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Intermediate Math Circles

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Sequences

What is a sequence?

A sequence is an ordered list of numbers.

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$$

This is a finite sequence.

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{50}$$

The “...” indicates the sequence continues on up until the last term $\frac{1}{50}$. This is a finite sequence.

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$$

The “...” indicates the sequence continues without ever ending. This is an infinite sequence.

Notation:

We denote the i th term by a_i

$$\text{Eg. } a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_5 = \frac{1}{5}$$

The **general term** of a sequence is a formula that expresses each term in the sequence as a function of its position.

The general term for this sequence is $a_n = \frac{1}{n}$

1. What is the next term? What is the general term?

- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$

- $1, 5, 9, 13, 17, \dots$

- $3, 6, 12, 24, 48, \dots$

- $1, 8, 27, 64, 125, \dots$

- $1, -2, 4, -8, 16, \dots$

- $2, \frac{5}{2}, 3, \frac{7}{2}, 4, \dots$

- $\frac{3}{4}, \frac{5}{9}, \frac{7}{16}, \frac{9}{25}, \frac{11}{36}, \dots$

- $1, 2, 6, 24, 120, \dots$

- $1, 9, 29, 67, 129, \dots$

Arithmetic Sequences:

An arithmetic sequence is a sequence where the difference between any consecutive pairs of numbers in the sequence is a constant. This difference is called the **common difference**.

For example: $1, 3, 5, 7, 9, \dots$ is an arithmetic sequence with first term 1 and common difference 2.

Are any of the sequences above arithmetic sequences? Which ones?

The terms in an arithmetic sequence with first term a and common difference d are:

$$a_1 = a, \quad a_2 = a + d, \quad a_3 = (a + d) + d = a + 2d, \quad a_4 = (a + 2d) + d = a + 3d, \dots$$

In general, the n th term of an arithmetic sequence with first term a and common difference d is:

$$a_n = a + (n - 1)d$$

Eg., in the sequence $1, 3, 5, 7, 9, \dots$, the 5th term is $a_5 = 1 + (5 - 1)2 = 9$

Geometric Sequences:

A geometric sequence is a sequence where there is the same ratio between any consecutive terms. This ratio is called the **common ratio**.

For example: 1, 2, 4, 8, 16, ... is an infinite geometric sequence with first term 1 and common ratio 2.

Are any of the sequences above geometric sequences? Which ones?

The terms in a geometric sequence with first term a and common ratio r are:

$$a_1 = a, \quad a_2 = ar, \quad a_3 = (ar)r = ar^2, \quad a_4 = (ar^2)r = ar^3, \dots$$

In general, the n th term of a geometric sequence with first term a and common ratio r is:

$$a_n = ar^{n-1}$$

Eg., in the sequence 1, 2, 4, 8, 16, ..., the 5th term is $a_5 = 1 \times (2)^{5-1} = 16$

4. State the first 4 terms of the geometric sequence with general term $a_n = 2 \times (-3)^{n-2}$

5. What is the 11th term of the sequence $9, 3, 1, \frac{1}{3}, \dots$?

6. The 5th term of a geometric sequence is 35 and the sixth is 140, what is the 7th term?

Exercise Solutions:

- | | |
|-----------------------------|----------------------------------------------|
| 1. next term: $\frac{1}{6}$ | general term: $a_n = \frac{1}{n}$ |
| next term: 21 | general term: $a_n = 1 + 4(n - 1)$ |
| next term: 96 | general term: $a_n = 3 \times 2^{n-1}$ |
| next term: 216 | general term: $a_n = n^3$ |
| next term: -32 | general term: $a_n = (-2)^{n-1}$ |
| next term: $\frac{9}{2}$ | general term: $a_n = 2 + \frac{1}{2}(n - 1)$ |
| next term: $\frac{13}{49}$ | general term: $a_n = \frac{2n+1}{(n+1)^2}$ |
| next term: 720 | general term: $a_n = n!$ |
| next term: 221 | general term: $a_n = n^3 + n - 1$ |

2. Using the formula for an arithmetic sequence we get:

$$a_{23} = 18 + (22)(-7) = -136$$

3. Set up a system of equations:

$$a_4 = 11 = a + 3d$$

$$a_8 = 31 = a + 7d$$

from the first equations we get that $a = 11 - 3d$. Substituting into the second equation, we get $d = 5$, and thus $a = -4$.

Therefore the 100th term is $a_{100} = -4 + (99)5 = 491$

4. $\frac{-2}{3}, 2, -6, 18$

5. We note that the first term is 9 and the common ratio is $\frac{1}{3}$.

Therefore we have $a_{11} = 9\left(\frac{1}{3}\right)^{10} = \frac{1}{6561}$

6. From a_5 and a_6 we know that the common ratio is $\frac{140}{35} = 4$.

We know that $a_6 = 140$, thus $a_7 = a_6 \times r = 140 \times 4 = 560$.

Extra Problems:

- The first term of a sequence is 20.
If a term in the sequence is t and t is even, the next term is $\frac{1}{2}t$.
If a term in the sequence is t and t is odd, the next term is $3t + 1$.
Therefore, the first three terms in the sequence are 20, 10, 5.
What is the 10th term of the sequence?
- You start your new after-school job making \$8.90/hour. Your boss promises you an increase of \$0.15/hour after each month of work. How long will it take until you are making at least twice your starting wage?
- Determine the largest four-digit number to be found in the arithmetic sequence 1, 4, 7, 10, 13, 16, ...
- In a sequence of positive numbers, each term after the first two is the sum of *all* of the previous terms. If the first term is a , the second is 2, and the sixth is 56, then determine the value of a .
- If the first four terms of an arithmetic sequence are $a, 2a, b$, and $a - 6 - b$ for some numbers a and b , then what is the value of the 100th term?
- In the sequence of five numbers $x, _, 3, _, 18$, each number after the second is obtained by multiplying the two previous terms. Determine the value of x .
- Find the 11th term in the arithmetic sequence 17, 22, 27, 32, ...
 - Explain why there is no number which occurs in each of the following arithmetic sequences:
17, 22, 27, 32, ...
13, 28, 43, 58, ...
 - Find a number between 400 and 420 which occurs in both of the following arithmetic sequences:
17, 22, 27, 32, ...
16, 22, 28, 34, ...
- If $x - 1, 2x + 2$ and $7x + 1$ are the first three terms of an arithmetic sequence, determine the value of x .
 - For the value of x from (a), what is the middle term of the arithmetic sequence $x - 1, 2x + 2, 7x + 1, \dots, 72$?
- In the four term sequence 0.001, 0.02, 0.4, x , each term after the first is found by multiplying the previous term by the same number. What is the value of x ?
- A company is storing 3 kg of radioactive material until it becomes safe to the environment. After one year, 95% of the radioactive material remains. How much radioactive material will be left after 100 years?
- How many terms are there in the geometric sequence 1024, 256, 64, ..., $\frac{1}{64}$?
- If the 5th term of a geometric series is 35 and the eighth is 385, what is the 20th term?
- The sequence 6, $-9, x, y$ is such that the first three terms form an arithmetic sequence and the last three terms form a geometric sequence. Find the values of x and y .