University of Waterloo
Faculty of Mathematics

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Mathematics and Computing

Senior Math Circles
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Conics I

Conics

The most mathematically interesting objects usually have several equivalent definitions. Conics are one example.
A conic is. . .

1. The intersection of any plane with a cone.

2. Any set of points in $\mathbf{R}^2$ satisfying a degree-2 polynomial equality [e.g., $\{(x, y) : x^2 - xy + y^2 - 5 = 0\}$]

3. The image of one of the following objects under an affine transformation:
   Note: we will define affine transformations a little later.
   [$\{(x, y) : x^2 + y^2 = 1\}$] (ellipse)
   [$\{(x, y) : x^2 - y^2 = 1\}$] (hyperbola)
   [$\{(x, y) : y = x^2\}$] (parabola)

4. The set
   $\{P : \|PF\| = e\|Pd\|\}$
   where $e > 0$, F is a point (called the focus), and d is a line (called the directrix).
   This is read as: The set of all points P, such that the distance from P to F is equal to the distance from P to d, times the constant e.

5. The orbit of some celestial body in the presence of another. (e.g., orbit of Earth and Sun)

   Note: These five definitions are not equivalent in certain degenerate cases.

Transformations

The distance between points $P = (x, y)$ and $P' = (x', y')$ is
$\|PP'\| = \sqrt{(x - x')^2 + (y - y')^2}$

Affine transformations are defined to be transformations of the form $(x, y) \longmapsto (ax + by + h, cx + dy + k)$ with $a, b, c, d, h, k \in \mathbf{R}^2$
examples) translations, rotation, reflection.

Another example is a stretch: $(x, y) \longmapsto (\alpha x, \beta y) \quad \alpha, \beta \in \mathbf{R}$ which is a stretch of α in the x-direction and β in the y-direction.
(Note: If $0 < \alpha < 1$ or $0 < \beta < 1$, it is more like a compression in the x-direction or y-direction respectively.)
Reflection is just a special case of stretching;
When $\alpha < 0$ and/or $\beta < 0$ we get a reflection. E.g. taking $\alpha = 1, \beta = -1$ gives us a reflection across the x-axis.

Exercise:
Find the equation of the unit circle, and its image when stretched by the factor α in the x-direction.

Solution:
$x^2 + y^2 = 1$ is the equation of the unit circle.
$(\frac{x}{\alpha})^2 + y^2 = 1$ is the unit circle stretched by α in the x-direction.

Proof of second part:
Consider a point $P(x, y)$ on the unit circle. So $x^2 + y^2 = 1$. The image of point P is $P'(x', y')$, where

$$\begin{cases} x' &= \alpha x \\ y' &= y \end{cases}$$

Now we want $x^2 + y^2 = 1$ in terms of x', y'. Rearranging, we get $x = \frac{x'}{\alpha}$.
Substituting we get $(\frac{x'}{\alpha})^2 + (y')^2 = 1$, which is the equation of the image.$\square$
(Note that if we also had a stretch of β in the y-direction we would use a similar argument to get $(\frac{x}{\alpha})^2 + (\frac{y}{\beta})^2 = 1$).

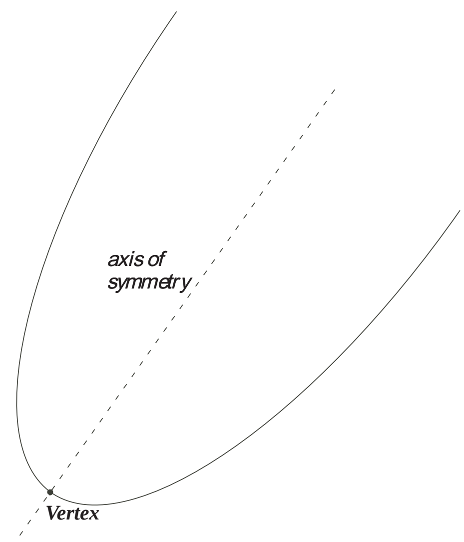
A translation is a transformation of the form $(x, y) \longmapsto (x + h, y + k)$. This moves the figure h units right and k units up.

Exercise:
What is the image of the curve $y = f(x)$ under this transformation?
Answer:
$y = k + f(x - h)$

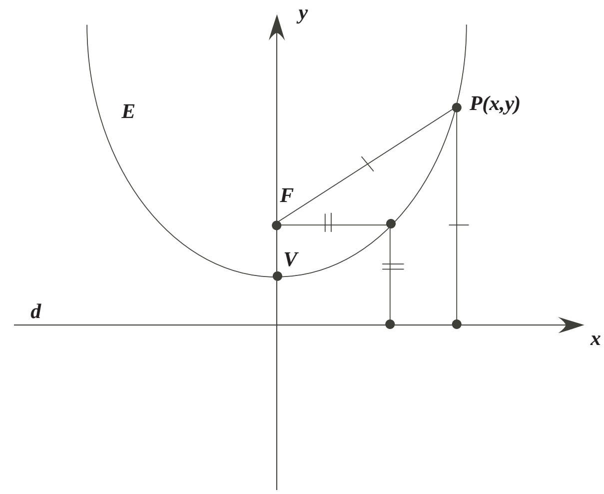
Parabolas

Given a point F called the focus and a line d called the directrix, the parabola they define is the curve
$\{P \in \mathbf{R}^2 : \|PF\| = \|Pd\|\}$
The locus of all points P such that the distance from P to F equals the distance from P to d.

A parabola has exactly one axis of symmetry. The intersection of that axis with the parabola is called the vertex.



Exercise:
We will verify that the locus definition of a parabola with a horizontal directrix does give the graph of a quadratic function.



Proof:
For simplicity let the directrix d be the x-axis, so the distance of $P(x, y)$ from d is $|y|$. Let the focus be $F = (0, t)$, with $t > 0$.
(Remark: These assumptions are without loss of generality by applying a translation plus possibly a reflection to the coordinate system.)
(Note: If we sketch out a few points P such that $\|Pd\| = \|PF\|$, it does look vaguely parabolic.)

Where is the vertex? It is on the axis of symmetry (the perpendicular line to the directrix through F, which in this case is the y-axis).
Let $V = (0, u)$ represent the vertex. We know from our definition that:
$\|dV\| = \|FV\|$
$|u| = |t - u|$ Solving, we get
$u = \frac{t}{2}$.
Therefore the vertex is at $(0, \frac{t}{2})$ i.e, half the distance from the focus to the origin as we could have intuitively guessed.

What is the equation of locus?
$\|dP\| = \|FP\| \iff |y| = \sqrt{(x-0)^2 + (y-t)^2}$
$\iff y^2 = x^2 + (y-t)^2$
$\iff 2ty = x^2 + t^2$
$\iff y = \frac{x^2}{2t} + \frac{t}{2}$
Which is the graph of a quadratic function, as required. $\square$
(One simple check for our answer is to see if the vertex $(0, \frac{t}{2})$ lies on the curve $y = \frac{x^2}{2t} + \frac{t}{2}$

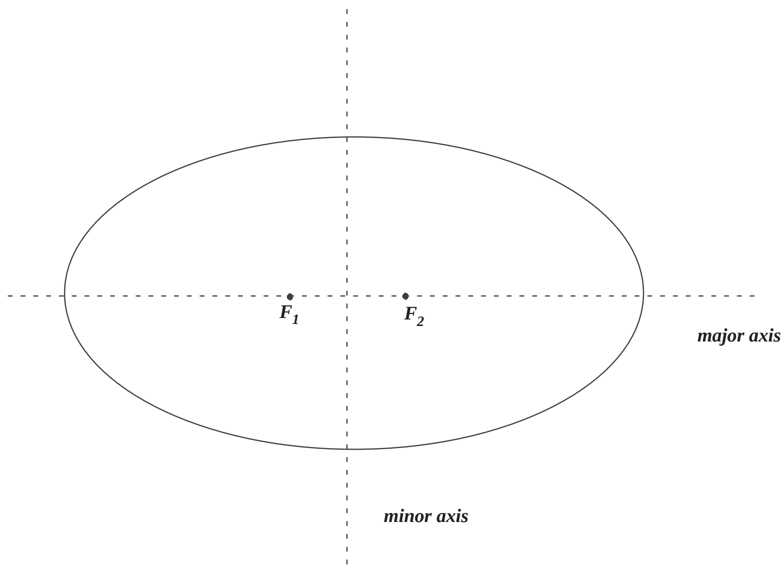
Exercise:
Given a parabola $y = \alpha x^2 + \beta x + \delta$, show that if we apply a transformation taking the vertex to the origin, we get the parabola $y = \alpha x^2$.

Reflections: If a ray of incoming light parallel to the axis of symmetry strikes the parabola, its reflection will pass through the focus.

Ellipses

Given two points F_1 and F_2 called foci and a real number a called the focal radius, the ellipse they define is the curve
$\{P \in \mathbf{R}^2 : \|PF_1\| + \|PF_2\| = 2a\}$

Every ellipse has two perpendicular axes of symmetry:
one contains the foci and is called the major axis and the other is called the minor axis.

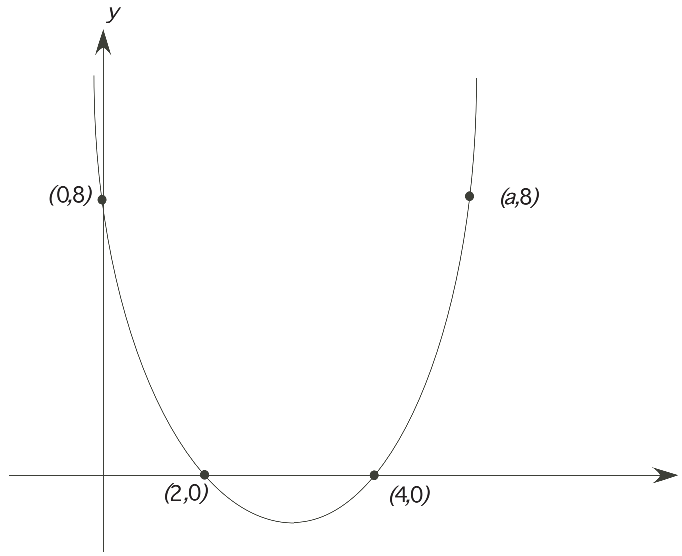


Exercise:
Let E be the curve defined by the locus of points
$\{P : \|PF_1\| + \|PF_2\| = 2a\}$
where $F_1 = (t, 0)$, $\quad F_2(-t, 0)$, $\quad$ and $a > t$
Show that E is a stretched out circle, i.e it can be expressed as
$(\frac{x}{\alpha})^2 + (\frac{y}{\beta})^2 = 1$.

Reflection property: If a light ray leaves one focus, after reflecting once off the ellipse it will hit the other focus.
eg: whisper chamber at Ontario Science Center

Problem Set

1. (a) In the diagram, the parabola cuts the y-axis at the point $(0, 8)$, cuts the x-axis at the points $(2, 0)$ and $(4, 0)$, and passes through the point $(a, 8)$. What is the value of a? [Euclid 2003]



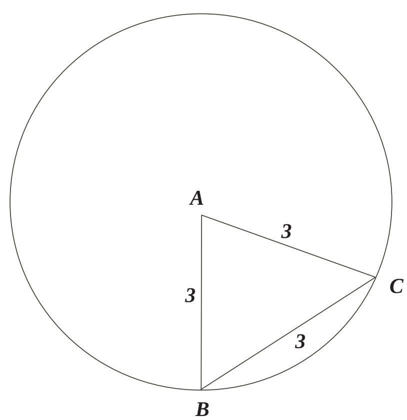
   (b) The quadratic equation $x^2+6x+k = 0$ has two real roots. What is the value of k? [Euclid 2003]

   (c) The line $y = 2x + 2$ intersects the parabola $y = x^2 - 3x + c$ at two points. One of these points is $(1, 4)$. Determine the coordinates of the second point of intersection. [Euclid 2003]

   (d) The parabola $y = x^2 - 4x + 3$ is translated 5 units to the right. In this new position, the equation of the parabola is $y = x^2-14x+d$. Determine the value of d. [Euclid 2002 #2]

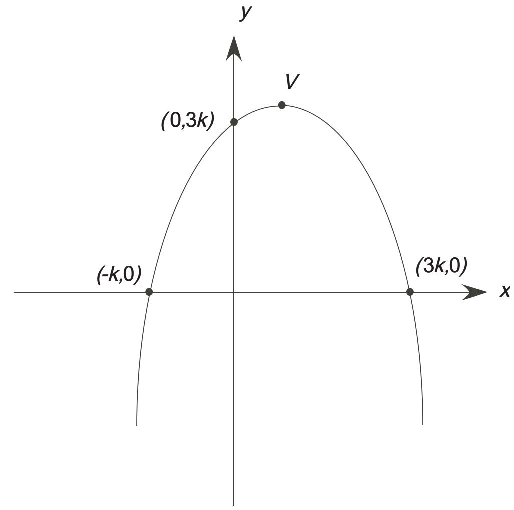
   (e) Determine all values of k with $k \neq 0$, for which the parabola
   $y = kx^2 + (5k + 3)x + (6k + 5)$
   has its vertex on the x-axis. [Euclid 2008]

2. (a) Equilateral triangle ABC has side length 3, with vertices B and C on a circle of radius 3, as shown. The triangle is then rotated clockwise inside the circle: first about C until A reaches the circle, then about A until B reaches the circle, and so on. Eventually the triangle returns to its original position and stops. What is the total distance travelled by the point B? [Euclid 2005]

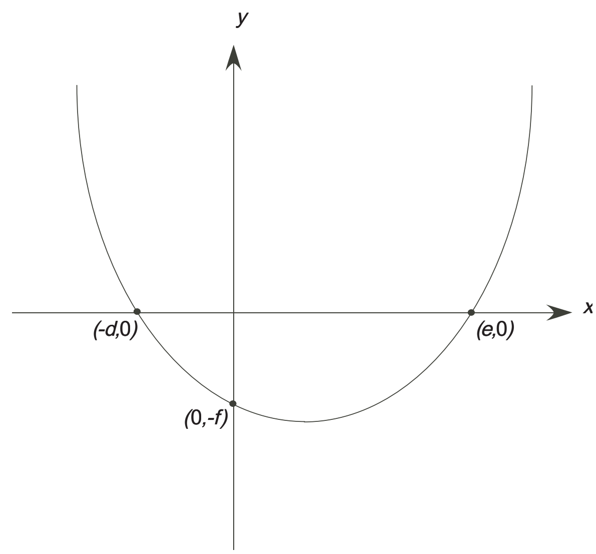


   (b) A circle passes through the origin and the points of intersection of the parabola $y = x^2 - 3$ and $y = -x^2 - 2x + 9$. Determine the coordinates of the centre of this circle. [Euclid 2002 #6]

(c) In the diagram, the parabola
$y = -\frac{1}{4}(x-r)(x-s)$
intersects the axes at three points. the vertex of this parabola is the point V. Determine the value of k and the coordinates of V. [Euclid 2005 #7]



3. In the graph, the parabola $y = x^2$ has been translated to the position shown. Prove that $de = f$. [Euclid 1998]



4. The parabola $y = f(x) = x^2 + bx + c$ has vertex P and the parabola $y = g(x) = -x^2 + dx + e$ has vertex Q, where P and Q are distinct points. The two parabolas also intersect at P and Q.
a) Prove that $2(e-c) = bd$.
b) Prove that the line through P and$Q$ has slope $\frac{1}{2}(b+d)$ and y-intercept $\frac{1}{2}(c+e)$. [Euclid 2007]