Adventures in Problem Solving

Ian VanderBurgh
(ian.vanderburgh@uwaterloo.ca)

Centre for Education in Mathematics and Computing
Faculty of Mathematics, University of Waterloo

cemc.uwaterloo.ca

Saturday 14 October 2017

Problem #1

Connie has a number of gold bars, all of different weights. She gives the 24 lightest bars, which weigh 45% of the total weight, to Brennan. She gives the 13 heaviest bars, which weigh 26% of the total weight, to Maya. She gives the rest of the bars to Blair. How many bars did Blair receive?

Problem #2

The product of N consecutive four-digit positive integers is divisible by 2010^2. What is the least possible value of N?

Problem #3

In a psychology experiment, an image of a cat or an image of a dog is flashed briefly onto a screen and then Anna is asked to guess whether the image showed a cat or a dog. This process is repeated a large number of times with an equal number of images of cats and images of dogs shown. If Anna is correct 95% of the time when she guesses "dog" and 90% of the time when she guesses "cat", determine the ratio of the number of times she *guessed* "dog" to the number of times she *guessed* "cat".

Problem #4

A class has 20 students. Of these 20 students, 18 have a dog, 16 have a cat, and 11 have a turtle.

(a) If x is the largest possible number of students that have a dog and a cat, and y is the smallest possible number of houses that have a dog and a cat, what are x and y?

(b) If x is the largest possible number of students that have a dog and a cat and a turtle, and y is the smallest possible number of houses that have a dog and a cat and a turtle, what are x and y?

Problem #5

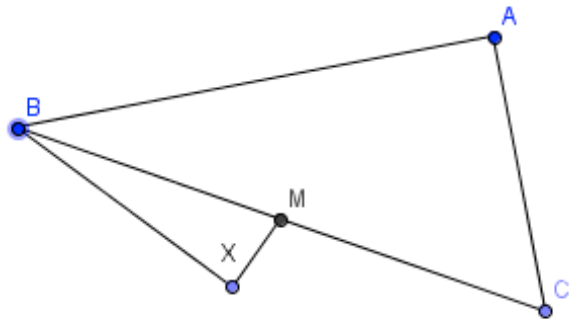
Residents were surveyed in order to determine which flowers to plant in the new Public Garden. A total of N people participated in the survey. Exactly $\frac{9}{14}$ of those surveyed said that the colour of the flower was important. Exactly $\frac{7}{12}$ of those surveyed said that the smell of the flower was important. In total, 753 people said that both the colour and smell were important. How many possible values are there for N?

Problem #6

A town has 2017 houses. Of these 2017 houses, 1820 have a dog, 1651 have a cat, and 1182 have a turtle. If x is the largest possible number of houses that have a dog, a cat, and a turtle, and y is the smallest possible number of houses that have a dog, a cat, and a turtle, what is $x - y$?

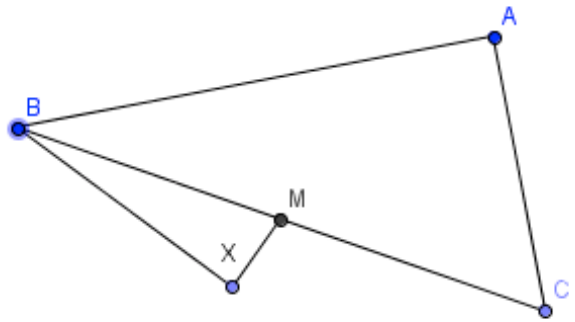
Problem #7

In the diagram, AC has length 30, M is the midpoint of BC, MX has length 7 and MC has length 25. CAB and MXB are right angles. Determine the distance from A to X.

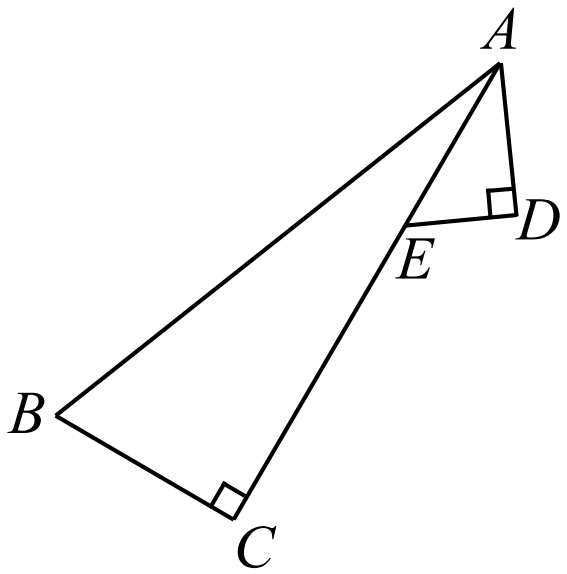


Problem #7

In the diagram, AC has length 30, M is the midpoint of BC, MX has length 7 and MC has length 25. CAB and MXB are right angles. Determine the distance from A to X.



In the diagram, $\angle ACB = \angle ADE = 90^\circ$. If $AB = 75$, $BC = 21$, $AD = 20$, and $CE = 47$, determine the exact length of BD.



Problem #8

Six soccer teams are competing in a tournament in Waterloo. Every team is to play three games, each against a different team. (Note that not every pair of teams plays a game together.) Judene is in charge of pairing up the teams to create a schedule of games that will be played. Ignoring the order and times of the games, how many different schedules are possible?

Problem #8

Six soccer teams are competing in a tournament in Waterloo. Every team is to play three games, each against a different team. (Note that not every pair of teams plays a game together.) Judene is in charge of pairing up the teams to create a schedule of games that will be played. Ignoring the order and times of the games, how many different schedules are possible?

An increasing list of two-digit positive integers is formed so that

- each integer in the list uses only digits from 1, 2, 3, 4, 5, 6,
- each of the digits 1, 2, 3, 4, 5, 6 appears in exactly three of the integers in the list, and
- each of the integers in the list has the property that its units digit is greater than its tens digit.

How many different lists are possible?

Problem #9

Consider the equation $x^2 - 2y^2 = 1$, which we label ①. There are many pairs of positive integers (x, y) that satisfy equation ①.

(a) Determine a pair of positive integers (x, y) with $x \leq 5$ that satisfies equation ①.

(b) Determine a pair of positive integers (u, v) such that

$$(3 + 2\sqrt{2})^2 = u + v\sqrt{2}$$

and show that (u, v) satisfies equation ①.

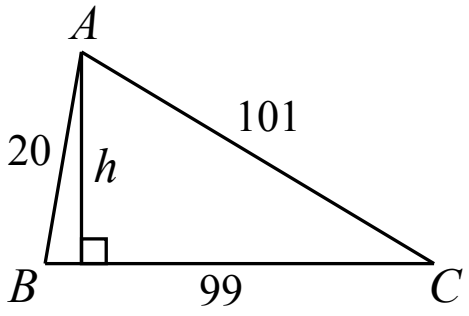
(c) Suppose that (a, b) is a pair of positive integers that satisfies equation ①. Suppose also that (c, d) is a pair of positive integers such that $(a + b\sqrt{2})(3 + 2\sqrt{2}) = c + d\sqrt{2}$. Show that (c, d) satisfies equation ①.

(d) Determine a pair of positive integers (x, y) with $y > 100$ that satisfies equation ①.

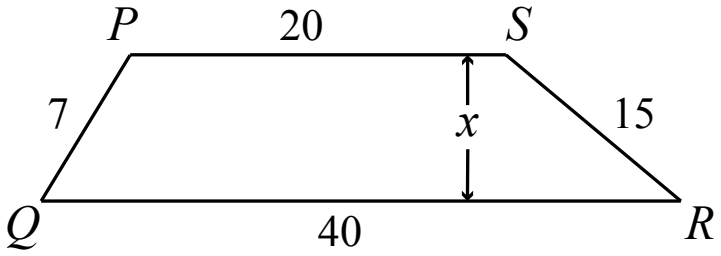
Problem #10

Heron's Formula says that if a triangle has side lengths a, b and c, then its area equals $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$ is called the *semi-perimeter* of the triangle.

(a) In the diagram, $\triangle ABC$ has side lengths $AB = 20$, $BC = 99$, and $AC = 101$. If h is the perpendicular distance from A to BC, determine the value of h.



(b) In the diagram, trapezoid $PQRS$ has PS parallel to QR. Also, $PQ = 7$, $QR = 40$, $RS = 15$, and $PS = 20$. If x is the distance between parallel sides PS and QR, determine the value of x.



Problem #10

(c) The triangle with side lengths 3, 4 and 5 has the following five properties:

- its side lengths are integers,
- the lengths of its two shortest sides differ by one,
- the length of its longest side and the semi-perimeter differ by one,
- its area is an integer, and
- its perimeter is less than 200.

Determine all triangles that have these five properties.