University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Intermediate Math Circles
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Number Theory I

Opening Problem I

Suppose that you are given a row consisting of three O's followed by three X's:

O O O X X X

Your task is to obtain a row in which the O's and X's alternate:

O X O X O X or X O X O X O

Here are the rules:

- You make a succession of moves.
- Each move takes a consecutive pair of letters and moves them to either end of the row or into a pair of adjacent vacant slots.
- The final row does not need to occupy the same slots as the beginning row, but there must be no gaps between adjacent letters.

Find a solution.

Next, find a solution that uses the fewest possible number of moves.

Is it possible to find a solution that ends with O X O X O X?

Repeat this process starting with O O O O X X X X.

Repeat again starting with O O O O O X X X X X.

Can you determine a general process when starting with $4m$ X's and $4m$ O's?

Opening Problem II

Seven towns P, Q, R, S, T, U, V lie (in that order) along a road. The table below is meant to give all the distances between pairs of towns, in km; for example, the distance from P to S is 23 km. Unfortunately, fifteen of the distances are missing. How many of the missing distances can be calculated from the given information?

P						
?	Q					
?	?	R				
23	?	?	S			
?	30	?	?	T		
58	?	40	?	?	U	
?	68	?	53	?	?	V

Problem Set

1. Mike bought ten calculators for $10 each and ten CDs for $19 each. How much change does he get from $400?

2. Kylie the clumsy koala is all fingers and thumbs. Like all koala bears, Kylie has two thumbs and three fingers on each front paw, and one thumb and four fingers on each rear paw. How many thumbs do Kylie and her nine brothers have between them?

3. Farmer Bob counts his chickens and cows in an unusual way. He counts 49 heads and 142 legs. How many chickens and how many cows does he have?

4. Each face of a cube is to be painted. Six different colours of paint are available. What is the smallest number of colours one must use if "adjacent" faces (those having an edge in common) never have the same colour?

5. Tickets for a school play cost $3 for adults and $1 for children. The total amount collected from ticket sales was $1320. The play was staged in a cafetorium seating 600, but the cafetorium was not full. What was the smallest possible number of adults at the play?

6. The Grand Old Duke of York, he had ten thousand men, he marched them up to the top of the hill... By 2 p.m., they were one-third of the way up. By 4 p.m., they were three-quarters of the way up. When did they start their march?

7. A square is divided into nine rectangles by two horizontal and two vertical lines. The area of the top left rectangle is 1, the area of the top right rectangle is 3, and the area of the bottom right rectangle is 6. The central rectangle is a square. What is the area of the bottom left rectangle? What is the perimeter of the bottom left rectangle?

Number Theory

What is a prime number?

A prime number is a positive integer p with $p > 1$ whose only positive divisors are 1 and p.

What is a positive integer called that is not prime?

A trick question – it's either called "1" or composite.

How do we find prime numbers?

Method 1 – Pick a number and look for divisors. Is there a good way to do this? What divisors do we have to check? How high do we need to go checking divisors until we are sure that the number is prime?

Method 2 – The Sieve of Erastothenes (276 BC to 194 BC):

1. Make a list of numbers from 2 to some maximum.
2. Cross out all multiples of 2 greater than 2 from the list.
3. The next highest uncrossed number in the list is prime.
4. Cross out all multiples of this number from the list. The crossing out can begin at the square of this number. (Why?)
5. Repeat Steps 3 and 4 until you reach a prime number greater than or equal to the square root of the highest number in the list; all numbers remaining are prime.

Problem:
Use the Sieve to find all primes from 1 to 196.

	2	3	4	5	6	7	8	9	10	11	12	13	14
15	16	17	18	19	20	21	22	23	24	25	26	27	28
29	30	31	32	33	34	35	36	37	38	39	40	41	42
43	44	45	46	47	48	49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80	81	82	83	84
85	86	87	88	89	90	91	92	93	94	95	96	97	98
99	100	101	102	103	104	105	106	107	108	109	110	111	112
113	114	115	116	117	118	119	120	121	122	123	124	125	126
127	128	129	130	131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150	151	152	153	154
155	156	157	158	159	160	161	162	163	164	165	166	167	168
169	170	171	172	173	174	175	176	177	178	179	180	181	182
183	184	185	186	187	188	189	190	191	192	193	194	195	196

How many prime numbers are there?

There are infinitely many.

Why is this true?

Imagine that there were only 4 prime numbers, namely 2, 3, 5 and 7.
Define $N = (2)(3)(5)(7) = 210$. If we do the Sieve as above, $N = 210$ is definitely crossed out for each prime.
What about 211?
If there were only 4 primes, then 211 would not be crossed out, since each prime crosses out 210 so would skip over 211.
This would mean that 211 wouldn't ever be crossed out, so there would be at least 1 more prime.
So there are more than 4 primes.

In fact, this process would work if there was any finite number of primes.

This is called Euclid's Theorem:
"There are infinitely many prime numbers".

What special types of primes exist?

Even primes – there is only one of these (why?)

Twin primes – two prime numbers that differ by 2.

List the twin primes less than 196.
It is not known if there are infinitely many twin prime pairs or not.

Mersenne primes – Primes of the form $2^n - 1$ where n is a positive integer.

Is 7 a Mersenne prime? Is 13?

List the Mersenne primes less than 196.

Join the Great Internet Mersenne Prime Search and become famous!
On 23 August 2008, the 45th known Mersenne prime was found to be $2^{43\,112\,609} - 1$. This has 12 978 189 digits and is the largest prime known.

Sophie Germain primes – Primes p for which $2p + 1$ is also prime.

List the Sophie Germain primes between 1 and 100.