.7CP$.
Since CQ is 0.7 times the length of CP, then DE is 0.7 times the length of AB, again by similar triangles, so $DE = 0.7(12) = 8.4$.

Solution 2
Let X and Y be the points where the folded portion of the triangle crosses AB, and Z be the location of the original vertex C after folding.



We are told that the area of $\triangle XYZ$ is 16% that of the area of $\triangle ABC$.
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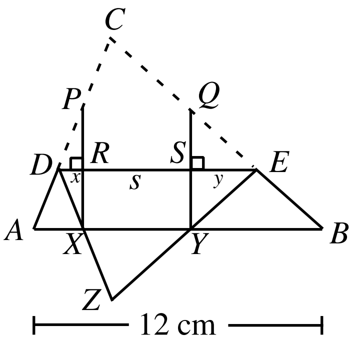
Since $\triangle XZY$ is similar to $\triangle ACB$ and its area is $0.16 = (0.4)^2$ that of $\triangle ACB$, then the sides of $\triangle XZY$ are 0.4 times as long as the sides of $\triangle ACB$.
Draw perpendiculars to AB at X and Y, intersecting AC and BC and P and Q, respectively, and DE at R and S, respectively.
By symmetry, PQ and RS are parallel to XY and the same length, so let $PQ = RS = XY = s$.
Since the sides of $\triangle XZY$ are 0.4 times as long as the sides of $\triangle ACB$, then $s = 0.4 \times 12 = 4.8$.
Since $\triangle CDE$ is congruent to $\triangle ZDE$ (since one is the folded over version of the other), then by symmetry, $PR = RX$ and $QS = SY$.
Let $DR = x$ and $ES = y$.



Then $AX = 2x$, since $\triangle PXA$ is similar to $\triangle PRD$ and has sides twice as long (since $PX = 2PR$. Similarly, $BY = 2y$.
Now looking at AB as a whole, we have $AB = 2x + s + 2y = 12$, so $x + y = \frac{1}{2}(12 - s) = 3.6$.
Looking at DE, we have $DE = s + x + y = 4.8 + 3.6 = 8.4$.

Answer: (B)

25. The first challenge in this problem is to find one set of numbers a, b, c that actually works.
Since this looks a bit similar to the Pythagorean Theorem, we can start with $3^2 + 4^2 = 5^2$ and try to manipulate this.
If we divide both sides by the least common multiple of 3^2, 4^2 and 5^2, which is $(3\times4\times5)^2 = 60^2$, we then obtain $\dfrac{3^2}{60^2} + \dfrac{4^2}{60^2} = \dfrac{5^2}{60^2}$ or $\dfrac{1}{20^2} + \dfrac{1}{15^2} = \dfrac{1}{12^2}$.
This gives us two possible triples: $(a, b, c) = (20, 15, 12)$ and $(a, b, c) = (15, 20, 12)$ (so two possible values for a so far).
How can we get more? We can multiply the equation $\dfrac{1}{20^2} + \dfrac{1}{15^2} = \dfrac{1}{12^2}$ by reciprocals of perfect squares.
Multiplying by $\dfrac{1}{2^2}$, we get $\dfrac{1}{40^2} + \dfrac{1}{30^2} = \dfrac{1}{24^2}$.
Multiplying by $\dfrac{1}{3^2}$, we get $\dfrac{1}{60^2} + \dfrac{1}{45^2} = \dfrac{1}{36^2}$.
Multiplying by $\dfrac{1}{4^2}$, we get $\dfrac{1}{80^2} + \dfrac{1}{60^2} = \dfrac{1}{48^2}$.
Multiplying by $\dfrac{1}{5^2}$, we get $\dfrac{1}{100^2} + \dfrac{1}{75^2} = \dfrac{1}{60^2}$.
Multiplying by $\dfrac{1}{6^2}$, we get $\dfrac{1}{120^2} + \dfrac{1}{90^2} = \dfrac{1}{72^2}$.
Multiplying by $\dfrac{1}{7^2}$, we get $\dfrac{1}{140^2} + \dfrac{1}{105^2} = \dfrac{1}{84^2}$.
At this point, the strategy will no longer work, since we are only looking for values of $a \leq 100$.
So far, the possible values of a are (from looking at each denominator of the left side of each the equations here): 20, 15, 40, 30, 60, 45, 80, 100, 75, 90. (Notice that 60 doesn't appear twice in the list!) The sum of these numbers is 555.

Can we find more starting equations by beginning with a different Pythagorean triple?
If we start with $5^2 + 12^2 = 13^2$ and divide both sides by the least common multiple of 5^2, 12^2 and 13^2 (ie. $(5 \times 12 \times 13)^2 = 780^2$), we get $\dfrac{1}{156^2} + \dfrac{1}{65^2} = \dfrac{1}{60^2}$ which gives us 65 as another possible value of a.
Therefore, our running total for values of a is $555 + 65 = 620$.
We can't generate more possible values for a using $\dfrac{1}{156^2} + \dfrac{1}{65^2} = \dfrac{1}{60^2}$ since multiplying both sides

by the reciprocal of any perfect square will make both of a and b at least 130, so bigger than 100.

Can we use $6^2 + 8^2 = 10^2$? Here, the least common multiple of 6^2, 8^2 and 10^2 is 120^2, and dividing by 120^2 gives us $\frac{1}{20^2} + \frac{1}{15^2} = \frac{1}{12^2}$, which we have already used.

Can we use any other Pythagorean triple? No, since any other Pythagorean triple is at least as big as 7-24-25, and so the smallest possible denominator that we will get on the left side by using this technique is $(7 \times 25)^2 = 175^2$, which would give an a larger than 100.

Also, any triple (a, b, c) that actually works does come from a Pythagorean triple, since we can multiply both sides of $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$ by $(abc)^2$ to get $(bc)^2 + (ac)^2 = (ab)^2$.

So every possible triple (a, b, c) comes from a Pythagorean triple, and no Pythagorean triples give any more allowable values of a, so we have found them all.

Therefore, the sum of all possible values of $a \leq 100$ is 620.

ANSWER: (E)

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2004 Solutions
Cayley Contest (Grade 10)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

2004 Cayley Contest Solutions

1. Calculating each term,
$$2^2+1^2+0^2+(-1)^2+(-2)^2=4+1+0+1+4=10$$
Answer: (E)

2. 25% of 2004 is $\frac{1}{4}$ of 2004, or 501.
50% of 4008 is $\frac{1}{2}$ of 4008, or 2004.
50% of 1002 is $\frac{1}{2}$ of 1002, or 501.
100% of 1002 is 1002.
10% of 8016 is $\frac{1}{10}$ of 8016, or 801.6.
20% of 3006 is $\frac{1}{5}$ of 3006, or 601.2.
Answer: (B)

3. To get from A to B, we must go 2 units to the right and 3 units up. Since B is the midpoint of AC, then to get from B to C, we must also go 2 units to the right and 3 units up. Therefore, C has coordinates $(5,7)$.
Answer: (E)

4. Since $x+1-2+3-4=5-6+7-8$, then simplifying both sides we get $x-2=-2$ so $x=0$.
Answer: (C)

5. The first figure has outer perimeter 4. The second figure has outer perimeter 8. The third figure has outer perimeter 12. So from one figure to the next, the outer perimeter increases by 4. So the outer perimeter of the fifth figure is 8 more than that of the third figure, or 20. (We could easily draw out the fifth figure, which would be made up of 9 small squares.)
Answer: (C)

6. Removing a common factor of 7,
$$7x+42y=7(x+6y)=7(17)=119$$
Answer: (E)

7. *Solution 1*
$3^2+3^2+3^2=3\times 3^2=3^1\times 3^2=3^3$. Thus, $a=3$.

Solution 2
$3^2+3^2+3^2=9+9+9=27$, so $3^a=27$ and so $a=3$, since $3^3=27$.
Answer: (B)

8. The circumference of a circle is equal to $2\pi r$, where r is the radius of the circle.
Since the circumference of the outer circle is 24π, then its radius is 12. Thus, $OB=12$.
Since the circumference of the inner circle is 14π, then its radius is 7. Thus, $OA=7$.
Therefore, $AB=OB-OA=5$. Answer: (B)

9. By the Pythagorean Theorem, the length of the rope joining B to C is $\sqrt{16^2+30^2}=\sqrt{256+900}=\sqrt{1156}=34\,\text{m}$.
We must also determine the length of rope joining A to C. To get from A to C we must go over 16 m and up 12 m (the difference between the heights of the towers), so the rope has length $\sqrt{16^2+12^2}=\sqrt{256+144}=\sqrt{400}=20\,\text{m}$.
Therefore, the total length of rope used is 54 m.

Answer: (A)

10. When the cube is folded, H and I share an edge. At one end of this edge is the face labelled G (so the faces G, H and I meet at a point), and at the other end of this edge is the face labelled J.
Thus, J is opposite G.

Answer: (D)

11. Since each number after the second is the product of the previous two numbers, then the 18 in the fifth position is the product of the 3 and the number in the fourth position. Thus, the number in the fourth position is 6.
So the sequence now reads, x, ____, 3, 6, 18.
Using the rule, 6 equals the product of 3 and the number in the second position. Thus, the sequence reads x, 2, 3, 6, 18.
This tells us that $2x=3$ or $x=\frac{3}{2}$.

Answer: (B)

12. The sum of the numbers in the 1st row is $2x+5$, so the sum of the numbers in any row, column or diagonal is also $2x+5$.
Therefore, the entry in the 2nd row, 1st column must be 5, using the 1st column.
Thus, the entry in the 2nd row, 2nd column is $2x+3$, using the 2nd row.
Looking at the sum of the 2nd column,

$$\begin{aligned}3+(2x+3)+x&=2x+5\\3x+6&=2x+5\\x&=-1\end{aligned}$$

so the sum of the numbers in any row is $2(-1)+5=3$.

Answer: (C)

13. Since the perimeter of the smaller square is 72 cm, its side length is $\frac{1}{4}(72)=18$ cm.
Therefore, the area of the smaller square is $(18\text{ cm})^2=324\text{ cm}^2$.
Since the area inside the larger square is the combined area of the smaller square and the shaded region, then the area of the larger square is $160\text{ cm}^2+324\text{ cm}^2=484\text{ cm}^2$.
Thus, the side length of the larger square is $\sqrt{484}=22\,\text{cm}$, and so the perimeter of the larger square is $4(22\text{ cm})=88\text{ cm}$.

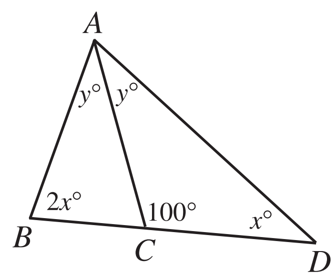
Answer: (B)

14. The average of 4, 20 and x is equal to $\frac{4+20+x}{3}$. The average of y and 16 is equal to $\frac{16+y}{2}$.
Since these averages are equal,

$$\begin{aligned}\frac{4+20+x}{3} &= \frac{16+y}{2}\\ 2(4+20+x) &= 3(16+y)\\ 48+2x &= 48+3y\\ 2x &= 3y\\ \frac{x}{y} &= \frac{3}{2}\end{aligned}$$

Answer: (A)

15. In triangle ACD, $x^\circ + y^\circ + 100^\circ = 180^\circ$, so $x + y = 80$ (*).
Since $\angle ACB$ and $\angle ACD$ are supplementary, then $\angle ACB = 180^\circ - \angle ACD = 80^\circ$.
Thus, in triangle ACB, $2x^\circ + y^\circ + 80^\circ = 180^\circ$, so $2x + y = 100$ (**).
Subtracting (*) from (**), we obtain $x = 20$.



Answer: (E)

16. When a player rolls two dice, there are 6 possibilities for the outcome on each die, so there are 36 possibilities for the outcomes when two dice are rolled.
Which possibilities give a score of 3 or less? These are: 1 and 1, 1 and 2, 1 and 3, 2 and 1, 2 and 2, 2 and 3, 3 and 1, 3 and 2, 3 and 3. So 9 of the 36 possibilities give a score of 3 or less.
Thus, the probability is $\frac{9}{36} = \frac{1}{4}$.

Answer: (A)

17. Putting each of the two fractions over a common denominator of mn, we get

$$\begin{aligned}\frac{1}{m}+\frac{1}{n} &= \frac{5}{24}\\ \frac{n}{mn}+\frac{m}{mn} &= \frac{5}{24}\\ \frac{m+n}{mn} &= \frac{5}{24}\\ \frac{20}{mn} &= \frac{5}{24}\\ mn &= \frac{24\times 20}{5}\\ mn &= 96\end{aligned}$$

Answer: (D)

18. This problem requires a fair amount of fiddling around. After some work, we can see that the ant can walk along 9 edges without walking along any edge for a second time. (For example, it could walk A to B to F to E to A to D to C to G to H to D. It is then stuck.)
In fact, 9 edges, or 108 cm, is the maximum.

Justifying this fact takes us into a really neat area of mathematics called "graph theory". (Graph theory was first developed by Euler when he was working on the Königsberg bridge problem.)
What if the ant could actually walk along 10 edges?
If the ant had done this, it would have walked in and out of a vertex 20 times in total (once at each end of each of the 10 edges). The cube has 8 vertices and each vertex has 3 edges meeting at it, so since the ant has not walked along the same edge twice, then it can only have been in and out of any given vertex at most 3 times.
But for 20 ins and outs with 8 vertices in total, there must be 4 vertices that have been used 3 times.
This is impossible, though, because other than the starting vertex and the ending vertex in the path, the ant must go both in and out of a vertex, so each vertex other than the starting and ending vertices must be used an even number of times. So we cannot have more than 2 vertices being used an odd number of times.
Therefore, the ant cannot walk along 10 edges, so 108 cm must indeed be the maximum.

Answer: (D)

19. Here we use the rule for manipulating exponents $\frac{2^a}{2^b} = 2^{a-b}$.

Therefore, each of the 2003 fractions after the first fraction is equal to $2^{-1} = \frac{1}{2}$.

This gives us 2004 copies of $\frac{1}{2}$ being added up, for a total of 1002.

Answer: (A)

20. Let a be the value of an arrow shot into ring A, let b be the value of an arrow shot into ring B, and let c be the value of an arrow shot into ring C.
From the given information about the three archers, we know that

$$c + a = 15$$
$$c + b = 18$$
$$b + a = 13$$

We are interested in calculating $2b$.
Adding up the second and third equations, we get $a + 2b + c = 31$ and so substituting the information from the first equation, we get $2b + 15 = 31$ or $2b = 16$.
(Notice that we could have found the values of a, b and c, but we did not need to do this.)

Answer: (C)

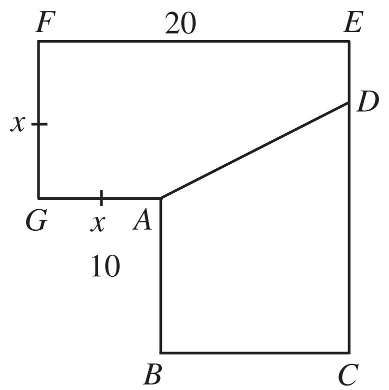
21. Suppose that Laura uses the last blue sheet on day number d.
Then the total number of blue sheets with which she started was d.
Since she uses 3 red sheets per day and has 15 red sheets left over, then she started with $3d+15$ red sheets.
Since the blue and red sheets were initially in the ratio 2 : 7, then

$$\begin{aligned}\frac{d}{3d+15} &= \frac{2}{7}\\ 7d &= 6d+30\\ d &= 30\end{aligned}$$

Thus, she started with 30 blue sheets and 105 red sheets, or 135 sheets of construction paper in total.

Answer: (C)

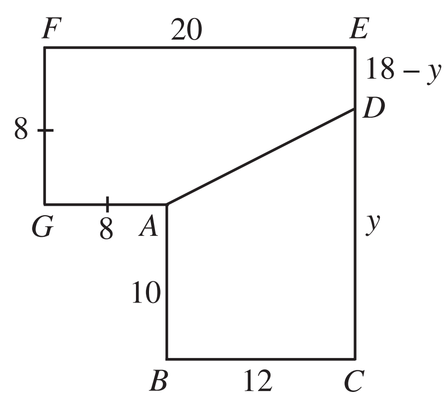
22. First, we determine the lengths of the sides of the rooms.
Suppose that $AG = x$. Then $FG = x$.
So the room can be thought of as a rectangular room of width $FE = 20$ and length $AB + FG = 10 + x$, with a rectangular corner of dimensions $AG = x$ by $AB = 10$ removed.
Equating the area of the entire room with this way of visualizing it,



$$\begin{aligned}20(10+x)-10x &= 280\\ 10x+200 &= 280\\ x &= 8\end{aligned}$$

Therefore, the lengths of the sides of the room are (starting from B and proceeding clockwise) 10, 8, 8, 20, 18 and 12.

Let y now be the distance from C to D, the point where the new wall touches CE.
Now $CBAD$ can be viewed as a trapezoid with base CD and parallel side AB (since the room has square corners).
Also, the height of the trapezoid is $BC = 12$. The area of this trapezoid is supposed to be 140, or half of the total area of the large room.
Therefore, since the area of the trapezoid is $\frac{1}{2}(BC)(AB+CD)$, we have



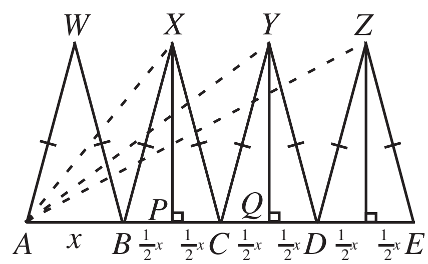
$$\begin{aligned}\tfrac{1}{2}(12)(10+y) &= 140\\ 6(10+y) &= 140\\ 10+y &= \tfrac{70}{3}\\ y &= \tfrac{40}{3}\end{aligned}$$

Also, the height of the trapezoid is $BC = 12$. The area of this trapezoid is supposed to be 140, or half of the total area of the large room.
Therefore, since the area of the trapezoid is $\frac{1}{2}(BC)(AB + CD)$, we have
Thus, the distance from C to D is $\frac{40}{3}$.

Answer: (E)

23. The ball is rolling towards Marcos at 4 m/s and he is running towards it at 8 m/s, so he gains 12 metres per second on the ball. Since he starts 30 m from the ball, it will take him $\frac{30}{12} = 2.5$ s to reach the ball.
The ball is rolling away from Michael at 4 m/s and he is running at 9 m/s, so he is gaining 5 m/s on the ball. Since he starts 15 m behind the ball, he would catch up to the ball in 3 s if it continued to roll.
Thus, Marcos gets to the ball first. After 2.5 s, Michael has gained $5(2.5) = 12.5$ m on the ball, so is 2.5 m from the ball when Marcos touches it first.

Answer: (C)



24. First, we determine the lengths of the sides of the new triangle, in terms of x.
We drop perpendiculars from X, Y and Z to points P, Q and R, respectively, on the line AE. Since each of the four triangles is isosceles, then
$BP = PC = CQ = QD = DR = RE = \frac{1}{2}x$.
Consider triangle ARZ, which is right-angled at R. Since $AZ = AE = 4x$, then by the Pythagorean Theorem,

$$AR^2 + RZ^2 = AZ^2$$
$$RZ^2 = (4x)^2 - \left(\tfrac{7}{2}x\right)^2$$
$$RZ^2 = \tfrac{15}{4}x^2$$

so the square of the height of each of the four isosceles triangles is $\frac{15}{4}x^2$.
Thus, $AY^2 = AQ^2 + QY^2 = \left(\frac{5}{2}x\right)^2 + \frac{15}{4}x^2 = 10x^2$, so $AY = \sqrt{10}x$, and
$AX^2 = AP^2 + PX^2 = \left(\frac{3}{2}x\right)^2 + \frac{15}{4}x^2 = 6x^2$, so $AX = \sqrt{6}x$.

Thus, the new triangle has side lengths $\sqrt{6}x$, $\sqrt{10}x$ and $4x$. Since $\left(\sqrt{6}x\right)^2 + \left(\sqrt{10}x\right)^2 = (4x)^2$, then this new triangle is right-angled, with hypotenuse $4x$, and so has area $\frac{1}{2}\left(\sqrt{6}x\right)\left(\sqrt{10}x\right) = \frac{1}{2}\sqrt{60}x^2 = \sqrt{15}x^2$.

We would like the area to be less than 2004, so $\sqrt{15}x^2 < 2004$ or $x < \sqrt{\dfrac{2004}{\sqrt{15}}} \approx 22.747$.

Therefore, the largest integer value of x that works is 22.

Answer: (E)

25. We start by rewriting each expression so that each has the same numerator:

$$\frac{7x+1}{2}=\frac{2+(7x-1)}{2}=1+\frac{7x-1}{2}$$

$$\frac{7x+2}{3}=\frac{3+(7x-1)}{3}=1+\frac{7x-1}{3}$$

$$\vdots$$

$$\frac{7x+300}{301}=\frac{301+(7x-1)}{301}=1+\frac{7x-1}{301}$$

For each of these expressions, the original fraction will be in lowest terms only when the fraction in the new expression is in lowest terms, ie. $\frac{7x+1}{2}$ is in lowest terms only when $\frac{7x-1}{2}$ is in lowest terms.

So the original problem is equivalent to determining the number of positive integers x with $x \leq 60$ such that each of

$$\frac{7x-1}{2}, \frac{7x-1}{3}, \ldots, \frac{7x-1}{301}$$

is in lowest terms.

This is equivalent to determine the number of positive integers x with $x \leq 60$ for which $7x-1$ has no common factor with any of the integers from 2 to 301, inclusive.

For x from 1 to 43, $7x-1$ will be actually equal to one of the integers from 2 to 301, so there will be a common factor.

So we must examine the integers from 44 to 60.

If x is odd, then $7x-1$ is even, and so has a common factor of 2, for example.

So we must examine the integers 44, 46, 48, 50, 52, 54, 56, 58, and 60.

The values of $7x-1$ for these integers are 307, 321, 335, 349, 363, 377, 391, 405, and 419, respectively. We would like to determine how many of these have no common factors with any of the integers from 2 to 301.

The integers 321, 363 and 405 are divisible by 3, so they can be removed. The integer 335 is divisible by 5 and so can be removed. This leaves us with 307, 349, 377, 391 and 419.

307 is a prime number.

349 is a prime number.

377 is divisible by 13, so can be removed.

391 is divisible by 17, so can be removed.

419 is a prime number.

Each of these prime numbers is divisible only 1 and itself, so has no common factor with any of the integers from 2 to 301.

Therefore, there are 3 integers x with $x \leq 60$ for which the fractions are all in lowest terms.

Answer: (C)

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University of Waterloo, Waterloo, Ontario

2003 Solutions
Cayley Contest (Grade 10)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Awards

2003 Cayley Contest Solutions

1. Evaluating,
$\frac{3-(-3)}{2-1}=\frac{6}{1}=6.$ ANSWER: (D)

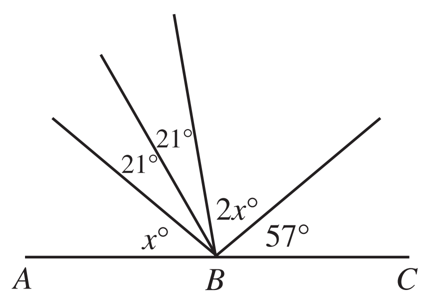
2. $17^2-15^2=289-225=64=8^2.$ ANSWER: (A)

3. Since $42=6\times 7$, the only possibility for the correct answer is that 42 is divisible by 7. (We can check that each of the remaining possibilities is not true.)
ANSWER: (D)

4. Since 25% of the number is 5 times 5% of that number, then 25% of the number is $5(8)=40$.
ANSWER: (A)

5. We could use a calculator to determine the value of the expression and then round to the nearest integer. Alternatively, we can calculate the value of the expression by hand:
$$\tfrac{3}{2}\times\tfrac{4}{9}+\tfrac{7}{2}=\tfrac{12}{18}+\tfrac{7}{2}=\tfrac{12}{18}+\tfrac{63}{18}=\tfrac{75}{18}=\tfrac{25}{6}=4+\tfrac{1}{6}$$
Therefore, the closest integer is 4.
ANSWER: (B)

6. Since ABC is a straight line, the sum of the 5 angles is 180°, and so
$$\begin{aligned} x^\circ+21^\circ+21^\circ+2x^\circ+57^\circ &= 180^\circ \\ 3x+99 &= 180 \\ 3x &= 81 \\ x &= 27 \end{aligned}$$



ANSWER: (A)

7. *Solution 1*
In the top right quarter, we have no unknowns, so we can calculate the sum of the three numbers to be $13+17+45=75$. Therefore, the sum of the numbers (including the unknowns) in each of the other 3 quarters is 75 as well, and so
$$\begin{aligned} (z+28+8)+(x+19+50)+(y+3+63) &= 3(75) \\ x+y+z+36+69+66 &= 225 \\ x+y+z &= 54 \end{aligned}$$

Solution 2
In the top right quarter, we have no unknowns, so we can calculate the sum of the three numbers to be $13+17+45=75$.
In the top left quarter, we then have $z+28+8=75$ and so $z=39$.

Similarly, $x = 6$ and $y = 9$.
Therefore, $x + y + z = 6 + 9 + 39 = 54$. ANSWER: (C)

8. An equilateral triangle with a side length of 20 must have a perimeter of 60.
A square with a perimeter of 60 must have a side length of 15.
A square with a side length of 15 must have an area of 225.

ANSWER: (C)

9. *Solution 1*
Taking the reciprocal of both sides and then solving,

$$\frac{1}{x+\frac{1}{5}} = \frac{5}{3}$$
$$x + \tfrac{1}{5} = \tfrac{3}{5}$$
$$x = \tfrac{2}{5}$$

Solution 2
Cross-multiplying,

$$\frac{1}{x+\frac{1}{5}} = \frac{5}{3}$$
$$5\left(x + \tfrac{1}{5}\right) = 3$$
$$5x + 1 = 3$$
$$x = \tfrac{2}{5}$$

ANSWER: (A)

10. *Solution 1*
When $\frac{5}{8}$ of the players are girls then $\frac{3}{8}$ of the players will be boys. Since the number of boys playing is 6 (and does not change), then after the additional girls join, there must be 16 players in total for $\frac{3}{8}$ of the players to be boys. Since there were 8 players initially, then 8 additional girls must have joined the game.

Solution 2
Let the number of additional girls be g.
Then

$$\frac{2+g}{8+g} = \frac{5}{8}$$
$$16 + 8g = 40 + 5g$$
$$3g = 24$$
$$g = 8$$

ANSWER: (D)

11. Since N is written out as a sum of powers of 10, then N can be written as $1\,111\,111\,000$, and so the sum of the digits is 7.

ANSWER: (E)

12. *Solution 1*

To get from point B to point C, we go to the left 12 units and up 4. Therefore, for each unit up, we have gone 3 to the left.

To get from B to A, we must go up 1, and since A is on BC, then we must have gone 3 to the left from 9, ie. $a = 9 - 3 = 6$.

Solution 2

Since the three points lie on the same line then

$$\text{Slope of } AB = \text{Slope of } BC$$

$$\frac{1-0}{a-9} = \frac{0-4}{9-(-3)}$$

$$\frac{1}{a-9} = \frac{-4}{12}$$

$$12 = -4a + 36$$

$$4a = 24$$

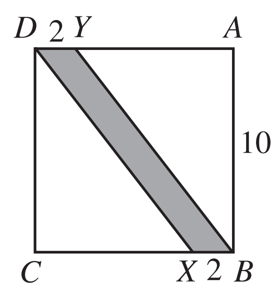
$$a = 6$$

ANSWER: (D)

13. *Solution 1*

Since $AY = CX = 8$, then we must have $DY = BX = 2$, and so $DYBX$ is a parallelogram.

If we rotate the picture by 90° clockwise, then we can see that $DYBX$ is a parallelogram with a base of length 2 and a height of 10, ie. it has an area of $bh = 2(10) = 20$.



Solution 2

The area of the shaded region is equal to the area of the entire square minus the areas of the two triangles. Each of the two triangles BAY and DCX is right-angled with one leg of length 10 and the other of length 8.

Therefore,

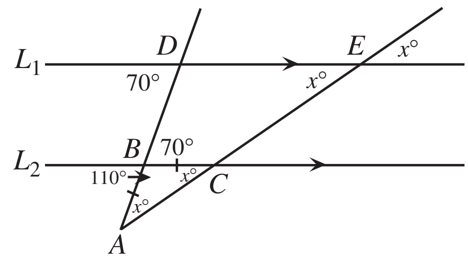
$$\begin{aligned}\text{Area of shaded region} &= \text{Area of square} - \text{Area of triangles}\\ &= (10)^2 - 2\left[\tfrac{1}{2}(8)(10)\right]\\ &= 100 - 2[40]\\ &= 20\end{aligned}$$

ANSWER: (B)

14. Since the distance covered by Jim in 4 steps is the same as the distance covered by Carly in 3 steps, then the distance covered by Jim in 24 steps is the same as the distance covered by Carly in 18 steps.
Since each of Carly's steps covers 0.5 m, she then covers 9 m in 18 steps, ie. Jim covers 9 m in 24 steps.

ANSWER: (B)

15. We label two more points on the diagram, as shown.
Then $\angle DEC = x^\circ$, since it is equal to its opposite angle.
Since line L_1 is parallel to line L_2, then $\angle DBC = 70^\circ$, since DB is a transversal, and $\angle BCA = x^\circ$ since EC is a transversal.
Since $\angle DBC = 70^\circ$, then $\angle ABC = 110^\circ$, since these angles are supplementary.
Since ΔABC is isosceles, then $\angle BAC = \angle BCA = x^\circ$, and so looking at the sum of the angles in ΔABC, we get



$$x^\circ + 110^\circ + x^\circ = 180^\circ$$
$$2x = 70$$
$$x = 35$$

ANSWER: (A)

16. Using exponent laws to write all of the factors as product of powers of 2 and 3,

$$\frac{\left(4^{2003}\right)\left(3^{2002}\right)}{\left(6^{2002}\right)\left(2^{2003}\right)} = \frac{\left(\left(2^2\right)^{2003}\right)\left(3^{2002}\right)}{\left((2\cdot 3)^{2002}\right)\left(2^{2003}\right)} = \frac{\left(2^{4006}\right)\left(3^{2002}\right)}{\left(2^{2002}\right)\left(3^{2002}\right)\left(2^{2003}\right)} = \frac{\left(2^{4006}\right)\left(3^{2002}\right)}{\left(2^{4005}\right)\left(3^{2002}\right)} = 2^{4006-4005} = 2^1 = 2$$

ANSWER: (B)

17. Since the largest circle has a radius of 4, its area is $\pi\left(4^2\right) = 16\pi$.
We must calculate the area of each of the shaded regions.
The innermost shaded region is a circle of radius 1, and so it has area $\pi\left(1^2\right) = \pi$.
The outermost shaded region is the region inside a circle of radius 3 and outside a circle of radius 2. Therefore its area is the *difference* between the areas of these two circles, or $\pi\left(3^2\right) - \pi\left(2^2\right) = 5\pi$.
Therefore, the total area of the shaded regions is $\pi + 5\pi = 6\pi$, and the required ratio is $6\pi : 16\pi = 6 : 16 = 3 : 8$.

ANSWER: (E)

18. Since 496 is less than 2^m, we might think to look for a power of 2 bigger than, but close to 496. $2^9 = 512$ works and in fact $496 = 512 - 16 = 2^9 - 2^4$ and so $m + n = 9 + 4 = 13$.
(This can also be done using an algebraic approach.)

ANSWER: (A)

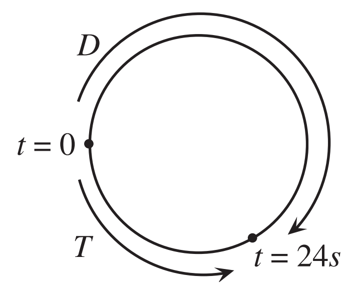
19. Suppose that the four digit number has digits *abcd*, ie. the product $abcd = 810$.
We must determine how to write 810 as the product of 4 *different* digits, none of which can be 0. So we must start by factoring 810, as $810 = 81 \times 10 = 3^4 \times 2 \times 5$.
So one of the digits must have a factor of 5. But the only non-zero digit having a factor of 5 is 5 itself, so 5 is a digit of the number.
Now we need to find 3 different digits whose product is $3^4 \times 2$.
The only digits with a factor of 3 are 3, 6, and 9, and since we need 4 factors of 3, we must use each of these digits (the 9 contributes 2 factors of 3; the others contribute 1 each). In fact, $3 \times 6 \times 9 = 3^4 \times 2 = 162$.
Therefore, the digits of the number are 3, 5, 6, and 9, and so the sum of the digits is 23.

ANSWER: (C)

20. The cost to modify the car's engine ($400) is the equivalent of the cost of $\frac{400}{0.80} = 500$ litres of gas. So the car would have to be driven a distance that would save 500 L of gas in order to make up the cost of the modifications.
Originally, the car consumes 8.4 L of gas per 100 km, and after the modifications the car consumes 6.3 L of gas per 100 km, a savings of 2.1 L per 100 km.
Thus, in order to save 500 L of gas, the car would have to be driven $\frac{500}{2.1} \times 100 = 23809.52$ km.

ANSWER: (D)

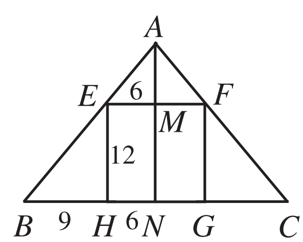
21. Let's say that time equals 0 seconds when Troye and Daniella first meet. Then at time 24 seconds, they will meet again.
In 24 seconds, how far does Troye get around the track? Since it takes her 56 seconds to complete one lap, then she has made it $\frac{24}{56} = \frac{3}{7}$ of the way around the track.



Since Daniella is running in the opposite direction, then she will go $\frac{4}{7}$ of the way around the track in 24 seconds, and so one complete lap will take her $\frac{7}{4}(24) = 42$ seconds.

ANSWER: (E)

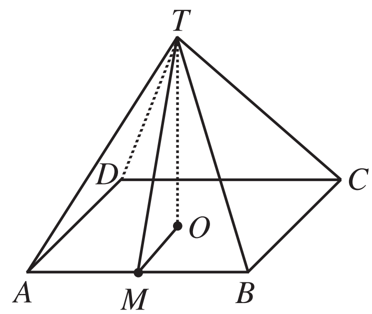
22. Let *M* be the midpoint of *EF* and *N* be the midpoint of *HG*. By symmetry, *N* is also the midpoint of *BC*. Also, the line through *A* and *M* will also pass through *N*, and will be perpendicular to both *EF* and *BC*.
Since the side length of the square is 12, then $EM = HN = 6$ and $EH = 12$.
Since we are told that $BC = 30$, then $BN = 15$ and so $BH = 9$.
Since *EFGH* is a square, then *EF* is parallel to *HG*, and so $\angle AEM = \angle EBH$, ie. ΔAME is similar to ΔEHB.
Therefore, $\frac{AM}{6} = \frac{12}{9}$ or $AM = 8$.



Thus, the area of ΔAEF is $\frac{1}{2}(12)(8) = 48 \text{ cm}^2$.

ANSWER: (D)

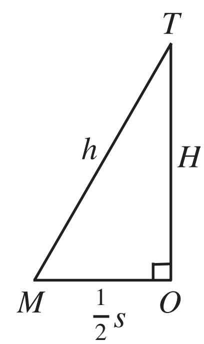
23. We label the pyramid with vertices A, B, C, and D (the square base) and T the "top" vertex. Let M be the midpoint of side AB on the base, and O the centre of the square base.
Since the pyramid has a square base and each of the four triangular faces is identical, then the "top" vertex of the pyramid lies directly above the centre of the base, by symmetry, and so each of the four triangular faces is isosceles.
Join T to O, T to M, and M to O.
Then TO is perpendicular to the square base by the symmetry of the pyramid, and so is perpendicular to OM.
Therefore, triangle TOM is right-angled at O.
Let H be the height of the pyramid, ie. $TO = H$.
Let s be the side length of the base of the pyramid. Then $MO = \frac{1}{2}s$, since O is the centre of the square and M is the midpoint of AB.



Let h be the length of MT. Since M is the midpoint of AB and ΔTAB is isosceles, then TM is perpendicular to AB.
So by Pythagoras, $H^2 = h^2 - \left(\frac{1}{2}s\right)^2 = h^2 - \frac{1}{4}s^2$.
But the base is square, so its area is $s^2 = 1440$, and the area of each of the triangular faces is $\frac{1}{2}sh = 840$, so

$$h^2 = \left(\frac{1680}{s}\right)^2 = \frac{1680^2}{1440} = 1960.$$

Therefore, $H^2 = 1960 - 360 = 1600$, and so $H = 40$.



ANSWER: (B)

24. Since we are looking at choosing four different numbers from the set $\{0, 1, 2, ..., 9\}$, then there is only one way to write them in increasing order. So we only need to look at the number of ways of choosing four numbers so that their sum is a multiple of 3 (that is, we do not need to worry about looking at the order of the choices).
If we take four numbers and add them up, then the fact that the sum is divisible by 3 (or not) is not affected when we subtract a multiple of 3 from any of the four numbers, since the difference between multiples of 3 is a multiple of 3.
Next, we can use the fact that every number can be written as a multiple of 3, or as one more or one less than a multiple of 3, ie. every integer can be written in the form $3n$, $3n+1$ or $3n-1$.
So combining these two facts, we can transform the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ into to the collection $\{0, 1, -1, 0, 1, -1, 0, 1, -1, 0\}$, for example by subtracting 6 from 5 to get –1. (A "set" cannot technically have more than one copy of the same element, whereas a "collection" can!)

So now we want to choose 4 numbers from the collection $\{0, 1, -1, 0, 1, -1, 0, 1, -1, 0\}$ whose sum is a multiple of 3 (including possibly 0).
How can we do this?
If we choose 4 zeros, then the sum is $0+0+0+0=0$, which is a multiple of 3.
If we choose 3 zeros, then the remaining number chosen is a 1 or a –1, so the sum is not a multiple of 3.
If we choose 2 zeros, then we can choose two 1's, two –1's, or 1 and –1. Only the third choice gives a multiple of 3.
If we choose 1 zero, then to get a multiple of 3, we must choose three 1's or three –1's. (You might want to check that no other combination works!)
If we choose 0 zeros, then to get a multiple of 3, we must choose two 1's and two –1's (otherwise we choose three of one kind and one of the other, which will not give a multiple of 3).
So now we must count the number of choices for each case:

Case 1: 0, 0, 0, 0

Since there are only four zeros, there is only 1 way to choose them. (Recall that this corresponds to choosing 0, 3, 6 and 9, whose sum is indeed divisible by 3.)

Case 2: 0, 0, 1, –1

We must choose two zeros from four zeros, and one each of 1's and –1's from collections of three.
If we have 4 objects A, B, C, D then the number of ways of choosing 2 objects is 6 (AB, AC, AD, BC, BD, CD), and if we have 3 objects, then the number of ways of choosing 1 object is 3. (This just means that there are 6 ways to choose 2 zeros from 4 possibilities.)
So the total number of choices here is $6 \times 3 \times 3 = 54$, since *for each* choice of two 0's, we have 3 choices for the 1, and 3 choices for the –1.

Case 3: 0, 1, 1, 1

We must choose one 0 from four zeros, and three 1's from 3. There are 4 ways to choose the zero and 1 way to choose the three 1's. Thus there are a total of 4 ways of making this selection.

Case 4: 0, –1, –1, –1

Similarly to Case 3, there are 4 possibilities.

Case 5: 1, 1, –1, –1

We must choose two 1's from three, and two –1's from three. There are 3 ways to make each of these choices, or $3 \times 3 = 9$ ways in total.

Therefore, there are $1 + 54 + 4 + 4 + 9 = 72$ ways in total of choosing the numbers.

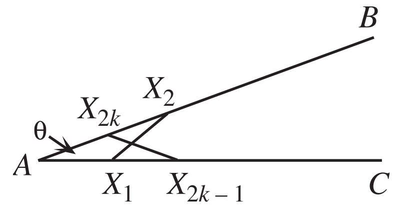
ANSWER: (E)

25. Suppose that the angle θ is an acute laceable angle with $2k$ points in the lacing. We need to determine what values of θ are possible.
First, we can note that the diagram must be symmetrical, since

$$AX_1 = X_1X_2 = X_{2k-1}X_{2k} = X_{2k}A$$

and so the two triangles ΔAX_1X_2 and $\Delta AX_{2k}X_{2k-1}$ are isosceles with equal base angles and equal legs, and thus congruent, so $AX_2 = AX_{2k-1}$.

Continuing this, we can show that each pair of corresponding points on the rays AB and AC are the same distance from A.



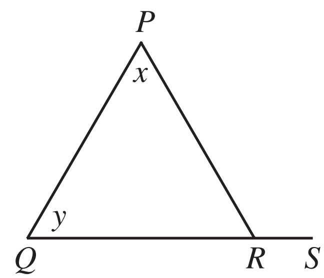
In particular, $AX_k = AX_{k+1}$. Thus ΔAX_kX_{k+1} is isosceles and so $\angle AX_kX_{k+1} = \angle AX_{k+1}X_k = \frac{1}{2}\left(180^\circ - \theta\right)$.

Next, we will develop a second expression for one of these two angles involving θ.

To do this, we need the fact that if we know two angles of a triangle, then we can calculate the "external angle" of the triangle, ie. in the diagram, $\angle PRS = x + y$, since $\angle PRS = 180^\circ - \angle PRQ = 180^\circ - \left(180^\circ - x - y\right) = x + y$.

Since ΔAX_1X_2 is isosceles, then $\angle X_1AX_2 = \angle AX_2X_1 = \theta$ and so by the external angle, $\angle X_2X_1C = 2\theta$.

Since $\Delta X_1X_2X_3$ is isosceles, then $\angle X_2X_1X_3 = \angle X_2X_3X_1 = 2\theta$, and so by the external angle in ΔAX_2X_3, $\angle X_3X_2C = 3\theta$.



Continuing in this way, $\angle X_3X_2X_4 = \angle X_3X_4X_2 = 3\theta$, and so on, eventually reaching

$$\angle X_kX_{k-1}X_{k+1} = \angle X_kX_{k+1}X_{k-1} = k\theta.$$

(Try this out in the diagram given in the problem.)

Therefore, comparing our two equations for angle $\angle AX_{k+1}X_k = \angle X_{k-1}X_{k+1}X_k$, we obtain

$$\frac{1}{2}\left(180^\circ - \theta\right) = k\theta$$

$$180^\circ = 2k\theta + \theta$$

$$\theta = \frac{180^\circ}{2k+1}$$

Since θ is an integer, $2k+1$ must be a divisor (an odd divisor that is at least 3) of 180.

The divisors of 180 are 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, and 180, with odd divisors 1, 3, 5, 9, 15, and 45.

Therefore, ignoring the 1, there are 5 possibilities for θ to be a laceable acute angle, namely 60°, 36°, 20°, 12°, and 4°.

ANSWER: (C)

Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2002 Solutions
Cayley Contest (Grade 10)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Awards

1. Expanding and simplifying,

$$\begin{aligned} 5x+2(4+x) &= 5x+(8+2x) \\ &= 7x+8 \end{aligned}$$

ANSWER: (C)

2. Evaluating,

$$\begin{aligned} (2+3)^2-\left(2^2+3^2\right) &= 5^2-(4+9) \\ &= 25-13 \\ &= 12 \end{aligned}$$

ANSWER: (A)

3. If $x=-3$,

$$\begin{aligned} x^2-4(x-5) &= (-3)^2-4(-3-5) \\ &= 9-4(-8) \\ &= 41 \end{aligned}$$

ANSWER: (D)

4. Since $n=\frac{5}{6}(240)$, then $\frac{2}{5}n=\frac{2}{5}\left(\frac{5}{6}\right)(240)=\frac{1}{3}(240)=80$. ANSWER: (B)

5. Using exponent laws,

$$2^{-2}\times 2^{-1}\times 2^{0}\times 2^{1}\times 2^{2}=2^{-2-1+0+1+2}=2^{0}=1.$$

Alternatively,

$$2^{-2}\times 2^{-1}\times 2^{0}\times 2^{1}\times 2^{2}=\frac{1}{4}\times\frac{1}{2}\times 1\times 2\times 4=1.$$

ANSWER: (B)

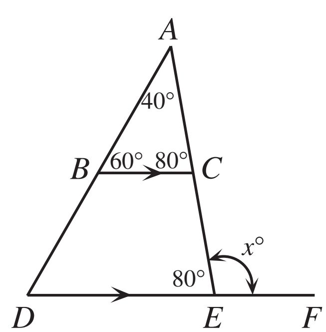
6. In ΔABC,

$$\begin{aligned} \angle ACB+40^\circ+60^\circ &= 180^\circ \\ \angle ACB &= 80^\circ \end{aligned}$$

Since BC is parallel to DE, $\angle AED=\angle ACB=80^\circ$.

So

$$\begin{aligned} x^\circ &= 180^\circ-\angle AED=100^\circ \\ x &= 100 \end{aligned}$$



ANSWER: (D)

7. Since the line has slope $\frac{1}{2}$, then for every 2 units we move to the right, the line rises by 1 unit. Therefore, $(-2+2,4+1)=(0,5)$ lies on the line. Thus the y-intercept is 5.

ANSWER: (A)

8. Since Megan's scoring average was 18 points per game after 3 games, she must have scored $3 \times 18 = 54$ points over her first three games. Similarly, she must have scored $4 \times 17 = 68$ points over her first 4 games. So in the fourth game, she scored $68 - 54 = 14$ points.

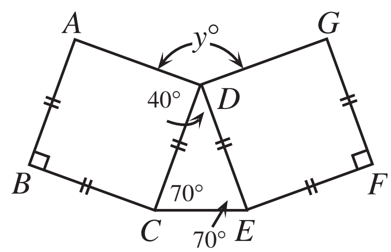
ANSWER: (E)

9. Since squares $ABCD$ and $DEFG$ have equal side lengths, then $DC = DE$, ie. ΔCDE is isosceles. Therefore, $\angle DEC = \angle DCE = 70^\circ$ and so

$$\angle CDE = 180^\circ - 70^\circ - 70^\circ = 40^\circ$$

and

$$y^\circ = 360^\circ - \angle ADC - \angle CDE - \angle GDE$$
$$y^\circ = 360^\circ - 90^\circ - 40^\circ - 90^\circ$$
$$y^\circ = 140^\circ$$
$$y = 140$$



ANSWER: (E)

10. Let the original number be x. Then

$$\frac{x-5}{4} = \frac{x-4}{5}$$
$$5(x-5) = 4(x-4)$$
$$5x - 25 = 4x - 16$$
$$x = 9$$

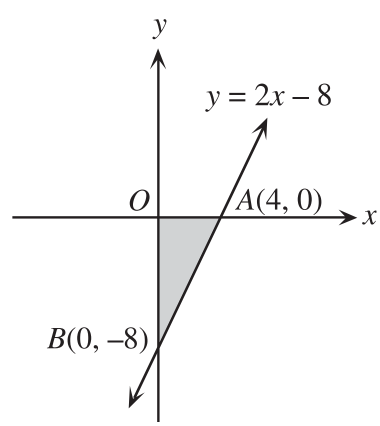
ANSWER: (C)

11. Point B is the y-intercept of $y = 2x - 8$, and so has coordinates $(0,-8)$ from the form of the line. Point A is the x-intercept of $y = 2x - 8$, so we set $y = 0$ and obtain

$$0 = 2x - 8$$
$$2x = 8$$
$$x = 4$$

and so A has coordinates $(4,0)$. This tells us that

$$\text{Area of } \Delta AOB = \tfrac{1}{2}(OA)(OB) = \tfrac{1}{2}(4)(8) = 16.$$



ANSWER: (B)

12. After the price is increased by 40%, the new price is 140% of \$10.00, or \$14.00. Then after this price is reduced by 30%, the final price is 70% of \$14.00, which is $0.7(\$14.00) = \$9.80$.

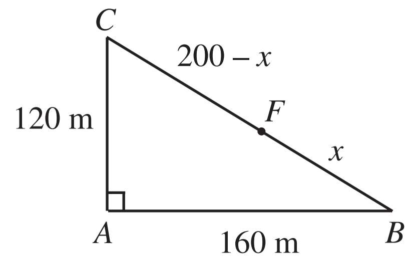
ANSWER: (A)

13. By Pythagoras,

$$BC^2 = 120^2 + 160^2 = 40\,000$$

so $BC = 200$ m.

Let $FB = x$. Then $CF = 200 - x$. From the given information,



$$\begin{aligned} AC + CF &= AB + BF \\ 120 + 200 - x &= 160 + x \\ 160 &= 2x \\ x &= 80 \end{aligned}$$

Thus, the distance from F to B is 80 m.

ANSWER: (D)

14. *Solution 1*

We factor out a common factor of a from the first two terms and a common factor of b from the last two terms, and so

$$\begin{aligned} ac + ad + bc + bd &= 68 \\ a(c+d) + b(c+d) &= 68 \\ a(4) + b(4) &= 68 \\ 4(a+b) &= 68 \\ a + b &= 17 \end{aligned}$$

using the fact that $c + d = 4$.

Then $a + b + c + d = (a+b) + (c+d) = 17 + 4 = 21$.

Solution 2

We let $c = d = 2$, since $c + d = 4$, and then substitute these values to find that

$$\begin{aligned} ac + ad + bc + bd &= 68 \\ 2a + 2a + 2b + 2b &= 68 \\ a + b &= 17 \end{aligned}$$

Thus $(a+b) + (c+d) = 17 + 4 = 21$.

ANSWER: (D)

15. From A, we can travel to D, E or B.

If we travel $A \to D$, we must then go $D \to E \to F$, following the arrows.

If we travel $A \to E$, we must then go $E \to F$.

If we travel $A \to B$, we can then travel from B to E, C or F.

From E or C, we must travel directly to F.

Thus, there are 5 different paths from A to F.

ANSWER: (B)

16. Let the four consecutive positive integers be n, $n+1$, $n+2$, $n+3$.

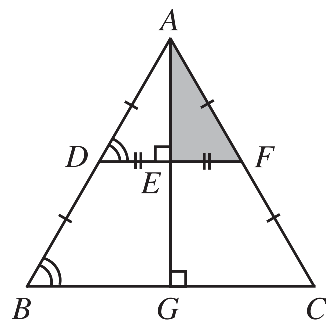
So we need to solve the equation $n(n+1)(n+2)(n+3)=358\ 800$.
Now $n(n+1)(n+2)(n+3)$ is approximately equal to n^4, so n^4 is close to $358\ 800$, and so n is close to $\sqrt[4]{358\ 800}\approx 24.5$.
So we try $24(25)(26)(27)=421\ 200$ which is too big, and so we try $23(24)(25)(26)=358\ 800$, which is the value we want.
So the sum is $23+24+25+26=98$.

ANSWER: (B)

17. A "double-single" number is of the form *aab*, where *a* and *b* are different digits. There are 9 possibilities for *a* (since *a* cannot be 0). For each of these possibilities, there are 9 possibilities for *b* (since *b* can be any digit from 0 to 9, provided that it doesn't equal *a*). So there are $9\times 9=81$ possibilities in total.

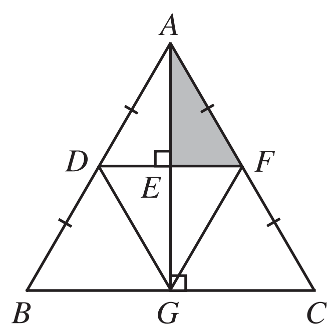
ANSWER: (A)

18. *Solution 1*



Since ΔADF and ΔABC share $\angle DAF$ and
$$\frac{AD}{AB}=\frac{AF}{AC}=\frac{1}{2}$$
then ΔADF is similar to ΔABC. So the ratio of their areas is the square of the ratio of their side lengths, that is
Area of $\Delta ADF=\left(\frac{1}{2}\right)^2\times$ Area of ΔABC.
Also from the similar triangles, $\angle ADF=\angle ABC$, so *DF* is parallel to *BC*. Therefore, since *AG* is perpendicular to *BC*, then *AG* is also perpendicular to *DE*, so *AE* is an altitude in ΔADF. Since ΔADF is isosceles, $DE=EF$.
Therefore,
$$\text{Area of } \Delta ADF=\tfrac{1}{2}\times\text{Area of }\Delta ADF$$
$$=\tfrac{1}{8}\times\text{Area of }\Delta ABC$$

Solution 2



Join *G* to *D* and *G* to *F*. This implies that we have four identical isosceles triangles: ΔADF, ΔGDF, ΔBDG, and ΔCFG. Each of these identical triangles has an equal area.
Thus,
$$\text{Area of }\Delta AEF=\tfrac{1}{2}\left(\tfrac{1}{4}\times\text{Area of }\Delta ABC\right)$$
$$=\tfrac{1}{8}\times\text{Area of }\Delta ABC$$

ANSWER: (E)

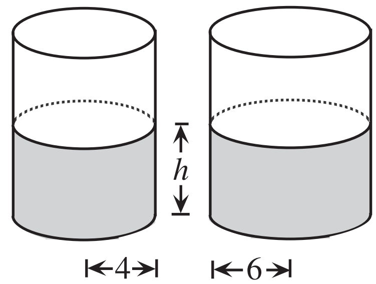
19. Let $x=777\ 777\ 777\ 777\ 777$ and $y=222\ 222\ 222\ 222\ 223$.

So the required quantity is $x^2 - y^2 = (x+y)(x-y)$.
Now $x + y = 1\,000\,000\,000\,000\,000$ and $x - y = 555\ 555\ 555\ 555\ 554$.
Thus, $x^2 - y^2 = 555\ 555\ 555\ 555\ 554\,000\,000\,000\,000\,000$, ie. the sum of the digits is $14 \times 5 + 4 = 74$.

ANSWER: (C)

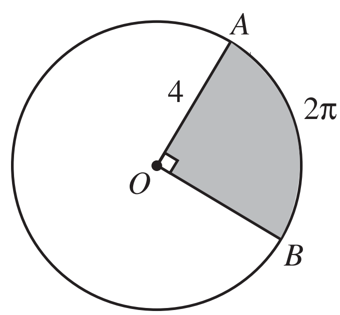
20. Let the final depth of the water be h.
The total initial volume of water is
$\pi(4\text{ m})^2(10\text{ m}) = 160\pi\text{ m}^3$.
When the depths of water are equal,

$$\pi(4\text{ m})^2 h + \pi(6\text{ m})^2 h = 160\pi\text{ m}^3$$
$$\left(52\pi\text{ m}^2\right)h = 160\pi\text{ m}^3$$
$$h = \frac{160}{52}\text{ m}$$
$$h = \frac{40}{13}\text{ m}$$



ANSWER: (E)

21. Let r be the radius of the circle.
Since $\angle AOB = 90^\circ$, then this sector is one quarter of the whole circle, so the circumference of the circle is $4 \times 2\pi = 8\pi$.
So $2\pi r = 8\pi$, or $r = 4$.
Thus the area of sector AOB is $\frac{1}{4}\pi r^2 = \frac{1}{4}\pi(16) = 4\pi$.



ANSWER: (A)

22. When we add up a number of consecutive integers, the sum of these integers will be equal to the number of integers being added times the average of the integers being added.
Let N be the number of consecutive integers being added, and let A be the average of the consecutive integers being added. Thus, $NA = 75$.
Before we determine the possibilities for N, we make the observation that N must be less than 12, since the sum of the first 12 positive integers is 78, and thus the sum of any 12 or more consecutive positive integers is at least 78.

<u>Case 1</u>: N is odd.

In this case, A (the average) is an integer (there is a "middle number" among the integers being added). Therefore, since $NA = 75$, N must be an odd positive factor of 75 which is bigger than 1 and less than 12, ie. N is one of 3 or 5. So there are two possibilities when N is odd, namely $24+25+26=75$ and $13+14+15+16+17=75$.

<u>Case 2</u>: N is even.

In this case A will be half-way between two integers.

Set $N = 2k$ and $A = \frac{2l+1}{2}$, where k and l are integers.

Then

$$2k\left(\frac{2l+1}{2}\right) = 75$$
$$k(2l+1) = 75$$

Thus k is a factor of 75, and so $N = 2k$ is 2 times a factor of 75, ie. N could be 2, 6 or 10. So the possibilities here are $37+38=75$, $10+11+12+13+14+15=75$, and $3+4+5+6+7+8+9+10+11+12=75$.

So there are 5 ways in total.

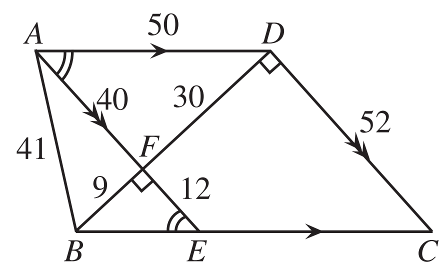
ANSWER: (E)

23. By Pythagoras in ΔAFB,

$$AF^2 + 9^2 = 41^2$$
$$AF = 40$$

By Pythagoras in ΔAFD,

$$FD^2 + 40^2 = 50^2$$
$$FD = 30$$



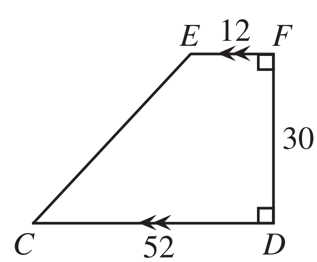
Since AD is parallel to BC, $\angle FAD = \angle FEB$. Therefore, ΔAFD is similar to ΔEFB, so $\frac{AF}{FD} = \frac{EF}{FB}$ or $\frac{40}{30} = \frac{EF}{9}$ or $EF = 12$.

Since AE is perpendicular to BD and DC is perpendicular to BD, then AE is parallel to DC, so $AECD$ is a parallelogram. Thus, $DC = AE = 52$.

Now consider quadrilateral $FECD$.

This quadrilateral is a trapezoid, so

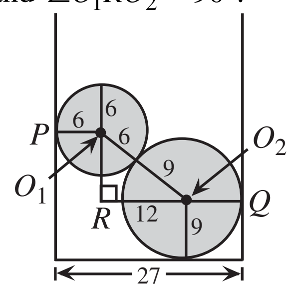
$$\text{Area of } FECD = \tfrac{1}{2}(EF+CD)(FD)$$
$$= \tfrac{1}{2}(12+52)(30)$$
$$= 960$$



ANSWER: (C)

24. We examine a vertical cross-section of the cylinder and the spheres that passes through the vertical axis of the cylinder and the centres of the spheres. (There is such a cross-section since the spheres will be pulled into this position by gravity.)
Let the centres of the spheres be O_1 and O_2, as shown.
Join the centres of the spheres to each other and to the respective points of tangency of the the spheres to the walls of the cylinder.
Then $O_1O_2 = 6+9 = 15$ (O_1O_2 passes through the point of tangency of the two spheres), $O_1P = 6$ and $O_2Q = 9$, using the radii of the spheres.
Next, draw ΔO_1RO_2 so that RO_1 is perpendicular to O_1P and $\angle O_1RO_2 = 90^\circ$.
Then looking at the width of the cylinder, since $PO_1 = 6$ and $O_2Q = 9$, we have

$$PO_1 + RO_2 + O_2Q = 27$$
$$RO_2 = 12$$



By Pythagoras in ΔO_1RO_2, we see that $O_1R = 9$.
Then the depth of the water will be

Radius of lower sphere $+ RO_1 +$ Radius of higher sphere $= 9+9+6 = 24$.

So

$$\text{Volume of water} = (\text{Volume of cylinder to height of } 24) - (\text{Volume of spheres})$$
$$= \pi\left(\tfrac{27}{2}\right)^2(24) - \tfrac{4}{3}\pi(6)^3 - \tfrac{4}{3}\pi(9)^3$$
$$= 4374\pi - 288\pi - 972\pi$$
$$= 3114\pi$$

Therefore, the volume of water required is 3114π cubic units.

ANSWER: (D)

25. Subtracting the first equation from the second,

$$k^2x - kx - 6 = 0$$
$$\left(k^2 - k\right)x = 6$$
$$[k(k-1)]x = 6$$

Since we want both k and x to be integers, then $k(k-1)$ is a factor of 6, ie. is equal to one of $\pm1, \pm2, \pm3, \pm6$. Now $k(k-1)$ is the product of two consecutive integers, so on this basis we can eliminate six of these eight possibilities to obtain

$k(k-1) = 2$ or $k(k-1) = 6$

which yields

$k^2 - k - 2 = 0$ or $k^2 - k - 6 = 0$

and so $k = 2, -1, 3, -2$.

We now make a table of values of k, x and y to check when y is also an integer. (We note that from the first equation, $y = \frac{1}{5}(kx + 7)$.)

k	x	y
2	3	$\frac{13}{5}$
-1	3	$\frac{4}{5}$
3	1	2
-2	1	1

So there are two values of k for which the lines intersect at a lattice point.

ANSWER: (B)

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2001 Solutions
Cayley Contest (Grade 10)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Awards

Part A

1. The value of $\dfrac{5(6)-3(4)}{6+3}$ is

(**A**) 1 (**B**) 2 (**C**) 6 (**D**) 12 (**E**) 31

Solution
By evaluating the numerator and denominator we have,

$$\frac{5(6)-3(4)}{6+3}=\frac{30-12}{9}=\frac{18}{9}=2.$$

ANSWER: (B)

2. When $\frac{1}{4}$ of 15 is multiplied by $\frac{1}{3}$ of 10, the answer is

(**A**) 5 (**B**) $\frac{25}{2}$ (**C**) $\frac{85}{12}$ (**D**) $\frac{99}{8}$ (**E**) $\frac{25}{7}$

Solution
Evaluating gives,

$$\left[\tfrac{1}{4}(15)\right]\left[\tfrac{1}{3}(10)\right]$$
$$=\frac{1}{\cancel{4}_{2}}(\cancel{15}^{5})\frac{1}{\cancel{3}_{1}}(\cancel{10}^{5})$$
$$=\frac{25}{2}$$

ANSWER: (B)

3. If $x=\frac{1}{4}$, which of the following has the largest value?

(**A**) x (**B**) x^2 (**C**) $\frac{1}{2}x$ (**D**) $\dfrac{1}{x}$ (**E**) $\sqrt{x}$

Solution
If we calculate the value of the given expressions, we get

(**A**) $\frac{1}{4}$

(**B**) $\left(\frac{1}{4}\right)^2=\frac{1}{16}$

(**C**) $\frac{1}{2}\left(\frac{1}{4}\right)=\frac{1}{8}$

(**D**) $\dfrac{1}{\frac{1}{4}}=1\times 4=4$

(**E**) $\sqrt{\frac{1}{4}}=\frac{1}{2}$

ANSWER: (D)

4. In a school, 30 boys and 20 girls entered the Cayley competition. Certificates were awarded to 10% of the boys and 20% of the girls. Of the students who participated, the percentage that received

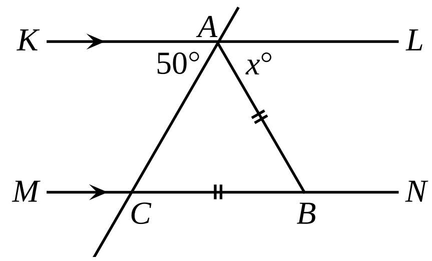
certificates was

(**A**) 14 (**B**) 15 (**C**) 16 (**D**) 30 (**E**) 50

Solution

If 30 boys entered the Cayley competition and 10% of them received certificates, this implies that $(0.1)(30)$ or 3 boys received certificates. Of the 20 girls who entered the competition $(0.2)(20)$ or 4 girls received certificates. This implies that 7 students in total out of 50 received certificates.
Thus 14% of the students in total received certificates. ANSWER: (A)

5. In the diagram, KL is parallel to MN, $AB = BC$, and $\angle KAC = 50°$. The value of x is



(**A**) 40 (**B**) 65 (**C**) 25
(**D**) 100 (**E**) 80

Solution

Since KL is parallel to MN, $\angle ACB$ and $\angle KAC$ are alternate angles. Thus, $\angle ACB = 50°$.
We are given that ΔBCA is isosceles, so $\angle BCA = \angle BAC = 50°$. We know that since angles KAC, BAC and LAB have a sum of $180°$,

$$50° + 50° + x° = 180°$$
$$x = 80.$$

ANSWER: (E)

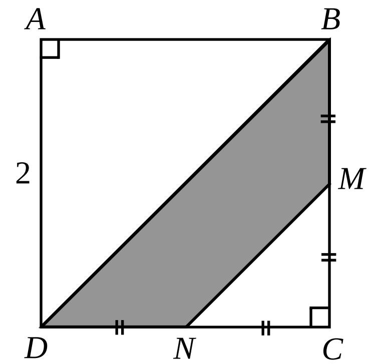
6. Dean scored a total of 252 points in 28 basketball games. Ruth played 10 fewer games than Dean. Her scoring average was 0.5 points per game higher than Dean's scoring average. How many points, in total, did Ruth score?

(**A**) 153 (**B**) 171 (**C**) 180 (**D**) 266 (**E**) 144

Solution

If Dean scored 252 points in 28 games this implies that he averages $\frac{252}{28}$ or 9 points per game.
Ruth must then have averaged 9.5 points in each of the 18 games she played. In total she scored 9.5×18 or 171 points. ANSWER: (B)

7. In the diagram, square $ABCD$ has side length 2, with M the midpoint of BC and N the midpoint of CD. The area of the shaded region $BMND$ is

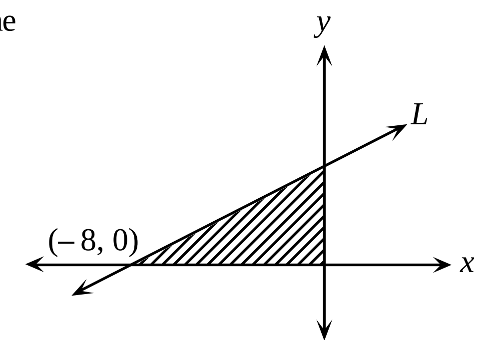


(**A**) 1 (**B**) $2\sqrt{2}$ (**C**) $\frac{4}{3}$
(**D**) $\frac{3}{2}$ (**E**) $4-\frac{3}{2}\sqrt{2}$

Solution
The area of ΔMNC is $\frac{1}{2}(1)(1)=\frac{1}{2}$. Since ΔBDC is half the square, it will have an area of 2.
Since the shaded region has an area equal to that of ΔBDC minus the area of ΔMNC, its area will be $2-\frac{1}{2}=\frac{3}{2}$.

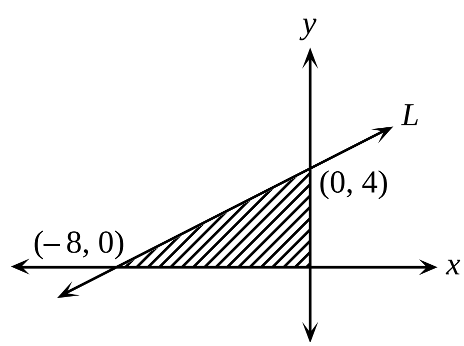
ANSWER: (D)

8. The line L crosses the x-axis at $(-8, 0)$. The area of the shaded region is 16. What is the slope of the line L?



(**A**) $\frac{1}{2}$ (**B**) 4 (**C**) $-\frac{1}{2}$
(**D**) 2 (**E**) -2

Solution
If the area of the shaded region is 16 and its base has a length of 8, its height must then be 4.
Thus we have the changes noted in the diagram.
Thus the slope is $\frac{4-0}{0-(-8)}=\frac{1}{2}$ or $\frac{1}{2}$ because the line slopes down from right to left and the line has a rise of 4 and a run of 8.



Area $=\frac{1}{2}|-8||4|=16$

ANSWER: (A)

9. If $\left[\left(10^3\right)\left(10^x\right)\right]^2=10^{18}$, the value of x is

(**A**) $\sqrt{2}$ (**B**) 12 (**C**) 6 (**D**) 1 (**E**) 3

Solution
If we simplify using laws of exponents, we will have,

$$\left(10^{3+x}\right)^2 = 10^{18}$$
$$10^{6+2x} = 10^{18}.$$

In order that the left and right side be equal, it is necessary that exponents be equal.
Thus, $6+2x=18$
$$2x=12$$
$$x=6.$$

ANSWER: (C)

10. The sum of five consecutive integers is 75. The sum of the largest and smallest of these five integers is

(**A**) 15 (**B**) 25 (**C**) 26 (**D**) 30 (**E**) 32

Solution
There are a variety of ways of approaching this problem. The easiest is to represent the integers as $x-2$, $x-1$, x, $x+1$, and $x+2$.
Thus, $(x-2)+(x-1)+x+(x+1)+(x+2)=75$
$$5x=75$$
$$x=15.$$
The five consecutive integers are 13, 14, 15, 16, and 17.
The required sum is $13+17=30$. ANSWER: (D)

Part B

11. When a positive integer N is divided by 60, the remainder is 49. When N is divided by 15, the remainder is

(**A**) 0 (**B**) 3 (**C**) 4 (**D**) 5 (**E**) 8

Solution
This problem can be done in a number of ways. The easiest way is to consider that if N is divided by 60 to achieve a remainder of 49, it must be a number of the form, $60k+49$, $k=0,1,2,\ldots$.
This implies that the smallest number to meet the requirements is 49 itself. If we divide 49 by 15 we get a remainder of 4. Or, if $k=1$ in our formula then the next number to satisfy the requirements is 109 which when divided by 15 gives 4 as the remainder. ANSWER: (C)

12. The 6 members of an executive committee want to call a meeting. Each of them phones 6 different

people, who in turn each calls 6 other people. If no one is called more than once, how many people will know about the meeting?

(**A**) 18 (**B**) 36 (**C**) 216 (**D**) 252 (**E**) 258

Solution

If 6 people each call 6 other people in the first round of calls, there will be 36 people making 6 calls each for an additional 216 calls. Altogether, there will be the original 6, followed by 36 who in turn phone another 216.

In total, there are $6+36+216=258$. ANSWER: (E)

13. The sequences 3, 20, 37, 54, 71, … and 16, 27, 38, 49, 60, 71, … each have 71 as a common term. The next term that these sequences have in common is

(**A**) 115 (**B**) 187 (**C**) 258 (**D**) 445 (**E**) 1006

Solution

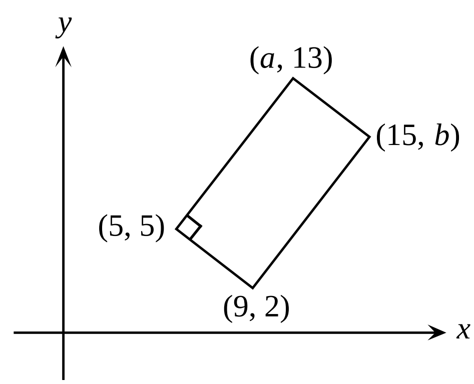
The first sequence increases by a constant value of 17 and the second by a constant value of 11.

After 71, the next common term will be 71 plus the Lowest Common Multiple of 11 and 17. Since the L.C.M. of 11 and 17 is 187 the next term will be $71+187=258$. ANSWER: (C)

14. In the rectangle shown, the value of $a-b$ is

(**A**) -3 (**B**) -1 (**C**) 0

(**D**) 3 (**E**) 1



Solution

To go from the point $(5,5)$ to the point $(9,2)$ we must move over 4 and down 3.

Since we are dealing with a rectangle, the same must be true for $(a,13)$ and $(15,b)$.

Thus, $a+4=15$ and $13-3=b$. From this, $a=11$ and $b=10$. So $a-b=11-10=1$.

ANSWER: (E)

15. A small island has $\frac{2}{5}$ of its surface covered by forest and $\frac{1}{4}$ of the remainder of its surface by sand dunes. The island also has 90 hectares covered by farm land. If the island is made up of only forest, sand dunes and farm land, what is the total area of the island, to the nearest hectare?

(**A**) 163 (**B**) 120 (**C**) 200 (**D**) 138 (**E**) 257

Solution
If $\frac{2}{5}$ of an island is covered by forest then $\frac{3}{5}$ of the island is made up of sand dunes and farm land. Since $\frac{1}{4}\times\frac{3}{5}=\frac{3}{20}$ is made up of sand dunes this implies that $\frac{2}{5}+\frac{3}{20}=\frac{11}{20}$ of the island is made up of forest and sand dunes. Thus $\frac{9}{20}$ of the island, or 90 hectares, is made up of farm land.

Thus, the island must be $\frac{20}{\cancel{9}_{1}} = \cancel{90}^{10}$ or 200 hectares in total. ANSWER: (C)

16. How many integer values of x satisfy $\dfrac{x-1}{3}<\dfrac{5}{7}<\dfrac{x+4}{5}$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

Solution
If we multiply all three fractions by $3(5)(7)$ we have,

$$(\cancel{3})(5)(7)\frac{(x-1)}{\cancel{3}}<(3)(5)(\cancel{7})\frac{5}{\cancel{7}}<(3)(\cancel{5})(7)\frac{x+4}{\cancel{5}}$$

$$35(x-1)<75<21(x+4)$$

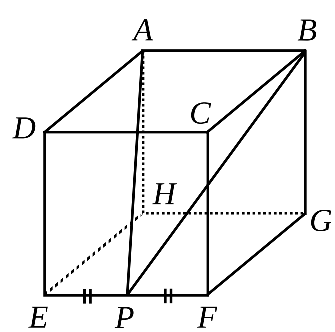
In order to satisfy this inequality then,

$$\begin{array}{rcr} 35(x-1)<75 & \text{and} & 21(x+4)>75 \\ 35x-35<75 & \text{and} & 21x+84>75 \\ 35x<110 & & 21x>-9 \\ x<3\frac{1}{7} & & x>-\frac{9}{21} \end{array}$$

The only integers to satisfy both conditions are then in the set $\{0,1,2,3\}$. ANSWER: (E)

17. $ABCDEFGH$ is a cube having a side length of 2. P is the midpoint of EF, as shown. The area of ΔAPB is

(**A**) $\sqrt{8}$ (**B**) 3 (**C**) 6
(**D**) $\sqrt{2}$ (**E**) $\sqrt{32}$



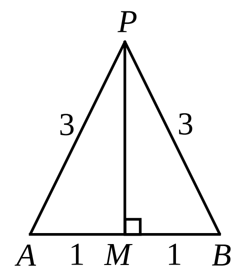
Solution

By symmetry, the lengths of AP and BP will be equal, and $AP = \sqrt{AD^2 + DE^2 + EP^2} = \sqrt{2^2 + 2^2 + 1^2} = 3$.

If M is the midpoint of AB, then PM is perpendicular to AB. By Pythagoras, $MP = \sqrt{3^2 - 1^2} = \sqrt{8}$.

So the area of ΔAPB is

$$\text{Area } = \tfrac{1}{2}(2)\left(\sqrt{8}\right) = \sqrt{8}.$$



ANSWER: (A)

18. How many five-digit positive integers, divisible by 9, can be written using only the digits 3 and 6?

(**A**) 5 (**B**) 2 (**C**) 12 (**D**) 10 (**E**) 8

Solution

If a five-digit number is composed of only 3's and 6's, there are just two cases to consider because these are the only ways that the digital sum can be a multiple of 9.

Case 1 one 6, four 3's

In this case, there are five numbers: $63\,333$, $36\,333$, $33\,633$, $33\,363$ and $33\,336$.

Case 2 one 3, four 6's

In this case there are five numbers: $36\,666$, $63\,666$, $66\,366$, $66\,636$ and $66\,663$.

In total there are 10 possible numbers. ANSWER: (D)

19. Three different numbers are chosen such that when each of the numbers is added to the average of the remaining two, the numbers 65, 69 and 76 result. The average of the three original numbers is

(**A**) 34 (**B**) 35 (**C**) 36 (**D**) 37 (**E**) 38

Solution

Let the three numbers be a, b and c.

We construct the first equation to be,

$$a + \frac{b+c}{2} = 65.$$

Or, $2a + b + c = 130$.

Similarly we construct the two other equations to be,

$$a + 2b + c = 138$$

and $$a + b + 2c = 152.$$

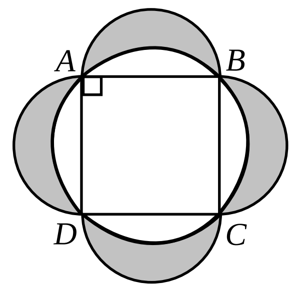
If we add the three equations we obtain,

$$4a + 4b + 4c = 420.$$

The average is $\frac{4(a+b+c)}{12} = \frac{420}{12}$.

Or, $\frac{a+b+c}{3} = 35$. ANSWER: (B)

20. Square $ABCD$ with side length 2 is inscribed in a circle, as shown. Using each side of the square as a diameter, semi-circular arcs are drawn. The area of the shaded region outside the circle and inside the semi-circles is



(**A**) π (**B**) 4 (**C**) $2\pi - 2$
(**D**) $\pi + 1$ (**E**) $2\pi - 4$

Solution

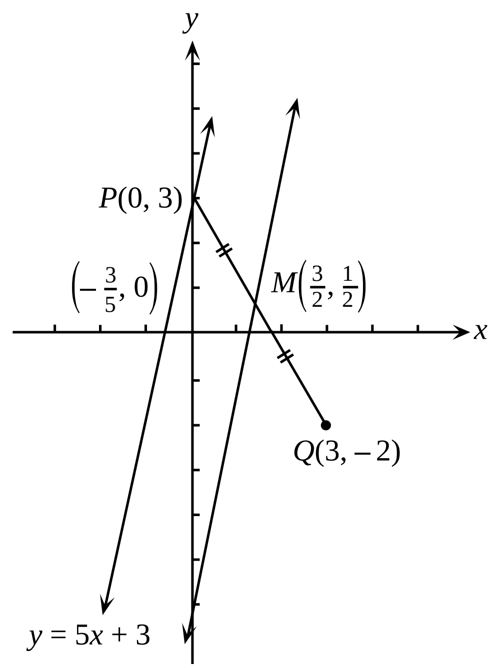
The side length of the square is 2 and thus the diameter of the circle has length $2\sqrt{2}$ which is also the length of the diagonal AC (or BD). The area of the circle is thus $\pi\left(\sqrt{2}\right)^2 = 2\pi$. Since the side length of the square is 2, it will have an area of 4. From this, we calculate the area of the circle outside the square to be $2\pi - 4$. To calculate the shaded area, we first calculate the area of each semi-circle. Each of the semi circles has a radius of 1 meaning that each semi-circle will have an area of $\frac{1}{2}\left[\pi(1)^2\right] = \frac{1}{2}\pi$. In total, the four semi-circles have an area of 2π. Thus the shaded area has an area of $2\pi - (2\pi - 4) = 4$. ANSWER: (B)

Part C

21. Point P is on the line $y = 5x + 3$. The coordinates of point Q are $(3, -2)$. If M is the midpoint of PQ, then M must lie on the line

(**A**) $y = \frac{5}{2}x - \frac{7}{2}$ (**B**) $y = 5x + 1$ (**C**) $y = -\frac{1}{5}x - \frac{7}{5}$ (**D**) $y = \frac{5}{2}x + \frac{1}{2}$ (**E**) $y = 5x - 7$

We start by drawing a diagram and labelling the intercepts.



Solution 1
Since the point P is on the line $y = 5x + 3$, select $P(0, 3)$ as a point on this line.
The midpoint of PQ is $M\left(\frac{3+0}{2}, \frac{-2+3}{2}\right) = M\left(\frac{3}{2}, \frac{1}{2}\right)$.
The required line must contain M and be midway between the given point and $y = 5x + 3$. The only possible line meeting this requirement is the line containing $M\left(\frac{3}{2}, \frac{1}{2}\right)$ and which has a slope of 5. The required line will this have as its equation

$$y - \tfrac{1}{2} = 5\left(x - \tfrac{3}{2}\right)$$

or, $$y = 5x - 7.$$

Solution 2
Let a general point on the line $y = 5x + 3$ be represented by $(a, 5a + 3)$. Also, let a point on the required line be $M(x, y)$. Since $M(x, y)$ is the midpoint of PQ then

(1) $x = \dfrac{a+3}{2}$ and (2) $y = \dfrac{(5a+3)+(-2)}{2}$

$$y = \frac{5a+1}{2}$$

Solving (1) for a, we have $a = 2x - 3$ and solving (2) for a, we have $\dfrac{2y-1}{5} = a$.

Equating gives, $2x - 3 = \dfrac{2y-1}{5}$

$$10x - 15 = 2y - 1$$

or, $$y = 5x - 7.$$

ANSWER: (E)

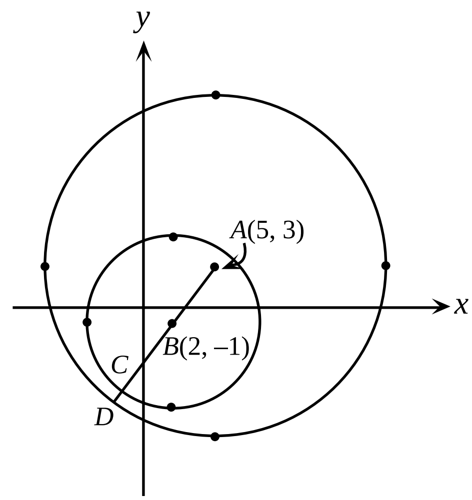
22. What is the shortest distance between two circles, the first having centre $A(5, 3)$ and radius 12, and the other with centre $B(2, -1)$ and radius 6?

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution

We start by drawing the two circles where the larger circle has centre $A(5,3)$ and the smaller circle has centre $B(2,-1)$. A line is drawn from *A*, through *B* to meet the circumference of the smaller circle at *C* and the circumference of the larger circle at *D*. The length *CD* is the desired length. The length from *A* to *D* is given to be 12 and the length from *B* to *C* is 6. We calculate the length of *AB* to be, $\sqrt{[3-(-1)]^2+(5-2)^2}=\sqrt{16+9}=5$. To find *CD*, we calculate as follows,

$$\begin{aligned} CD &= AD-(AB+BC) \\ &= 12-(5+6) \\ &= 1. \end{aligned}$$



ANSWER: (A)

23. A sealed bottle, which contains water, has been constructed by attaching a cylinder of radius 1 cm to a cylinder of radius 3 cm, as shown in Figure A. When the bottle is right side up, the height of the water inside is 20 cm, as shown in the cross-section of the bottle in Figure B. When the bottle is upside down, the height of the liquid is 28 cm, as shown in Figure C. What is the total height, in cm, of the bottle?

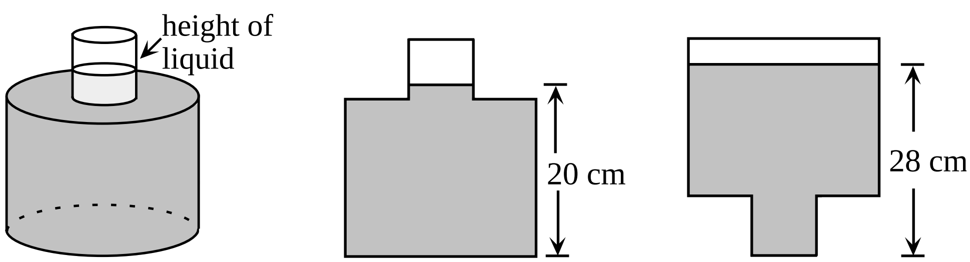


Figure A Figure B Figure C

(**A**) 29 (**B**) 30 (**C**) 31 (**D**) 32 (**E**) 48

Solution

We'll start by representing the height of the large cylinder as h_1 and the height of the small cylinder as h_2. For simplicity, we'll let $x=h_1+h_2$.

If the bottom cylinder is completely filled and the top cylinder is only partially filled the top cylinder will have a cylindrical space that is not filled. This cylindrical space will have a height equal to $x-20$ and a volume equal to, $\pi(1)^2(x-20)$.

Similarly, if we turn the cylinder upside down there will be a cylindrical space unfilled that will have a height equal to $x-28$ and a volume equal to, $\pi(3)^2(x-28)$.

Since these two unoccupied spaces must be equal, we then have,

$$\begin{aligned}\pi(1)^2(x-20) &= \pi(3)^2(x-28)\\ x-20 &= 9x-252\\ 8x &= 232\\ x &= 29.\end{aligned}$$

Therefore, the total height is 29. ANSWER: (A)

24. A palindrome is a positive integer whose digits are the same when read forwards or backwards. For example, 2882 is a four-digit palindrome and 49194 is a five-digit palindrome. There are pairs of four-digit palindromes whose sum is a five-digit palindrome. One such pair is 2882 and 9339. How many such pairs are there?

(**A**) 28 (**B**) 32 (**C**) 36 (**D**) 40 (**E**) 44

Solution

Our first observation is that since we are adding two four-digit palindromes to form a five-digit palindrome then the following must be true,

$$\begin{array}{r} a\,b\,b\,a \\ \underline{c\,d\,d\,c} \\ 1\,e\,f\,e\,1 \end{array}.$$

(i.e. the first digit of the 5-digit palindrome is 1.)

From this, we can see that $a+c=11$ since $a+c$ has a units digit of 1 and $10<a+c<20$.

We first note that there are four possibilities for *a* and *c*. We list the possibilities:

a	2	3	4	5
c	9	8	7	6

Note that there are only four possibilities here.

(If we extended this table, we would get an additional four possibilities which would be duplicates of these four, with *a* and *c* reversed.)

Let us consider one case, say $a=2$ and $c=9$.

$$\begin{array}{r} 2\,b\,b\,2 \\ \underline{9\,d\,d\,9} \\ 1\,e\,f\,e\,1 \end{array}$$

From this, we can only get palindromes in two ways. To see this we note that *e* is either 1 or 2 depending on whether we get a carry from the previous column (we see this looking at the thousands digit of 1 *e f e* 1). If $e=1$, then $b+d$ has no carry and so looking at the tens digit $e=1$, we see that $b+d=0$ to get this digit.

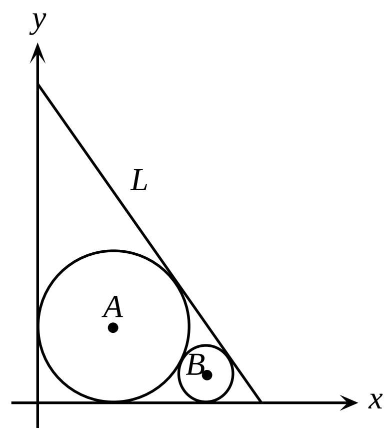
If $e = 2$, we do get a carry from $b + d$, so looking again at the tens digit $e = 2$, we see that $b + d = 11$.

Possibility 1 $b = d = 0$
Since there are only four possibilities for *a* and *c* and just one way of selecting *b* and *d* so that $b + d = 0$ for each possibility, there are just four possibilities.

Possibility 2 $b + d = 11$
For each of the four possible ways of choosing *a* and *c*, there are eight ways of choosing *b* and *d* so that $b + d = 11$ thus giving 32 possibilities.
This gives a total of $4 + 32 = 36$ possibilities.

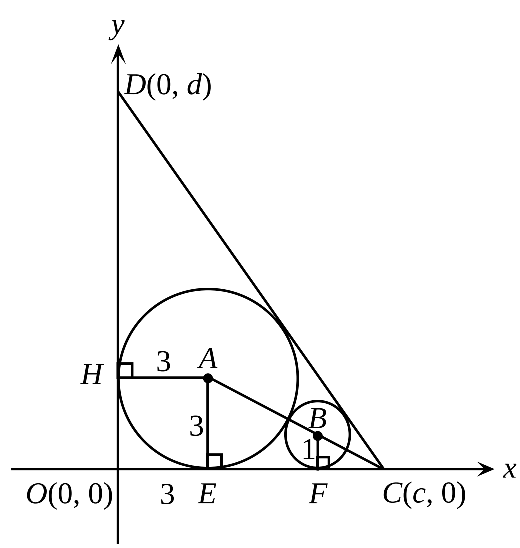
ANSWER: (C)

25. The circle with centre *A* has radius 3 and is tangent to both the positive *x*-axis and positive *y*-axis, as shown. Also, the circle with centre *B* has radius 1 and is tangent to both the positive *x*-axis and the circle with centre *A*. The line *L* is tangent to both circles. The *y*-intercept of line *L* is

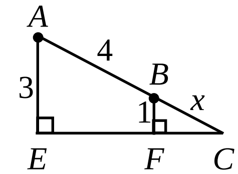


(**A**) $3 + 6\sqrt{3}$ (**B**) $10 + 3\sqrt{2}$ (**C**) $8\sqrt{3}$
(**D**) $10 + 2\sqrt{3}$ (**E**) $9 + 3\sqrt{3}$

Solution
We start by drawing a line from point *C* that will pass through *A* and *B*. From *A* and *B*, we drop perpendiculars to the points of tangency on the *x*-axis and label these points as *E* and *F* as shown. We also drop a perpendicular from *A* to the *y*-axis which makes $AH = AE = 3$.



Extracting ΔCAE from the diagram and labelling with the given information we would have the following noted in the diagram.



If we represent the distance from C to B as x and recognize that ΔCBF is similar to ΔCAE,

$$\frac{x}{1} = \frac{x+4}{3}$$

$$x = 2.$$

In ΔCBF, $FC^2 = 2^2 - 1^2 = 3$

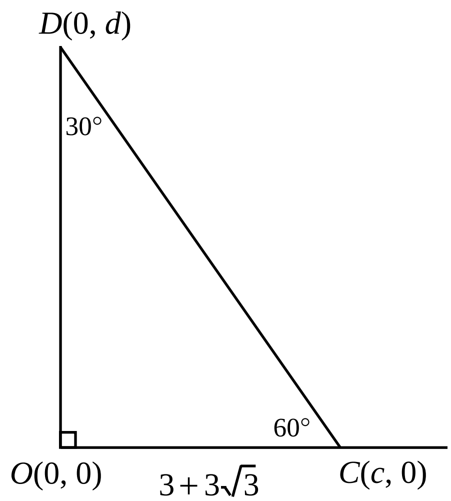
$$FC = \sqrt{3},\ (FC > 0).$$

This implies that $\angle BCF = 30°$ and $\angle OCD = 60°$. Therefore $EF = 2\sqrt{3}$, from similar triangles again.

This now gives us the diagram shown.

Thus, $d = \sqrt{3}\left(3 + 3\sqrt{3}\right)$

$$= 3\sqrt{3} + 9.$$



ANSWER: (E)

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2000 Solutions
Cayley Contest (Grade 10)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Awards

Part A:

1. The value of $2(5-2)-5^2$ is

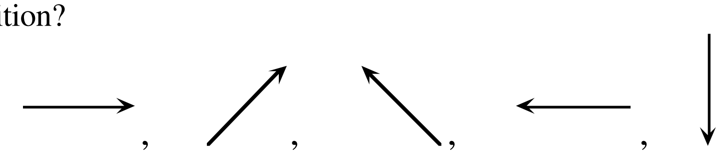
(**A**) -19 (**B**) -4 (**C**) 1 (**D**) -11 (**E**) -17

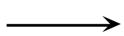
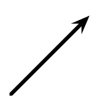
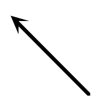
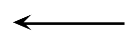
Solution
When evaluating this expression, we use ‘order of operations’ in the standard way. Doing so, we find,

$$\begin{aligned}&2(5-2)-5^2\\&=2(3)-25\\&=6-25\\&=-19.\end{aligned}$$

ANSWER: (A)

2. If the following sequence of five arrows repeats itself continuously, what arrow would be in the 48th position?

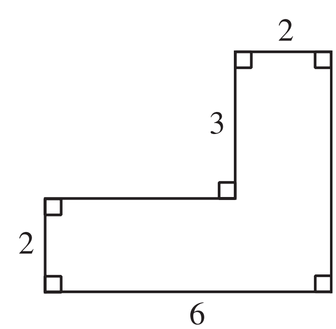


(**A**)  (**B**)  (**C**)  (**D**)  (**E**) 

Solution
Since this sequence repeats itself, once it has completed nine cycles it will be the same as starting at the beginning. Thus the 48th arrow will be the same as the third one.

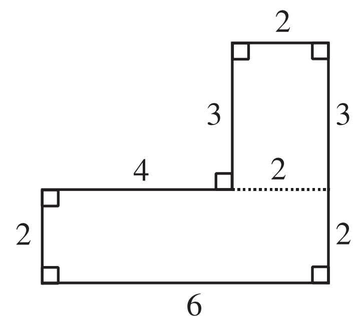
ANSWER: (C)

3. In the given diagram, the numbers shown are the lengths of the sides. What is the perimeter of the figure?



(**A**) 13 (**B**) 18 (**C**) 22
(**D**) 21 (**E**) 19

Solution
The easiest way to determine the length of the missing side is by drawing in a line to form two rectangles. Since opposite sides in a rectangle are equal we can label the sides as shown. The required perimeter is thus $2+3+2+6+2+4+3=22$.



ANSWER: (C)

4. A farmer has 7 cows, 8 sheep and 6 goats. How many more goats should be bought so that half of her animals will be goats?

(**A**) 18 (**B**) 15 (**C**) 21 (**D**) 9 (**E**) 6

Solution 1
If the cows and sheep were themselves goats we would have 15 goats. This means that she would need nine extra goats.

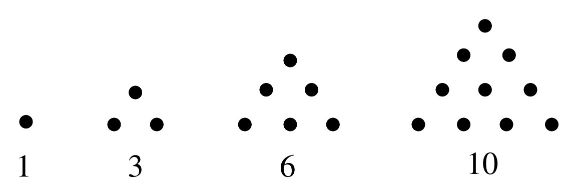
Solution 2
Let the number of goats added be x.
Therefore, $\frac{6+x}{21+x}=\frac{1}{2}$.
Cross multiplying gives,

$$\begin{aligned} 2(6+x) &= 21+x \\ 12+2x &= 21+x \\ x &= 9. \end{aligned}$$

As in solution 1, she would add 9 goats.

ANSWER: (D)

5. The first four triangular numbers 1, 3, 6, and 10 are illustrated in the diagram. What is the tenth triangular number?



(**A**) 55 (**B**) 45 (**C**) 66
(**D**) 78 (**E**) 50

Solution
From our diagram, it can be seen that the fifth triangular number is found by adding a row with five dots. This number is thus, $1+2+3+4+5=\frac{5\times6}{2}=15$. The sixth triangular number is $1+2+3+4+5+6=\frac{6\times7}{2}=21$. If we follow this to its conclusion, the nth triangular number is

$1+2+3+\ldots+n=\dfrac{(n)(n+1)}{2}$. The tenth triangular number will be $\dfrac{(10)(11)}{2}=55$.

ANSWER: (A)

6. The sum of the digits of an even ten digit integer is 89. The last digit is

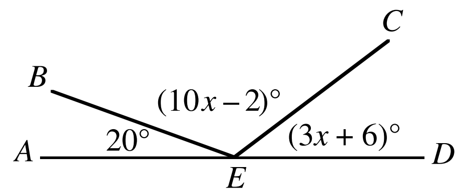
(**A**) 0 (**B**) 2 (**C**) 4 (**D**) 6 (**E**) 8

Solution

89 is a large number to be the sum of the digits of a ten digit number. In fact, the largest possible digital sum is 10×9 or 90. Since 89 is only 1 less than 90, the number in question must be composed of nine 9's and one 8. In order that the number be divisible by 2, the last digit must be 8.

ANSWER: (E)

7. If AD is a straight line segment and E is a point on AD, determine the measure of $\angle CED$.



(**A**) 20° (**B**) 12° (**C**) 42°
(**D**) 30° (**E**) 45°

Solution

Since there are $180°$ in a straight line, we can form the equation,

$$20°+(10x-2)°+(3x+6)°=180°$$
$$20+10x-2+3x+6=180 \text{ (in degrees)}$$
$$13x+24=180$$
$$13x=156$$
$$x=12.$$

Therefore $\angle CED=\big(3(12)+6\big)°=42°$.

ANSWER: (C)

8. On a 240 kilometre trip, Corey's father drove $\frac{1}{2}$ of the distance. His mother drove $\frac{3}{8}$ of the total distance and Corey drove the remaining distance. How many kilometres did Corey drive?

(**A**) 80 (**B**) 40 (**C**) 210 (**D**) 30 (**E**) 55

Solution

If Corey's father and mother drove $\frac{1}{2}$ and $\frac{3}{8}$ the total distance, respectively, altogether they drove $\frac{1}{2}+\frac{3}{8}$ or $\frac{7}{8}$th the total distance. Thus Corey must have driven $\frac{1}{8}\times 240$ or 30 kilometres.

ANSWER: (D)

9. Evaluate $(-50)+(-48)+(-46)+...+54+56$.

(**A**) 156 (**B**) 10 (**C**) 56 (**D**) 110 (**E**) 162

Solution

If we add some terms to this series, we would have the following:

$$(-50)+(-48)+(-46)+\ldots+48+50+52+54+56.$$

Each of the negative integers has its opposite included in the sum and each pair of these sums is 0. This implies that, $(-50)+(-48)+(-46)+...+46+48+50$ is 0. The overall sum is now just $52+54+56$ or 162. ANSWER: (E)

10. The ages of three contestants in the Cayley Contest are 15 years, 9 months; 16 years, 1 month; and 15 years, 8 months. Their average (mean) age is

(**A**) 15 years, 8 months (**B**) 15 years, 9 months (**C**) 15 years, 10 months
(**D**) 15 years, 11 months (**E**) 16 years

Solution 1

Consider one of the ages, say the youngest, as a base age. The other two contestants are one month and five months older respectively. Since $\frac{0+1+5}{3}=2$, this implies that the average age is two months greater than the youngest. This gives an average age of 15 years, 10 months.

Solution 2

This second solution involves a little more calculation but still gives the same correct answer. Since there are twelve months in a year, the age of the first contestant, in months, is $15\times12+9$ or 189 months. Similarly, the ages of the other two students would be 193 and 188 months. The average age would thus be $\frac{189+193+188}{3}$ or 190 months. The average age is then 15 years, 10 months because $190=12\times15+10$. ANSWER: (C)

Part B: Each correct answer is worth 6.

11. A store had a sale on T-shirts. For every two T-shirts purchased at the regular price, a third T-shirt was bought for \$1.00. Twelve T-shirts were bought for \$120.00. What was the regular price for one T-shirt?

(**A**) \$10.00 (**B**) \$13.50 (**C**) \$14.00 (**D**) \$14.50 (**E**) \$15.00

Solution
We will start this question by representing the regular price of one T-shirt as x dollars. If a person bought a 'lot' of three T-shirts, they would thus pay $(2x+1)$ dollars. Since the cost of twelve T-shirts is \$120.00, this implies that a single 'lot' would cost \$30. This allows us to write the equation, $2x+1=30$ or $x=14.50$. The regular price of a T-shirt is \$14.50.

ANSWER: (D)

12. Natural numbers are equally spaced around a circle in order from 1 to n. If the number 5 is directly opposite the number 14, then n is

(**A**) 14 (**B**) 15 (**C**) 16 (**D**) 18 (**E**) 20

Solution
If 5 is opposite 14 then each of the eight numbers between and including 6 and 13 are each opposite a natural number. These eight numbers would be matched giving a total of 2×8 or 16 numbers. If we add 5 and 14 the total is 18.

ANSWER: (D)

13. The average of 19 consecutive integers is 99. The largest of these integers is

(**A**) 118 (**B**) 108 (**C**) 109 (**D**) 117 (**E**) 107

Solution
If the average of the 19 consecutive numbers is 99 the middle number is 99 which is the tenth number. If the tenth number is 99, the nineteenth number will be 108.

ANSWER: (B)

14. A positive integer is to be placed in each box. The product of any four adjacent integers is always 120. What is the value of x?

		2			4		x				3		

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution
Since the product of any four integers is 120, $a_1a_2a_3a_4 = a_2a_3a_4a_5 = 120$ where a_n represents the number in the nth box. Therefore, $a_1 = a_5$ and similarly $a_2 = a_6$, $a_3 = a_7$, $a_4 = a_8$ or more generally, $a_n = a_{n+4}$. Thus the boxes can be filled as follows:

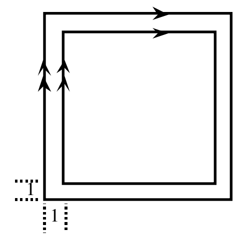
x	4	2	3	x	4	2	3	x	4	2	3	x	4

Therefore, $(4)(2)(3)(x) = 120$

$$x = \frac{120}{24} = 5.$$

ANSWER: (E)

15. Eight squares with the same centre have parallel sides and are one unit apart. The two largest squares are shown. If the largest square has a perimeter of 96, what is the perimeter of the smallest square?



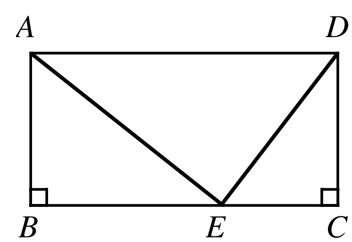
(**A**) 40 (**B**) 68 (**C**) 32
(**D**) 64 (**E**) 89

Solution

Since the largest square has perimeter 96, it has a side length of $\frac{96}{4}$ or 24. From the diagram, the side length of the next square is $24 - 2$ or 22. Continuing thus, the side lengths of the eight squares form the sequence: 24, 22, 20, 18, 16, 14, 12, 10. The side length of the eighth square will be 10 giving a perimeter of $4 \times 10 = 40$.

ANSWER: (A)

16. In the diagram, $ABCD$ is a rectangle with $AD = 13$, $DE = 5$ and $EA = 12$. The area of $ABCD$ is

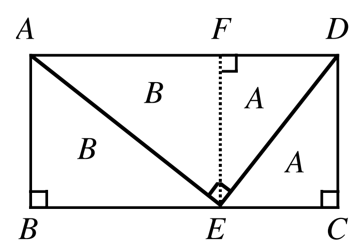


(**A**) 39 (**B**) 60 (**C**) 52
(**D**) 30 (**E**) 25

Solution

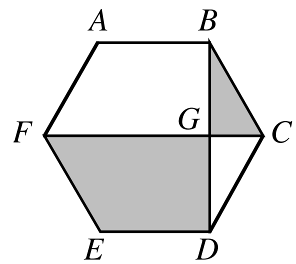
Since $13^2 = 12^2 + 5^2$ we use the converse of Pythagorus' Theorem to conclude that $\angle AED = 90°$.

The area of ΔAED is then $\frac{1}{2}(5)(12) = 30$. Through E, we draw a line parallel to CD and BA. Since the area of ΔFDE equals the area of ΔCDE we label each of these areas A. Similarly, the area of ΔAFE equals the area of ΔBAE and so each of these areas can be labelled B. Since $A + B = 30$, the area of the rectangle is $2(A + B)$ or $2(30) = 60$.



ANSWER: (B)

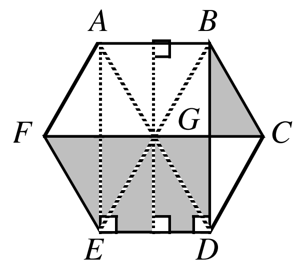
17. In the regular hexagon $ABCDEF$, two of the diagonals, FC and BD, intersect at G. The ratio of the area of quadrilateral $FEDG$ to ΔBCG is



(**A**) $3\sqrt{3}:1$ (**B**) 4:1 (**C**) 6:1
(**D**) $2\sqrt{3}:1$ (**E**) 5:1

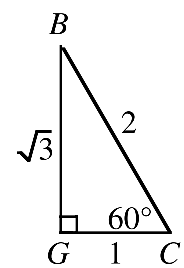
Solution 1

Join E to B and D to A as shown. Also join E to A and draw a line parallel to AE through the point of intersection of BE and AD. Quadrilateral $FEDG$ is now made up of five triangles each of which has the same area as ΔBCG. The required ratio is 5:1.



Solution 2

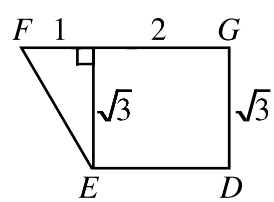
For convenience, assume that each side of the hexagon has a length of 2 units. Each angle in the hexagon equals 120° so $\angle BCG = \frac{1}{2}(120^\circ) = 60^\circ$. Now label ΔBCG as shown. Using the standard ratios for a $30^\circ - 60^\circ - 90^\circ$ triangle we have $BG = \sqrt{3}$ and $GC = 1$.



The area of $\Delta BCG = \frac{1}{2}(1)\sqrt{3} = \frac{\sqrt{3}}{2}$. Dividing the quadrilateral $FGDE$ as illustrated, it will have an area of

$$2\left(\sqrt{3}\right) + \frac{1}{2}(1)\left(\sqrt{3}\right) = \frac{5\sqrt{3}}{2}.$$

The required ratio is $\frac{5\sqrt{3}}{2} : \frac{\sqrt{3}}{2}$ or 5:1, as in solution 1.



ANSWER: (E)

18. If a, b and c are distinct positive integers such that $abc = 16$, then the largest possible value of $a^b - b^c + c^a$ is

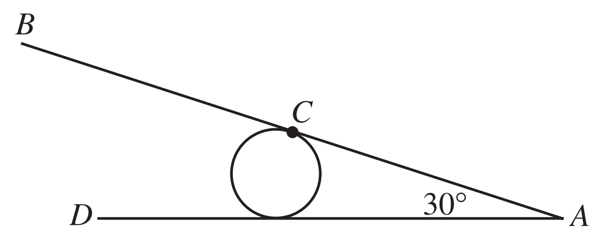
(**A**) 253 (**B**) 63 (**C**) 249 (**D**) 263 (**E**) 259

Solution

If a, b and c are distinct then the correct factorization is $16 = 1 \times 2 \times 8$. Since a, b and c must be some permutation of 1, 2 and 8 there are exactly six possibilities which give the values –247, -61, 65, 249, 263, and 63. Of these, $8^1 - 1^2 + 2^8$ or 263 is the largest.

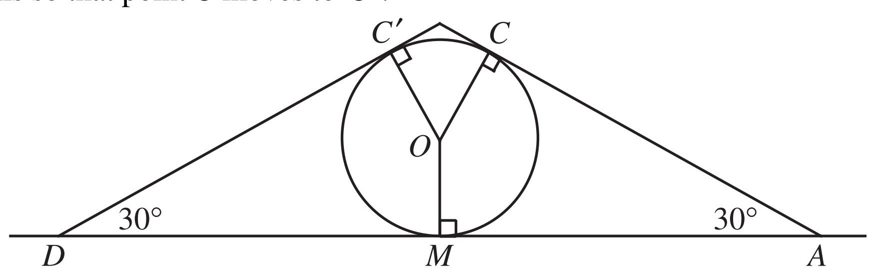
ANSWER: (D)

19. A metal rod with ends A and B is welded at its middle, C, to a cylindrical drum of diameter 12. The rod touches the ground at A making a $30°$ angle. The drum starts to roll along AD in the direction of D. How far along AD must the drum roll for B to touch the ground?



(**A**) π (**B**) 2π (**C**) 3π (**D**) 4π (**E**) 5π

Solution
The drum rolls so that point C moves to C'.



Since a line drawn from the centre of the circle makes angles of $90°$ with tangents drawn to a circle, $\angle COM = 360° - 90° - 90° - 30° = 150°$. By symmetry, $\angle C'OM = 150°$ and thus $\angle C'OC = 360° - 150° - 150° = 60°$. Since $\angle C'OC = 60°$, this implies that point C will have to travel $\frac{1}{6}$th the circumference of the circle or $\frac{1}{\cancel{6}_1}(\cancel{12}^{2}\pi) = 2\pi$.

ANSWER: (B)

20. Twenty pairs of integers are formed using each of the integers 1, 2, 3, ..., 40 once. The positive difference between the integers in each pair is 1 or 3. (For example, 5 can be paired with 2, 4, 6 or 8.) If the resulting differences are added together, the greatest possible sum is

(**A**) 50 (**B**) 54 (**C**) 56 (**D**) 58 (**E**) 60

Solution

Since we have twenty pairings, it is possible to have nineteen differences of 3 and one difference of 1. The maximum sum of these differences is thus, $3(19)+1=58$. The pairings can be achieved in the following way: $\{(1,4), (2,5), (3,6), (7,10), (8,11), (9,12), (13,16), (14,17), (15,18), (19,22), (20,23), (21,24), (25,28), (26,29), (27,30), (31,34), (32,35), (33,36), (37,40), (38,39)\}$. Note that there is just one pair, $(38,39)$, that differs by one.

ANSWER: (D)

Part C: Each correct answer is worth 8.

21. A wooden rectangular prism has dimensions 4 by 5 by 6. This solid is painted green and then cut into 1 by 1 by 1 cubes. The ratio of the number of cubes with exactly two green faces to the number of cubes with three green faces is

(**A**) 9:2 (**B**) 9:4 (**C**) 6:1 (**D**) 3:1 (**E**) 5:2

Solution

The cubes with two green faces are the cubes along the edges, not counting the corner cubes. In each dimension, we lost two cubes to the corners so we then have four edges with 4 cubes, four with 3 cubes and four with 2 cubes. The total number of cubes with paint on two edges is then $4(4)+4(3)+4(2)=36$. The number of cubes that have paint on three sides are the corner cubes of which there are eight. The required ratio is then 36:8 or 9:2.

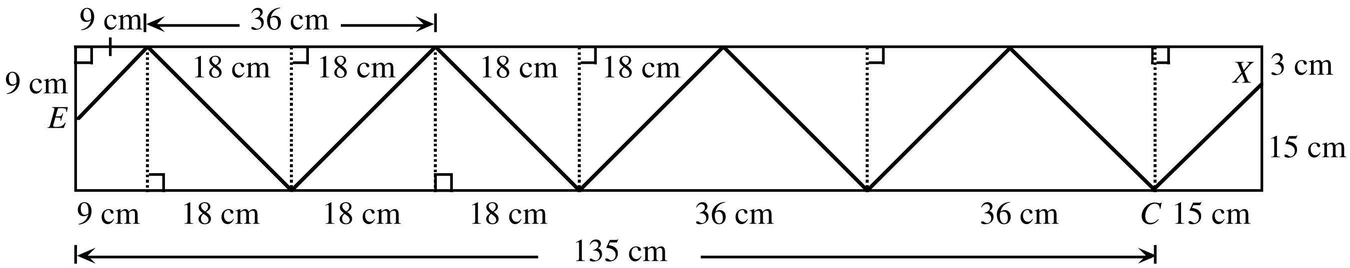
ANSWER: (A)

22. An ant walks inside a 18 cm by 150 cm rectangle. The ant's path follows straight lines which always make angles of $45°$ to the sides of the rectangle. The ant starts from a point X on one of the shorter sides. The first time the ant reaches the opposite side, it arrives at the midpoint. What is the distance, in centimetres, from X to the nearest corner of the rectangle?

(**A**) 3 (**B**) 4 (**C**) 6 (**D**) 8 (**E**) 9

Solution

If we took a movie of the ant's path and then played it backwards, the ant would now start at the point E and would then end up at point X. Since the ant now 'starts' at a point nine cm from the corner, the 'first' part of his journey is from E to B. This amounts to nine cm along the length of the rectangle since ΔBAE is an isosceles right-angled triangle. This process continues as illustrated, until the ant reaches point C. By the time the ant has reached C, it has travelled $9+18+3\times 36$ or 135 cm along the length of the rectangle. To travel from C to X, the ant must travel 15 cm along the length of the rectangle which puts the ant 3 cm from the closest vertex.



ANSWER: (A)

23. The left most digit of an integer of length 2000 digits is 3. In this integer, any two consecutive digits must be divisible by 17 or 23. The 2000th digit may be either 'a' or 'b'. What is the value of $a+b$?

(**A**) 3 (**B**) 7 (**C**) 4 (**D**) 10 (**E**) 17

Solution

We start by noting that the two-digit multiples of 17 are 17, 34, 51, 68, and 85. Similarly we note that the two-digit multiples of 23 are 23, 46, 69, and 92. The first digit is 3 and since the only two-digit number in the two lists starting with 3 is 34, the second digit is 4. Similarly the third digit must be 6. The fourth digit, however, can be either 8 or 9. Let's consider this in two cases.

Case 1

If the fourth digit is 8, the number would be 3468517 and would stop here since there isn't a number in the two lists starting with 7.

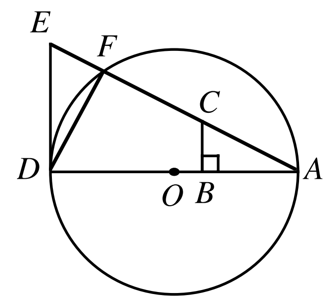
Case 2

If the fourth digit is 9, the number would be 34692 34692 34 ... and the five digits '34692' would continue repeating indefinitely as long as we choose 9 to follow 6.

If we consider a 2000 digit number, its first 1995 digits must contain 399 groups of '34692'. The last groups of five digits could be either 34692 or 34685 which means that the 2000th digit may be either 2 or 5 so that $a+b=2+5=7$.

ANSWER: (B)

24. In the diagram shown, $\angle ABC = 90°$, $CB \parallel ED$, $AB = DF$, $AD = 24$, $AE = 25$ and O is the centre of the circle. Determine the perimeter of $CBDF$.



(**A**) 39 (**B**) 40 (**C**) 42
(**D**) 43 (**E**) 44

Solution
We start by showing that $\Delta ABC \cong \Delta DFE$.
Since $ED \parallel CB$ this implies that $\angle DEF = \angle BCA$ because of corresponding angles. Also, $\angle DFA = 90°$ because it is an angle in a semicircle which also means that $\angle DFE$ is $90°$. Thus the two triangles are equiangular. Since $AB = DF$, $\Delta ABC \cong \Delta DFE$ (ASA). Therefore, $EF = CB$ and $DF = BA$. Using Pythagorus in ΔADE, $DE^2 = 25^2 - 24^2 \Rightarrow DE = 7 = CA$.
Thus $CE = 25 - 7 = 18$.
The required perimeter is,

$$\begin{aligned} CB + BD + DF + FC = EF + (BD + DF) + FC &= (EF + FC) + (BD + DF) \\ &= (EF + FC) + (BD + BA), \text{ since } DF = BA \\ &= CE + AD \\ &= 18 + 24 = 42\,. \end{aligned}$$

ANSWER: (C)

25. For the system of equations $x^2 + x^2y^2 + x^2y^4 = 525$ and $x + xy + xy^2 = 35$, the sum of the real y values that satisfy the equations is

(**A**) 20 (**B**) 2 (**C**) $\frac{3}{2}$ (**D**) $\frac{55}{2}$ (**E**) $\frac{5}{2}$

Consider the system of equations $x^2 + x^2y^2 + x^2y^4 = 525$ (1)
and $x + xy + xy^2 = 35$ (2)

The expression on the left side of equation (1) can be rewritten as,

$$\begin{aligned} x^2 + x^2y^2 + x^2y^4 &= \left(x + xy^2\right)^2 - x^2y^2 \\ &= \left(x + xy^2 - xy\right)\left(x + xy^2 + xy\right) \end{aligned}$$

Thus, $\left(x + xy^2 - xy\right)\left(x + xy^2 + xy\right) = 525$

Substituting from (2) gives, $\left(x + xy^2 - xy\right)(35) = 525$

or, $x + xy^2 - xy = 15$ (3)

Now subtracting (3) from (2), $2xy = 20$, $x = \frac{10}{y}$

Substituting for x in (3) gives,

$$\frac{10}{y}+10y-10=15$$
$$10y^2-25y+10=0$$
$$2y^2-5y+2=0$$
$$(2y-1)(y-2)=0$$
$$y=\frac{1}{2} \quad \text{or} \quad y=2$$

The sum of the real y values satisfying the system is $\frac{5}{2}$.

ANSWER: (E)

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

1999 Solutions
Cayley Contest (Grade 10)

for the
NATIONAL BANK OF CANADA
Awards

Part A

1. The value of $3^2 + 7^2 - 5^2$ is

(**A**) 75 (**B**) 83 (**C**) 33 (**D**) 25 (**E**) 10

Solution
$3^2 + 7^2 - 5^2 = 9 + 49 - 25 = 33$ ANSWER: (C)

2. If 8 is added to the square of 5 the result is divisible by

(**A**) 5 (**B**) 2 (**C**) 8 (**D**) 23 (**E**) 11

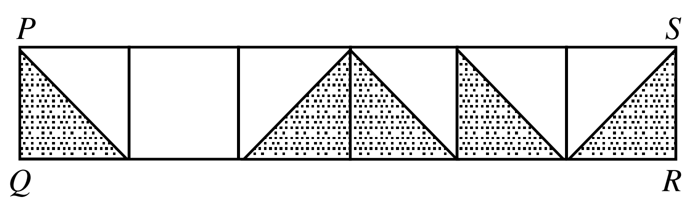
Solution
$8 + 5^2 = 33$
Of the given numbers, 11 is the only possible divisor of 33. ANSWER: (E)

3. Today is Wednesday. What day of the week will it be 100 days from now?

(**A**) Monday (**B**) Tuesday (**C**) Thursday (**D**) Friday (**E**) Saturday

Solution
Since there are 7 days in a week it will be Wednesday in 98 days.
In 100 days it will be Friday. ANSWER: (D)

4. The rectangle $PQRS$ is divided into six equal squares and shaded as shown. What fraction of $PQRS$ is shaded?



(**A**) $\frac{1}{2}$ (**B**) $\frac{7}{12}$ (**C**) $\frac{5}{11}$ (**D**) $\frac{6}{11}$ (**E**) $\frac{5}{12}$

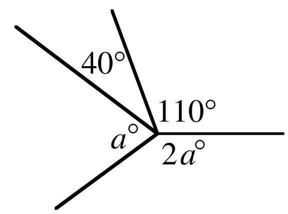
Solution
There are 5 half-squares shaded out of a total possible of 12 half-squares, hence $\frac{5}{12}$ of the area is shaded. ANSWER: (E)

5. If $x = 4$ and $y = 3x$ and $z = 2y$, then the value of $y + z$ is

(**A**) 12 (**B**) 20 (**C**) 40 (**D**) 24 (**E**) 36

Solution
If $x = 4$ this makes $y = 12$ and $z = 24$.
Thus $y + z = 36$. ANSWER: (E)

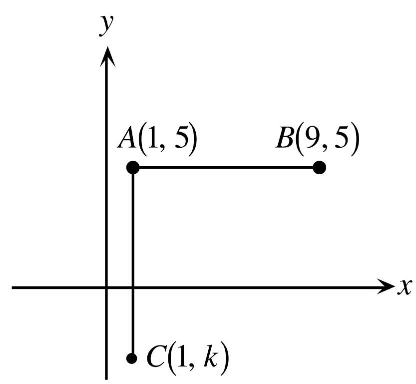
6. In the diagram, the value of a is



(**A**) 50 (**B**) 65 (**C**) 70
(**D**) 105 (**E**) 110

Solution
Since $3a° + 150° = 360°$
$3a° = 210°$
Therefore $a = 70$. ANSWER: (C)

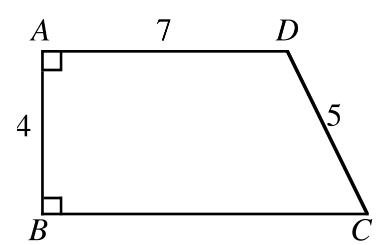
7. In the diagram, AB and AC have equal lengths. What is the value of k?



(**A**) -3 (**B**) -4 (**C**) -5
(**D**) -7 (**E**) -8

Solution
Since $AB = AC = 8$, $5 - k = 8$
$k = -3$. ANSWER: (A)

8. In the diagram, $AD < BC$. What is the perimeter of $ABCD$?

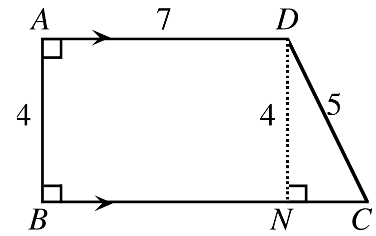


(**A**) 23 (**B**) 26 (**C**) 27
(**D**) 28 (**E**) 30

Solution

From D we draw a line perpendicular to BC that meets BC at N. Since $ADNB$ is a rectangle and $AD \| BC$, $DN = 4$. We use Pythagoras to find $NC = 3$. We now know that $BC = BN + NC = 7 + 3 = 10$.

The required perimeter is $7 + 5 + 10 + 4 = 26$.



ANSWER: (B)

9. Three CD's are bought at an average cost of \$15 each. If a fourth CD is purchased, the average cost becomes \$16. What is the cost of the fourth CD?

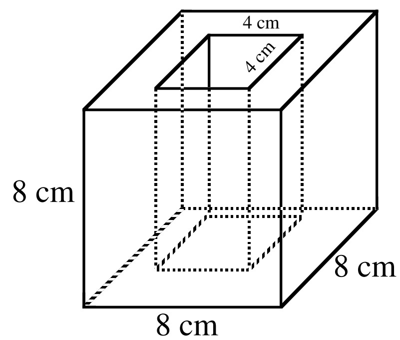
(**A**) \$16 (**B**) \$17 (**C**) \$18 (**D**) \$19 (**E**) \$20

Solution

If four C.D.'s have an average cost of \$16 this implies that \$64 was spent in purchasing the four of them. Using the same reasoning, \$45 was spent buying the first three. Thus, the fourth C.D. must have cost $\$64 - \$45 = \$19$. ANSWER: (D)

10. An 8 cm cube has a 4 cm square hole cut through its centre, as shown. What is the remaining volume, in cm^3?

(**A**) 64 (**B**) 128 (**C**) 256
(**D**) 384 (**E**) 448



Solution

$$\begin{aligned}\text{Remaining volume } &= 8\times8\times8 - 8\times4\times4 \text{ (in cm}^3\text{)}\\ &= 8(64-16)\\ &= 8\times48\\ &= 384\end{aligned}$$

ANSWER: (D)

Part B

11. The time on a digital clock is 5:55. How many minutes will pass before the clock next shows a time with all digits identical?

(**A**) 71 (**B**) 72 (**C**) 255 (**D**) 316 (**E**) 436

Solution

The digits on the clock will next be identical at 11:11. This represents a time difference of 316 minutes. (Notice that times like 6:66, 7:77 etc. are not possible.) ANSWER: (D)

12. The numbers 49, 29, 9, 40, 22, 15, 53, 33, 13, 47 are grouped in pairs so that the sum of each pair is the same. Which number is paired with 15?

(**A**) 33 (**B**) 40 (**C**) 47 (**D**) 49 (**E**) 53

Solution

If we arrange the numbers in ascending order we would have: 9, 13, 15, 22, 29, 33, 40, 47, 49, 53. If the sum of each pair is equal they would be paired as: $9 \leftrightarrow 53$, $13 \leftrightarrow 49$, $15 \leftrightarrow 47$, $22 \leftrightarrow 40$, $29 \leftrightarrow 33$. ANSWER: (C)

13. The units digit in the product $\left(5^2+1\right)\left(5^3+1\right)\left(5^{23}+1\right)$ is

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 5 (**E**) 6

Solution

Since 5^2, 5^3 and 5^{23} all end in 5, then 5^2+1, 5^3+1 and $5^{23}+1$ all end in 6. When we multiply these three numbers together their product must also end in a 6.

ANSWER: (E)

14. In an election for class president, 61 votes are cast by students who are voting to choose one of four candidates. Each student must vote for only one candidate. The candidate with the highest number of votes is the winner. The smallest number of votes the winner can receive is

(**A**) 15 (**B**) 16 (**C**) 21 (**D**) 30 (**E**) 31

Solution

After 60 votes are cast, theoretically it is possible for each candidate to have 15 votes.
The final vote, the 61st, would mean that the winning candidate would need just 16 votes to have the minimum number possible. ANSWER: (B)

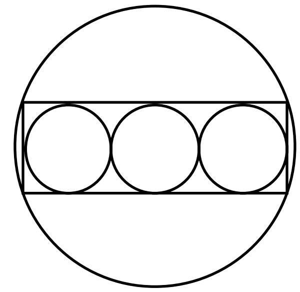
15. A chocolate drink is 6% pure chocolate, by volume. If 10 litres of pure milk are added to 50 litres of this drink, the percent of chocolate in the new drink is

(**A**) 5 (**B**) 16 (**C**) 10 (**D**) 3 (**E**) 26

Solution

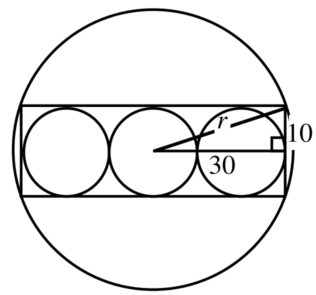
If 6% of the 50 litres is pure chocolate, this means that there will be three litres of pure chocolate in the final mixture. If the final mixture contains sixty litres of which three litres are pure chocolate this represents $\frac{3}{60}$ or 5% of the total. ANSWER: (A)

16. Three circles, each with a radius of 10 cm, are drawn tangent to each other so that their centres are all in a straight line. These circles are inscribed in a rectangle which is inscribed in another circle. The area of the largest circle is



(**A**) 1000π (**B**) 1700π (**C**) 900π
(**D**) 1600π (**E**) 1300π

Solution
By symmetry, the centre of the large circle is the centre of the smaller middle circle. If the constructions are made as shown and with the appropriate representation of the lengths we find, $r^2 = 30^2 + 10^2 = 1000$.
Thus, $A = \pi r^2 = \pi(1000) = 1000\pi$.



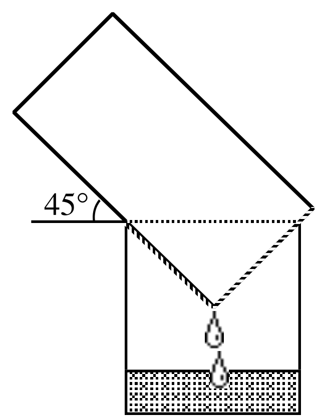
ANSWER: (A)

17. Let N be the smallest positive integer whose digits have a product of 2000. The sum of the digits of N is

(**A**) 21 (**B**) 23 (**C**) 25 (**D**) 27 (**E**) 29

Solution
Since $2000 = 2^4 \cdot 5^3$, the smallest possible positive integer satisfying the required conditions is $25\,558$ which gives the sum $2+5+5+5+8=25$. A natural answer might be 23 since $44\,555$ satisfies the given conditions. However, since $25\,558 < 44\,555$ and the question requires the smallest number then the answer must be 25 and not 23. ANSWER: (C)

18. A cylindrical pail containing water drains into a cylindrical tub 40 cm across and 50 cm deep, while resting at an angle of 45° to the horizontal, as shown. How deep is the water in the tub when its level reaches the pail?



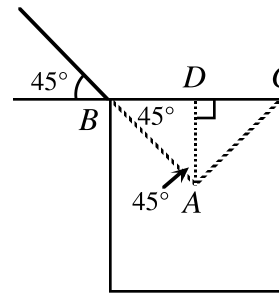
(**A**) 10 cm (**B**) 20 cm (**C**) 30 cm
(**D**) 35 cm (**E**) 40 cm

Solution

We label the points A, B and C as shown in the diagram. From symmetry, we note that ΔBAC is an isosceles right-angled triangle.

From A we draw a line perpendicular to BC meeting the line at point D. This construction allows us to conclude that ΔABD is also a right-angled isosceles triangle and specifically that $BD = DA$. Since $BD = DA$ and $BD = DC = 20$, we find $DA = 20$.

This makes the depth of the water $50 - 20$ or 30.



ANSWER: (C)

19. A number is *Beprisque* if it is the only natural number between a prime number and a perfect square (e.g. 10 is Beprisque but 12 is not). The number of *two-digit* Beprisque numbers (including 10) is

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution

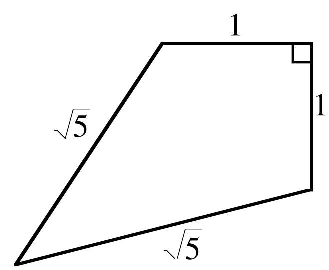
We start with the observation that it is necessary to consider only the odd perfect squares and the integers adjacent to them. It is not necessary to consider the even perfect squares because if we add 2 or subtract 2 from an even number the result is even and it is required by the conditions set out in the question that this number be prime. Considering then the odd perfect squares we have: $\{9, 10, 11\}$, $\{23, 24, 25, 26, 27\}$, $\{47, 48, 49, 50, 51\}$, $\{79, 80, 81, 82, 83\}$.

The Beprisque numbers are those that are circled. ANSWER: (E)

20. The area of the given quadrilateral is

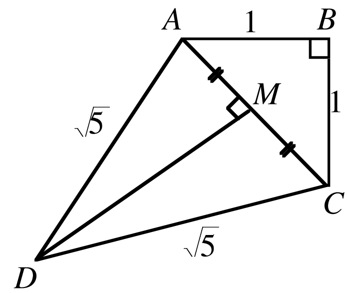
(**A**) $\frac{3}{2}$ (**B**) $\sqrt{5}$ (**C**) $\frac{1+\sqrt{10}}{2}$

(**D**) 2 (**E**) 3



Solution

Label the quadrilateral as shown. We join A to C and using Pythagoras in ΔABC we calculate BC to be $\sqrt{2}$. Since ΔACD is isosceles, we can draw a line perpendicular to AC which passes through M, the mid-point of AC. Since $AC = \sqrt{2}$, $AM = \frac{\sqrt{2}}{2}$. Since $AM = \frac{\sqrt{2}}{2}$, $DA = \sqrt{5}$ and ΔADM is right-angled, we can once again use Pythagoras to find that $DM = \frac{3}{\sqrt{2}}$. The line segment DM is also the height of ΔDAC. The area of ΔADC is $\frac{1}{2}\left(\frac{3}{\sqrt{2}}\right)\left(\sqrt{2}\right) = \frac{3}{2}$ and because the area of $\Delta ABC = \frac{1}{2}(1)(1) = \frac{1}{2}$, the total area of the quadrilateral is $\frac{1}{2} + \frac{3}{2} = 2$ square units.



ANSWER: (D)

Part C

21. A number is formed using the digits 1, 2, ..., 9. Any digit can be used more than once, but adjacent digits cannot be the same. Once a pair of adjacent digits has occurred, that pair, in that order, cannot be used again. How many digits are in the largest such number?

(**A**) 72 (**B**) 73 (**C**) 144 (**D**) 145 (**E**) 91

Solution

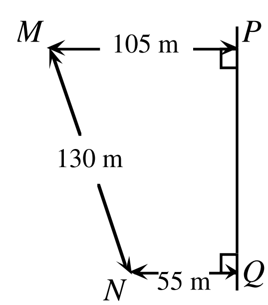
Since there are $9(8) = 72$ ordered pairs of consecutive digits, and since the final digit has no successor, we can construct a 73 digit number by adding a 9. The question is, of course, can we actually construct this number? The answer is ‘yes’ and the largest such number is,

98 97 96 95 94 93 92 91 87 86 85 84 83 82 81 76 75 74 73 72 71 65 64 63 62 61
54 53 52 51 43 42 41 32 31 21 9.

If we count the numbers in the string we can see that there are actually 73 numbers contained within it.

ANSWER: (B)

22. A main gas line runs through P and Q. From some point T on PQ, a supply line runs to a house at point M. A second supply line from T runs to a house at point N. What is the minimum total length of pipe required for the two supply lines?



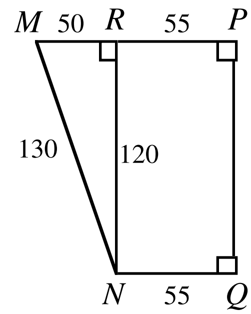
(**A**) 200 (**B**) 202 (**C**) 198
(**D**) 210 (**E**) 214

Solution

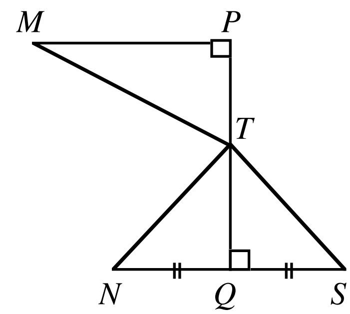
We start by choosing point R so that $RPQN$ is a rectangle.

Thus, $MR = 105 - 55$
$= 50$.

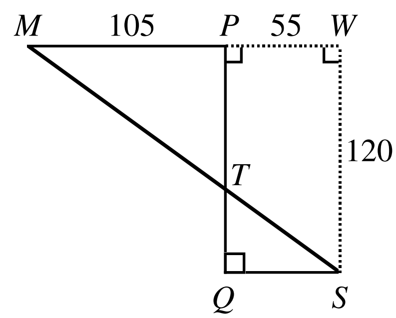
Using Pythagoras, $RN = \sqrt{130^2 - 50^2} = 120$.



Let S be the image of N reflected in PQ.
Join M to T, T to S and T to N.
Since $\Delta TNQ \equiv \Delta TSQ$, it follows that $TN = TS$.
The length of the supply line is $MT + TN = MT + TS$.



Clearly the length $MT + TS$ is a minimum when M, T and S are collinear. In that case, $MT + TS = MS$.
Create ΔMSW as shown.
By Pythagoras, $MS = \sqrt{160^2 + 120^2}$
$= 200$.



ANSWER: (A)

23. How many integers can be expressed as a sum of three distinct numbers chosen from the set $\{4, 7, 10, 13, ..., 46\}$?

(**A**) 45 (**B**) 37 (**C**) 36 (**D**) 43 (**E**) 42

Solution

Since each number is of the form $1 + 3n$, $n = 1, 2, 3, ..., 15$, the sum of the three numbers will be of the form $3 + 3k + 3l + 3m$ where k, l and m are chosen from $\{1, 2, 3, ..., 15\}$. So the question is equivalent to the easier question of, 'How many distinct integers can be formed by adding three numbers from, $\{1, 2, 3, ..., 15\}$?'

The smallest is $1 + 2 + 3 = 6$ and the largest is $13 + 14 + 15 = 42$.

It is clearly possible to get every sum between 6 and 42 by:

(a) increasing the sum by one replacing a number with one that is 1 larger or,

(b) decreasing the sum by one by decreasing one of the addends by 1.

Thus all the integers from 6 to 42 inclusive can be formed.

This is the same as asking, 'How many integers are there between 1 and 37 inclusive?' The answer, of course, is 37. ANSWER: (B)

24. The sum of all values of x that satisfy the equation $\left(x^2-5x+5\right)^{x^2+4x-60}=1$ is

(**A**) -4 (**B**) 3 (**C**) 1 (**D**) 5 (**E**) 6

Solution

We consider the solution in three cases.

Case 1 It is possible for the base to be 1.

Therefore, $x^2-5x+5=1$

$$x^2-5x+4=0$$

$$(x-1)(x-4)=0$$

Therefore $x=1$ or $x=4$.

Both these values are acceptable for x.

Case 2 It is possible that the exponent be 0.

Therefore, $x^2+4x-60=0$

$$(x+10)(x-6)=0$$

$$x=-10 \text{ or } x=6$$

Note: It is easy to verify that neither $x=-10$ nor $x=6$ is a zero of x^2-5x+5, $x^2-5x+5=0$, so that the indeterminate form 0° does not occur.

Case 3 It is possible that the base is –1 and the exponent is even.

Therefore, $x^2-5x+5=-1$ but $x^2+4x-60$ must also be even.

$$x^2-5x+5=-1$$

$$x^2-5x+6=0$$

$$(x-2)(x-3)=0$$

$$x=2 \text{ or } x=3$$

If $x=2$, then $x^2-4x-60$ is even, so $x=2$ is a solution.

If $x=3$, then $x^2-4x-60$ is odd, so $x=3$ is ***not*** a solution.

Therefore the sum of the solutions is $1+4-10+6+2=3$. ANSWER: (B)

25. If $a=3^p$, $b=3^q$, $c=3^r$, and $d=3^s$ and if p, q, r, and s are positive integers, determine the smallest value of $p+q+r+s$ such that $a^2+b^3+c^5=d^7$.

(**A**) 17 (**B**) 31 (**C**) 106 (**D**) 247 (**E**) 353

Solution

If we rewrite the given expression by substituting we arrive at the new expression

$3^{2p} + 3^{3q} + 3^{5r} = 3^{7s}$. (This is derived by replacing a with 3^p, b with 3^q and so on.)
On the left side we remove the lowest power of 3 (whatever it is), $3^{2p}\left(1 + 3^{3q-2p} + 3^{5r-2p}\right) = 3^{7s}$.

Both factors on the left side must be multiples of 3 but $1 + 3^{3q-2p} + 3^{5r-2p}$ cannot be a multiple of 3 unless 3^{3q-2p} and 3^{5r-2p} are both exactly 1. This means that $2p = 3q = 5r$ or that the exponents are themselves multiples of 30, say 30m.

We now have, $3^{30m} + 3^{30m} + 3^{30m} = 3^{7s}$

or, $$(3)\left(30^{30m}\right) = 3^{7s}$$

$$3^{30m+1} = 3^{7s}.$$

We are now looking for the smallest integers, m and s, such that $30m + 1 = 7s$.
If we try $m = 1, 2, 3, 4, \ldots$ we find that $m = 3$ and $s = 13$. Thus $2p = 90$, $p = 45$; $3q = 90$, $q = 30$, $5r = 90$, $r = 18$ and $7s = 91$, $s = 13$.
From this, $p + q + r + s = 45 + 30 + 18 + 13 = 106$. ANSWER: (C)

35th Anniversary 1963 – 1998

Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

1998 Solutions
Cayley Contest (Grade 10)

for the
NATIONAL BANK OF CANADA
Awards

PART A:

1. The value of $(0.3)^2 + 0.1$ is

(**A**) 0.7 (**B**) 1 (**C**) 0.1 (**D**) 0.19 (**E**) 0.109

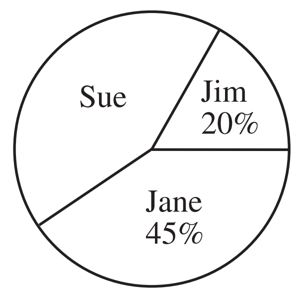
Solution

$$(0.3)^2 + 0.1 = 0.09 + 0.1$$
$$= 0.19$$

ANSWER: (D)

2. The pie chart shows a percentage breakdown of 1000 votes in a student election. How many votes did Sue receive?

(**A**) 550 (**B**) 350 (**C**) 330
(**D**) 450 (**E**) 935



Solution

Sue received $100 - (20 + 45) = 35$ percent of the total number of votes. Since there was a total of 1000 votes, Sue received $0.35(1000) = 350$ votes. ANSWER: (B)

3. The expression $\dfrac{a^9 \times a^{15}}{a^3}$ is equal to

(**A**) a^{45} (**B**) a^8 (**C**) a^{18} (**D**) a^{14} (**E**) a^{21}

Solution

$$\frac{a^9 \times a^{15}}{a^3} = \frac{a^{24}}{a^3}$$
$$= a^{21}$$

ANSWER: (E)

4. The product of two positive integers p and q is 100. What is the largest possible value of $p + q$?

(**A**) 52 (**B**) 101 (**C**) 20 (**D**) 29 (**E**) 25

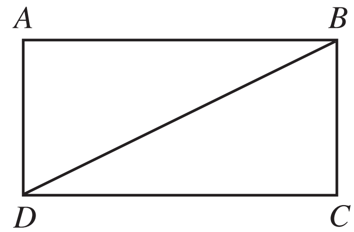
Solution

The pairs of positive integers whose product is 100 are: 1 and 100, 2 and 50, 4 and 25, 5 and 20, 10 and 10. The pair with the largest sum is 1 and 100. The sum is 101.

ANSWER: (B)

5. In the diagram, $ABCD$ is a rectangle with $DC = 12$. If the area of triangle BDC is 30, what is the perimeter of rectangle $ABCD$?

(**A**) 34 (**B**) 44 (**C**) 30
(**D**) 29 (**E**) 60



Solution

Since the area of ΔBDC is 30, we know

$$\frac{1}{2}(12)(BC) = 30$$
$$6(BC) = 30$$
$$BC = 5$$

Thus, the perimeter of rectangle $ABCD$ is $2(12) + 2(5) = 34$. ANSWER: (A)

6. If $x = 2$ is a solution of the equation $qx - 3 = 11$, the value of q is

(**A**) 4 (**B**) 7 (**C**) 14 (**D**) -7 (**E**) -4

Solution

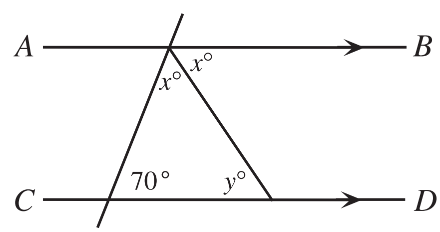
If $x = 2$ is a solution of $qx - 3 = 11$, then

$$q(2) - 3 = 11$$
$$2q = 14$$
$$q = 7$$

ANSWER: (B)

7. In the diagram, AB is parallel to CD. What is the value of y?

(**A**) 75 (**B**) 40 (**C**) 35
(**D**) 55 (**E**) 50

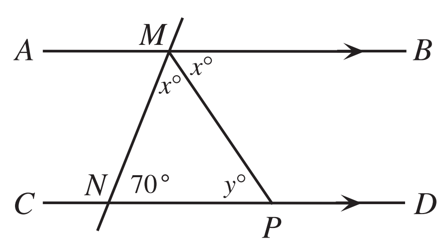


Solution

Since AB is paralled to CD, then $\angle BMN + \angle MND = 180$.

Thus, $2x + 70 = 180$
$$2x = 110$$
$$x = 55$$

Using ΔMNP, $y = 180 - (70 + 55)$
$$= 55$$



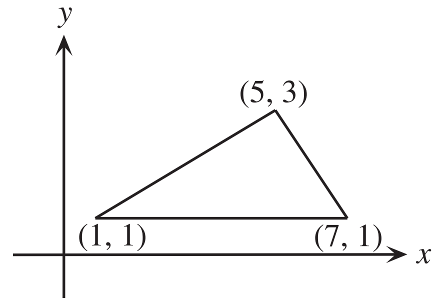
ANSWER: (D)

8. The vertices of a triangle have coordinates $(1,1)$, $(7,1)$ and $(5,3)$. What is the area of this triangle?

(**A**) 12 (**B**) 8 (**C**) 6 (**D**) 7 (**E**) 9

Solution

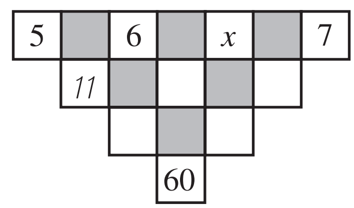
Draw the triangle on the coordinate axes. This triangle has a base of 6 and a height of 2. Its area is $\frac{1}{2}(6)(2)=6$.



ANSWER: (C)

9. The number in an unshaded square is obtained by adding the numbers connected to it from the row above. (The '11' is one such number.) The value of x must be

(**A**) 4 (**B**) 6 (**C**) 9
(**D**) 15 (**E**) 10



Solution

The three entries in row two, from left to right, are 11, $6+x$, and $x+7$. The two entries in row three, from left to right, are $11+(6+x)=17+x$ and $(6+x)+(x+7)=2x+13$. The single entry in row four is $(17+x)+(2x+13)=3x+30$.

Thus,
$$\begin{aligned} 3x+30&=60\\ 3x&=30\\ x&=10 \end{aligned}$$

ANSWER: (E)

10. The sum of the digits of a five-digit positive integer is 2. (A five-digit integer cannot start with zero.) The number of such integers is

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution

If the sum of the digits of a five-digit positive integer is 2, then the only possible integers are 20 000, 11 000, 10 100, 10 010, and 10 001. There are 5 such integers. ANSWER: (E)

PART B:

11. If $x+y+z=25$, $x+y=19$ and $y+z=18$, then y equals

(**A**) 13 (**B**) 17 (**C**) 12 (**D**) 6 (**E**) -6

Solution

We are given that $x + y + z = 25$ (1)

$x + y = 19$ (2)

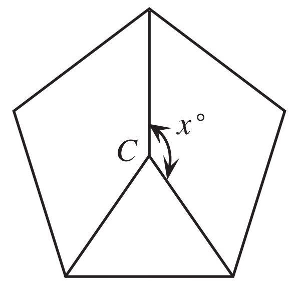
$y + z = 18$ (3)

Add equations (2) and (3) to get $x + 2y + z = 37$ (4). Subtract equation (1) from equation (4) to get $y = 12$.

ANSWER: (C)

12. A regular pentagon with centre C is shown. The value of x is

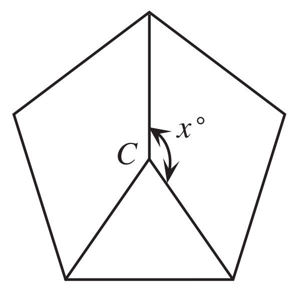
(**A**) 144 (**B**) 150 (**C**) 120
(**D**) 108 (**E**) 72



Solution

Join C to each of the remaining vertices, as shown. Since the pentagon is regular, each of the small angles at C has measure $360^\circ \div 5 = 72^\circ$.

Thus, the value of x is $2(72) = 144$.



ANSWER: (A)

13. If the surface area of a cube is 54, what is its volume?

(**A**) 36 (**B**) 9 (**C**) $\frac{81\sqrt{3}}{8}$ (**D**) 27 (**E**) $162\sqrt{6}$

Solution

Let each edge of the cube have length x. Then each face of the cube has area x^2. Since a cube has six faces, the surface area of the cube is $6x^2$.

We know $6x^2 = 54$

$x^2 = 9$

$x = 3$

Since each edge of the cube has length 3, the volume of the cube is $3^3 = 27$.

ANSWER: (D)

14. The number of solutions (x, y) of the equation $3x + y = 100$, where x and y are positive integers, is

(**A**) 33 (**B**) 35 (**C**) 100 (**D**) 101 (**E**) 97

Solution

Rewrite the given equation as $x = \frac{100-y}{3}$. Since x must be a positive integer, $100-y$ must be divisible by 3. Since y must also be a positive integer, the only possible values of y are 1, 4, 7, 10, 13, ... , 94, and 97. Thus, there are 33 possible values for y, and 33 solutions (x, y) that meet the given conditions.

ANSWER: (A)

15. If $\sqrt{y-5} = 5$ and $2^x = 8$, then $x+y$ equals

(**A**) 13 (**B**) 28 (**C**) 33 (**D**) 35 (**E**) 38

Solution

Since $\sqrt{y-5} = 5$, then

$$\left(\sqrt{y-5}\right)^2 = 5^2$$
$$y-5 = 25$$
$$y = 30$$

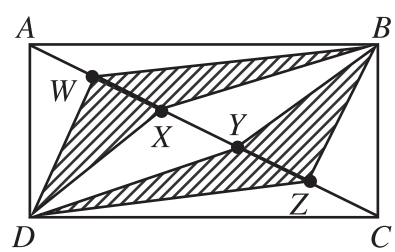
Also, since $2^x = 8$, then

$$2^x = 2^3$$
$$x = 3$$

Thus, $x+y = 33$.

ANSWER: (C)

16. Rectangle $ABCD$ has length 9 and width 5. Diagonal AC is divided into 5 equal parts at W, X, Y, and Z. Determine the area of the shaded region.



(**A**) 36 (**B**) $\frac{36}{5}$ (**C**) 18

(**D**) $\frac{4\sqrt{106}}{5}$ (**E**) $\frac{2\sqrt{106}}{5}$

Solution

Triangle ABC has area $\frac{1}{2}(9)(5) = \frac{45}{2}$. Triangles ABW, WBX, XBY, YBZ, and ZBC have equal bases and altitudes, so the area of each of these small triangles is $\frac{1}{5}\left(\frac{45}{2}\right) = \frac{9}{2}$. Similarly, triangles ADW, WDX, XDY, YDZ, and ZDC each have area $\frac{9}{2}$.

Thus, the shaded region has area $4\left(\frac{9}{2}\right) = 18$.

ANSWER: (C)

17. If $N = \left(7^{p+4}\right)\left(5^q\right)\left(2^3\right)$ is a perfect cube, where p and q are positive integers, the smallest possible value of $p+q$ is

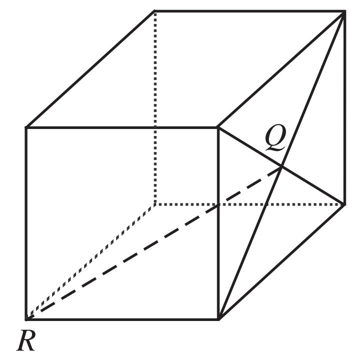
(**A**) 5 (**B**) 2 (**C**) 8 (**D**) 6 (**E**) 12

Solution

In order for N to be a perfect cube, each prime factor of N must have an exponent that is divisible by 3. Since p and q must be positive integers, the smallest value of p is 2 and the smallest value of q is 3. Thus, the smallest value of $p+q$ is 5. ANSWER: (A)

18. Q is the point of intersection of the diagonals of one face of a cube whose edges have length 2 units. The length of QR is

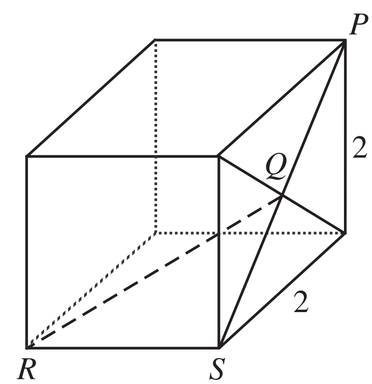
(**A**) 2 (**B**) $\sqrt{8}$ (**C**) $\sqrt{5}$
(**D**) $\sqrt{12}$ (**E**) $\sqrt{6}$



Solution

Label point S as shown. Since each face of the cube is a square with sides of length 2, use the Pythagorean Theorem to find the length of diagonal PS.

$$\begin{aligned} PS^2 &= 2^2 + 2^2 \\ &= 8 \\ PS &= 2\sqrt{2} \end{aligned}$$



Then QS has length $\sqrt{2}$, as Q is the midpoint of diagonal PS.

Because we are working with a cube, $\angle QSR = 90^\circ$ and ΔQRS is a right-angled triangle. Use the Pythagorean Theorem in ΔQRS to get

$$\begin{aligned} QR^2 &= 2^2 + \left(\sqrt{2}\right)^2 \\ &= 6 \\ QR &= \sqrt{6} \end{aligned}$$

ANSWER: (E)

19. Mr. Anderson has more than 25 students in his class. He has more than 2 but fewer than 10 boys and more than 14 but fewer than 23 girls in his class. How many different class *sizes* would satisfy these conditions?

(**A**) 5 (**B**) 6 (**C**) 7 (**D**) 3 (**E**) 4

Solution

Let b represent the number of boys and g represent the number of girls in Mr. Anderson's class. We know that $b+g>25$. We also know $2<b<10$ and $14<g<23$.

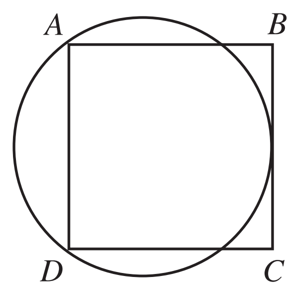
The following pairs (b,g) satisfy all three conditions: (4,22), (5,21), (5,22), (6,20), (6,21), (6,22), (7,19), (7,20), (7,21), (7,22), (8,18), (8,19), (8,20), (8,21), (8,22), (9,17), (9,18), (9,19), (9,20), (9,21), (9,22).

All of the different pairs (b,g) lead to the following different class sizes: 26, 27, 28, 29, 30, 31.

Thus, only 6 different class sizes are possible. ANSWER: (B)

20. Each side of square $ABCD$ is 8. A circle is drawn through A and D so that it is tangent to BC. What is the radius of this circle?

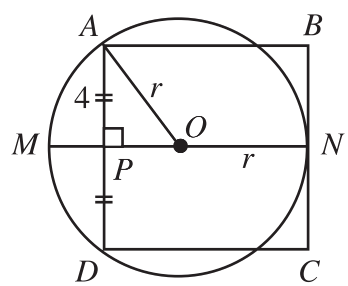
(**A**) 4 (**B**) 5 (**C**) 6
(**D**) $4\sqrt{2}$ (**E**) 5.25



Solution

Let r represent length of the radius and let O represent the centre of the circle. Draw diameter MN that bisects chord AD perpendicularly at P. Join OA.

ΔOAP is a right-angled triangle with $\angle APO = 90°$. The length of AP is 4, since it is half of a side of square $ABCD$. The length of OA is r, and the length of PO is $PN - ON = 8 - r$.



Using the Pythagorean Theorem we get

$$
\begin{aligned}
r^2 &= 4^2 + (8-r)^2 \\
r^2 &= 16 + 64 - 16r + r^2 \\
16r &= 80 \\
r &= 5
\end{aligned}
$$

Thus, the radius of the circle is 5. ANSWER: (B)

PART C:

21. When Betty substitutes $x=1$ into the expression $ax^3 - 2x + c$ its value is -5. When she substitutes $x=4$ the expression has value 52. One value of x that makes the expression equal to zero is

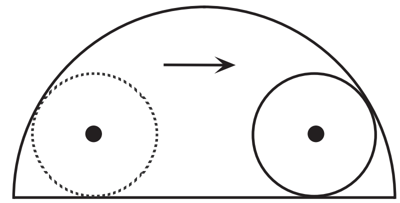
(**A**) 2 (**B**) $\frac{5}{2}$ (**C**) 3 (**D**) $\frac{7}{2}$ (**E**) 4

Solution

When $x=1$, we are told that $a(1)^3 - 2(1) + c = -5$, or $a+c=-3$ (1). Similarly, when $x=4$, $a(4)^3 - 2(4) + c = 52$, or $64a + c = 60$ (2). Subtracting equation (1) from equation (2) gives $63a = 63$, or $a=1$. Substituting $a=1$ into equation (1) gives $c=-4$. The original expression

$ax^3 - 2x + c$ becomes $x^3 - 2x - 4$. By trial and error, using divisors of 4, when $x = 2$ we get $2^3 - 2(2) - 4 = 0$. ANSWER: (A)

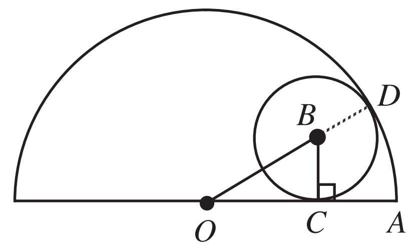
22. A wheel of radius 8 rolls along the diameter of a semicircle of radius 25 until it bumps into this semicircle. What is the length of the portion of the diameter that cannot be touched by the wheel?



(**A**) 8 (**B**) 12 (**C**) 15
(**D**) 17 (**E**) 20

Solution

Draw ΔOBC, where O is the centre of the large circle, B is the centre of the wheel, and C is the point of tangency of the wheel and the diameter of the semicircle. Since BC is a radius of the wheel, $\angle OBC = 90^\circ$ and ΔOBC is right-angled at C.



Extend OB to meet the semicircle at D. Then $BD = BC = 8$, since they are both radii of the wheel, and $OB = 25 - 8 = 17$.

Use the Pythagorean Theorem in ΔOBC to find OC.

$$OC^2 = 17^2 - 8^2$$
$$OC^2 = 225$$
$$OC = 15$$

Then $AC = 25 - 15 = 10$. The length of the portion of the diameter that cannot be touched by the wheel is a length equivalent to $2AC$ or 20. ANSWER: (E)

23. There are four unequal, positive integers a, b, c, and N such that $N = 5a + 3b + 5c$. It is also true that $N = 4a + 5b + 4c$ and N is between 131 and 150. What is the value of $a + b + c$?

(**A**) 13 (**B**) 17 (**C**) 22 (**D**) 33 (**E**) 36

Solution

We are told that $N = 5a + 3b + 5c$ (1) and $N = 4a + 5b + 4c$ (2). Multiply equation (1) by 4 to get $4N = 20a + 12b + 20c$ (3). Similarly, multiply equation (2) by 5 to get $5N = 20a + 25b + 20c$ (4). Subtract equation (3) from equation (4) to get $N = 13b$.

Since N and b are both positive integers with $131 < N < 150$, N must be a multiple of 13. The only possible value for N is 143, when $b = 11$.

Substitute $N = 143$ and $b = 11$ into equation (1) to get

$$143 = 5a + 3(11) + 5c$$
$$110 = 5a + 5c$$
$$22 = a + c$$

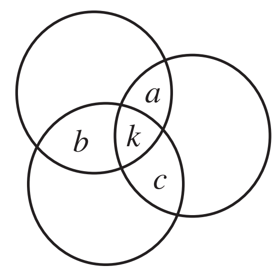
Thus, the value of $a+b+c$ is 22+11=33. ANSWER: (D)

24. Three rugs have a combined area of 200 m^2. By overlapping the rugs to cover a floor area of 140 m^2, the area which is covered by exactly two layers of rug is 24 m^2. What area of floor is covered by three layers of rug?

(**A**) 12 m^2 (**B**) 18 m^2 (**C**) 24 m^2 (**D**) 36 m^2 (**E**) 42 m^2

Solution

Draw the rugs in the following manner, where $a+b+c$ represents the amount of floor covered by exactly two rugs and k represents the amount of floor covered by exactly three rugs. We are told that $a+b+c=24$ (1).



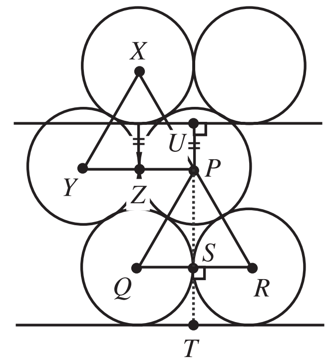
Since the total amount of floor covered when the rugs do not overlap is 200 m^2 and the total covered when they do overlap is 140 m^2, then 60 m^2 of rug is wasted on double or triple layers. Thus, $a+b+c+2k=60$ (2). Subtract equation (1) from equation (2) to get $2k=36$ and solve for $k=18$. Thus, the area of floor covered by exactly three layers of rug is 18 m^2. ANSWER: (B)

25. One way to pack a 100 by 100 square with 10 000 circles, each of diameter 1, is to put them in 100 rows with 100 circles in each row. If the circles are repacked so that the centres of any three tangent circles form an equilateral triangle, what is the maximum number of additional circles that can be packed?

(**A**) 647 (**B**) 1442 (**C**) 1343 (**D**) 1443 (**E**) 1344

Solution

Remove one circle from every second row and shift to form the given configuration. Label the diagram as shown. Since each circle has diameter 1, ΔPQR and ΔPXY are equilateral triangles with sides of length 1.



In ΔPQR, altitude PS bisects side QR. Use the Pythagorean Theorem to find PS.

$$PS^2 = (1)^2 - \left(\frac{1}{2}\right)^2$$
$$= \frac{3}{4}$$
$$PS = \frac{\sqrt{3}}{2}$$

Also, $XZ = \frac{\sqrt{3}}{2}$

Since all radii have length $\frac{1}{2}$, then $PU = \frac{\sqrt{3}}{2} - \frac{1}{2}$ and $TU = \frac{1}{2} + \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \sqrt{3}$. This tells us that two rows of circles require a height of $\sqrt{3}$ before a third row begins.

Since $\frac{100}{\sqrt{3}} = 57.7$, we can pack 57 double rows, each containing 100+99=199 circles.

Can we pack one final row of 100 circles? Yes. The square has sides of length 100 and our configuration of 57 double rows requires a height of $57\sqrt{3}$ before the next row begins. Since $100 - 57\sqrt{3} > 1$, and since the circles each have diameter 1, there is room for one final row of 100 circles.

The number of circles used in this new packing is 57(199)+100=11 443

Thus, the maximum number of extra circles that can be packed into the square is

11 443 − 10 000 = 1443. ANSWER: (D)

Canadian
Mathematics
Competition

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1997 Solutions
Cayley Contest (Grade 10)

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Awards

PART A:

1. ***Solution***

$$2\tfrac{1}{10} + 3\tfrac{11}{100} = 2.1 + 3.11$$
$$= 5.21$$

ANSWER: (D)

2. ***Solution***

$$(1)^{10} + (-1)^{8} + (-1)^{7} + (1)^{5} = 1 + 1 - 1 + 1$$
$$= 2$$

ANSWER: (C)

3. ***Solution***

Since the final result contains a factor of 10, it must have at least one zero at the end. The only listed possibility is 30. ANSWER: (E)

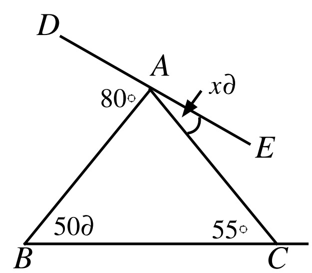
4. ***Solution***

If the first day is a Monday, then every seventh day is also a Monday, and Monday falls on the following numbered days: 1, 8, 15, 22, 29, 36, 43.
In 45 consecutive days, the maximum number of Mondays is seven. ANSWER: (C)

5. ***Solution***

The value of $-BAC$, in degrees, is $180 - 50 - 55 = 75$.
Since D, A, and E lie on a straight line,

$$80 + 75 + x = 180$$
$$x = 25$$



ANSWER: (A)

6. ***Solution***

The first ten balloons are popped in the following order: C, F, I, L, D, H, A, G, B, and K.
The two remaining balloons are E and J. ANSWER: (D)

7. ***Solution***

In rectangle $ABCD$, side AB has length $4 - (-3) = 7$. Since the area of the rectangle is 70, the length of side AD must be $\frac{70}{7} = 10$.
Thus, the value of k is $1 + 10 = 11$. ANSWER: (D)

8. ***Solution***
Rearranging and combining the inequalities yields $p < q < t < r < s$. The greatest of these numbers is s. ANSWER: (B)

9. ***Solution***
Since the sum of the seven integers is 77, their average is $\frac{77}{7} = 11$. Because there is an odd number of consecutive integers, 11 is the middle integer and the smallest is 8. ANSWER: (C)

10. ***Solution***
The greatest possible value of p^q is $3^4 = 81$.
Thus, the greatest possible value of $p^q + r^s$ is $3^4 + 2^1 = 83$. ANSWER: (E)

PART B:

11. ***Solution***
Since x, y, z, and w are integers, then y must divide evenly into both 6 and 25. The only possible value of y is 1. Thus, $x = 6$ and $w = 25$. The value of xw is $(6)(25) = 150$. ANSWER: (A)

12. ***Solution***
Let the depth of each cut be d.
Then,
$$\begin{aligned} 80(15) - 5d - 15d - 10d &= 990 \\ 1200 - 30d &= 990 \\ 30d &= 210 \\ d &= 7 \end{aligned}$$
The depth of each cut is 7. ANSWER: (B)

13. ***Solution***
Using Pythagoras in ΔABC gives BC to be 6. Since $BC = 3DC$, $DC = 2$.
$$\begin{aligned} AD^2 &= 2^2 + 8^2 \\ &= 68 \end{aligned}$$
Using Pythagoras again in ΔADC, $AD = \sqrt{68}$ ANSWER: (E)

14. ***Solution***
The first twelve numbers in the list begin with either the digit 1 or 2. The next six begin with the digit 3. In order, these six numbers are 3124, 3142, 3214, 3241, 3412, 3421.
We see that the number 3142 is in the fourteenth position. ANSWER: (B)

15. ***Solution***

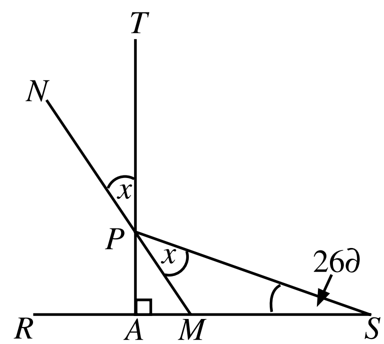
Since each factor of 10 produces a zero at the end of the integer we want to know how many 10's occur in the product.

The product of 20^{50} and 50^{20} can be rewritten as follows:

$$\begin{aligned}\left(20^{50}\right)\left(50^{20}\right) &= \left(2^2 \times 5\right)^{50}\left(5^2 \times 2\right)^{20} \\ &= 2^{100} \times 5^{50} \times 5^{40} \times 2^{20} \\ &= 2^{120} \times 5^{90} \\ &= 2^{30}\left(2^{90} \times 5^{90}\right) \\ &= 2^{30} \times 10^{90}\end{aligned}$$

From this, we see that there are 90 zeros at the end of the resulting integer. ANSWER: (C)

16. ***Solution***



In the diagram, extend TP to meet RS at A. Since $AT \perp RS$,

$$\angle SPA = 180° - 90° - 26°$$

then $$= 64°$$

Label points M and N. Since $\angle TPN$ and $\angle MPA$ are vertically opposite angles, they are equal, so $\angle MPA = x$.

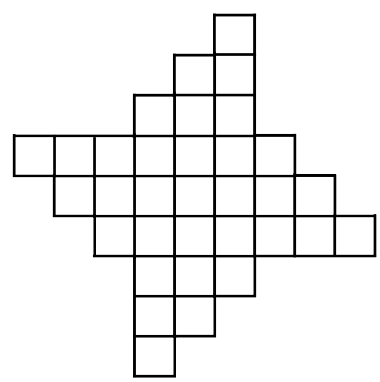
Since

$$\angle SPA = 2x, \quad 2x = 64°$$
$$x = 32°$$

Thus, the value of x is $32°$.

ANSWER: (A)

17. ***Solution***



Since all of the shorter edges are equal in length, the diagram can be subdivided into 33 small squares, as shown. Each of these squares has area $\frac{528}{33} = 16$ and the length of each side is $\sqrt{16} = 4$.

By counting, we find 36 sides and a perimeter of 144.

ANSWER: (E)

18. ***Solution***

Rewrite $\frac{30}{7}$ as

$$4+\frac{2}{7}=4+\frac{1}{\left(\frac{7}{2}\right)}$$

$$=4+\frac{1}{3+\frac{1}{2}}$$

By comparison, $x=4$, $y=3$ and $z=2$.

Thus, $x+y+z=9$. ANSWER: (B)

19. ***Solution***

By multiplying the given equations together we obtain

$$\left(x^2yz^3\right)\left(xy^2\right)=\left(7^4\right)\left(7^5\right)$$

$$x^3y^3z^3=7^9$$

Taking the cube root of each side gives $xyz=7^3$. ANSWER: (C)

20. ***Solution***

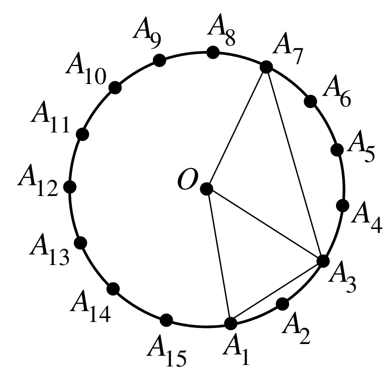
Join A_1, A_3 and A_7 to O, the centre of the circle, as shown. Since the points $A_1, A_2, A_3, \ldots, A_{15}$ are evenly spaced, they generate equal angles at O, each of measure $\frac{360^\circ}{15}=24^\circ$.

Thus, $\angle A_1OA_3=48^\circ$ and $\angle A_3OA_7=96^\circ$.

Since $OA_1=OA_3$ (radii), $\triangle A_1OA_3$ is isosceles, and

$$\angle A_1A_3O=\frac{(180^\circ-48^\circ)}{2}$$

$$=66^\circ.$$



Similarly, $\triangle A_3OA_7$ is isosceles, and

$$\angle OA_3A_7=\frac{(180^\circ-96^\circ)}{2}$$

$$=42^\circ.$$

$$\angle A_1A_3A_7=\angle A_1A_3O+\angle OA_3A_7$$

$$=66^\circ+42^\circ$$

Thus, $=108^\circ$. ANSWER: (D)

PART C:

21. ***Solution***

Simplify the expression $\dfrac{\left(\frac{a}{c}+\frac{a}{b}+1\right)}{\left(\frac{b}{a}+\frac{b}{c}+1\right)}=11$ as follows:

$$\frac{\left(\frac{ab+ac+bc}{bc}\right)}{\left(\frac{bc+ab+ac}{ac}\right)}=11$$

$$\frac{ac}{bc}=11$$

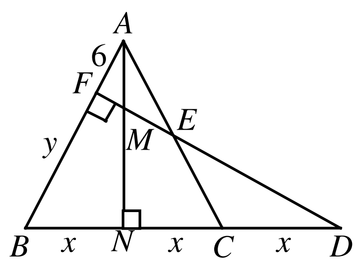
$$\frac{a}{b}=11 \quad (\text{since } c\neq 0)$$

$$a=11b$$

By substitution, the condition $a+2b+c\leq 40$ becomes $13b+c\leq 40$.
Since b and c are positive integers, then b can only take on the values 1, 2, or 3. The values of a correspond directly to the values of b, since $a=11b$.
If $b=3$, there is one corresponding value of c. When $b=2$, there are 14 possible values of c. Finally if $b=1$, there are 27 possible values of c.
Therefore, the number of different ordered triples satisfying the given conditions is $1+14+27=42$.

ANSWER: (D)

22. ***Solution***



Drop a perpendicular from A to BC, and label as shown.
Since ΔABC is equilateral, $BN=NC=CD$. Let $BN=x$ and $BF=y$.
Then $6+y=2x$. (1)
Also, $\angle FAM=30°$, and ΔAMF is a $30°-60°-90°$ triangle with sides in the ratio $1:\sqrt{3}:2$.
Thus, $AM=4\sqrt{3}$ and $FM=2\sqrt{3}$.
Use similar triangles DBF and AMF to find

$$\frac{DB}{BF}=\frac{AM}{MF}$$

$$\frac{3x}{y}=\frac{4\sqrt{3}}{2\sqrt{3}}$$

$$3x=2y \qquad (2)$$

Solving equations (1) and (2) yields $x=12$ and $y=18$.
Find $AN=12\sqrt{3}$; the area of ΔABC is thus $144\sqrt{3}$.
Use similar triangles EAF and AMF to find $FE=6\sqrt{3}$; the area of ΔAFE is $18\sqrt{3}$.
The area of quadrilateral $FBCE$ is $126\sqrt{3}$.

ANSWER: (C)

23. ***Solution***

First count the number of integers between 3 and 89 that can be written as the sum of exactly two elements. Since each element in the set is the sum of the two previous

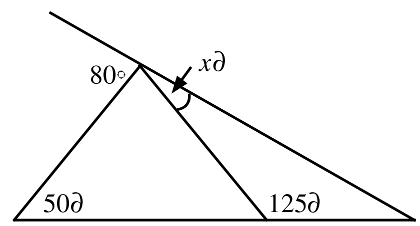
elements, 55 can be added to each of the seven smallest elements to form seven unique integers smaller than 89.
In the same way, 34 can be added to each of the seven smaller elements, 21 can be added to each of the six smaller elements, and so on.
The number of integers between 3 and 89 that can be written as the sum of two elements of the set is $7+7+6+5+4+3+2=34$. Since there are 85 integers between 3 and 89, then $85-34=51$ integers cannot be written as the sum of exactly two elements in the set.

ANSWER: (A)

24. ***Solution***

Since exactly five interior angles are obtuse, then exactly five exterior angles are acute and the remaining angles must be obtuse. Since we want the maximum number of obtuse angles, assume that the five acute exterior angles have a sum less than 90°. Since obtuse angles are greater than 90°, we can only have three of them.
Thus, the given polygon can have at most $3+5=8$ sides. ANSWER: (B)

25. ***Solution***



Join A to R and C to T. Label the diagram as shown. Let the area of $\triangle ABC$ be k.
Since triangles CRS and CRA have equal heights and bases that are in the ratio $3b:4b=3:4$, then
area of $\triangle CRS = \frac{3}{4}$ (area of $\triangle CRA$)
However, since triangles CRA and ABR also have equal heights and bases that are in the ratio $a:a=1:1$, then

$$\triangle CRS = \tfrac{3}{8} \text{ (area of } \triangle ABC)$$

area of $$= \frac{3k}{8}$$

Similarly, area of $$\triangle TBR = \frac{q}{p+q} \text{ (area of } \triangle ABR) = \frac{qk}{2(p+q)}$$

and area of $$\triangle ATS = \frac{1}{4} \text{ (area of } \triangle ATC) = \frac{pk}{4(p+q)}$$

Since (area of $\triangle RST$) = 2(area of $\triangle TBR$), then

$$k-\frac{3k}{8}-\frac{qk}{2(p+q)}-\frac{pk}{4(p+q)}=\frac{2qk}{2(p+q)}$$

$$1-\frac{3}{8}-\frac{q}{2(p+q)}-\frac{p}{4(p+q)}=\frac{2q}{2(p+q)} \text{ (since } k\,\Sigma 0\text{)}$$

Simplify this expression to get $\frac{p}{q}=\frac{7}{3}$. ANSWER: (E)