Senior Math Circles: Equations

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The Power of Symbols

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2. Are there non-zero rational numbers x, y, z such that $x^3 + y^3 = z^3$? If so, name some.

There are no non-zero rational numbers (x, y, z) that satisfy this equation. In 2004 it was proven that there are no non-zero solutions to the equations $x^n + y^n = z^n$ for $n \in \mathbf{N}$, $n \geq 3$. This result is called Fermat's Last Theorem.

Quadratic Equations

What about $ax^2 + bx + c = 0$? You've seen equations like these before: factored $x^2 - 5x + 6$, or $x^2 - 16 = 0$, or graphed $-3(x-1)^2 + 4$. When $ax^2 + bx + c$ is set equal to 0, we ask which x out there will make the equation true?

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The answer was discovered millenia ago. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. But what about $ax^3 + bx^2 + cx + d$, the cubic polynomial equation? Is there such a formula?

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There will always be 2 roots, 3 for the cubic equation. More on this later.

Cubic and Quartic Equations

It took until the 1500's to derive this formula, which gives 3 roots, just as the quadratic formula gives two roots. The first root is given by:

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Where $A = \frac{2b^3 - 9abc + 27a^2 d + \sqrt{(2b^3 - 9abc + 27a^2 d)^2 - 4(b^2 - 3ac)^3}}{2}$

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The formula for the 4^{th} degree polynomial equation $ax^4 + bx^3 + cx^2 + dx + e$ is even messier!

The Magic Number is 5

Soon the race was on for the formula that would give the roots of the 5^{th} degree polynomial $ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. Unfortunately no one would cross the finish: there is **no** formula for any polynomial of degree 5 or higher.

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There was an entire branch of mathematics developed in the 1800's to prove this. It is called Galois Theory after the French mathematician Galois. Keep in mind that 'no formula exists' is different from 'unable to find a formula'. Thus for 5^{th} degree or higher polynomials, we must use other tools to solve for the roots.

Factor Theorem

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The *Remainder Theorem* says that given polynomials $f(x)$ and $(x - r)$, we can write $f(x) = (x - r)g(x) + R(x)$, where $g(x)$ is the quotient and $R(x)$ is the remainder. This is similar to how we can write $23 = 4(5) + 3$.

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Let's do the work on the board. We know the degree of $g(x)$ is the degree of $f(x)$ minus that of $(x - 1)$, so it will be 2. After long division we will see $g(x) = 3x^2 + x + 5$. We could also use synthetic division, but I don't like it.

Factoring

Now we can write $f(x) = (x-1)(3x^2+x+5)$ and we know one of the solutions is $x = 1$. But what about the other two? Now that we have reduced the cubic to a product of a linear and a quadratic, it is easy: we use the Quadratic Formula or a factoring technique.

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Examining $3x^2+x+5$, we can see that the discriminant $b^2-4ac = 1-4(3)(5) = -59$, which is less than zero. Thus the roots are complex and the Quadratic Formula is our best bet. Here is the work:

$$
\begin{aligned}
& x = (-b \pm \sqrt{b^2+4ac})/2a \\
& = (-1 \pm \sqrt{-59})/6 \\
& = \tfrac{-1}{6} \pm \tfrac{i\sqrt{59}}{6}
\end{aligned}
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We can always use the factor theorem if we find one root.
$f(1) = 1 + 5 + 5 - 5 - 6 = 0$, thus 1 is a root and $(x - 1)$ is a factor.
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We must factor the quotient using the same approach. Let $g(x) = x^3 + 6x^2 + 11x + 6$. Then $g(1) \neq 0$ but $g(-1) = -1 + 6 - 11 + 6 = 0$, so that -1 is a root and $(x - (-1)) = (x + 1)$ is a factor. Long division of $g(x)$ by $(x + 1)$ gives us: $g(x) = (x + 1)(x^2 + 5x + 6)$.

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Now $x^2 + 5x + 6$ factors completely into $(x + 2)(x + 3)$, so putting this all together we have:

$f(x) = (x - 1)(x + 1)(x + 2)(x + 3)$. We say there are 4 linear factors of $f(x)$. The roots are: $1, -1, -2, -3$.

Exercises

7. Let $f(x) = x^5 + 8x^4 + 9x^3 - 26x^2 - 52x - 24$. How many roots does $f(x)$ have and must they all be real? What is your guess for the number of roots of a polynomial of highest power n?

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$f(x)$ will have 5 roots. They may not all be real, but all will lie in the Complex Numbers. A reasonable guess for a degree n polynomial is n roots, but some may be repeating.

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$f(1) = 1 + 8 + 9 - 26 - 52 - 24 \neq 0$, but
$f(-1) = -1 + 8 - 9 - 26 + 52 - 24 = 0$ so $(x + 1)$ is a factor. Division of $f(x)$ by $(x + 1)$ gives us: $f(x) = (x + 1)(x^4 + 7x^3 + 2x^2 - 28x - 24)$.

Let $g(x) = x^4 + 7x^3 + 2x^2 - 28x - 24$. $g(-1) = 0$, so $(x + 1)$ is a factor of $g(x)$. Division of $g(x)$ by $(x + 1)$ gives us:
$g(x) = (x + 1)(x^3 + 6x^2 - 4x - 24)$. We let $h(x) = x^3 + 6x^2 - 4x - 24$ and continue.

Exercises

Neither $h(1)$, $h(-1)$ or $h(2)$ are zero, so we will try $h(-2)$. $h(-2) = -8 + 24 + 8 - 24 = 0$. Thus $(x + 2)$ is a factor. Division of $h(x)$ by $(x + 2)$ gives us: $h(x) = (x + 2)(x^2 + 4x - 12)$.

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$x^2 + 4x - 12$ factors completely into $(x - 2)(x + 6)$. Putting everything together we have:

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Geometrically, this means the graph of $f(x)$ will hit the x-axis 4 times: at $-2, -1, 2$ and 6.

Root Extraction

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In these cases we must approximate that root using techniques like Newton's Method, which uses simple calculus to approximate the root of a given polynomial. Computer programs become very helpful in these cases.

Fundamental Theorem

We have seen that given any polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $a_i \in \mathbf{Q}$ we know it has n roots, all complex numbers. We can always write $f(x) = (x - r_1)(x - r_2) \cdots (x - r_n)$, where $r_1, r_2, ..., r_n$ are the n complex roots of $f(x)$.

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This result is known as the Fundamental Theorem of Algebra. We say $f(x)$ *splits completely* in $\mathbf{C}$.

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Use the $\log_2$ function to grab the x from the left hand side:

$$\log_2(2^x) = \log_2(5)$$
$$\rightarrow x = log_2(5)$$

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Use the $\log_2$ function to grab the x from the left hand side:

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$\rightarrow x^2 + 3x = \log_e 7$
$\rightarrow x^2 + 3x - \log_e 7 = 0$

Now we must use the Quadratic Formula to find the two solutions, with $a = 1, b = 3$ and $c = -\log_e 7$:

$x = (-3 \pm \sqrt{9 + 4\log_e 7})/2$ ...whatever those are.

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We can take $\log_{10}$ of both sides but let us first rewrite this equation: $10^{x^2} = 10^4$. The only way this equatoin can be true is if the exponent on the LHS equals the exponent on the RHS. Thus $x^2 = 4$, or $x^2 - 4 = 0$. This gives two solutions: $x = \pm 2$.

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The same approach would be used for the first equation, taking $\log_7$ of both sides twice. $x = \log_7(\log_7 93)$. The second equation cannot be solved with the material we have learned, however funny enough we could all take its derivative!

Linear Systems

Even linear equations can get complicated if there are more than one to solve at a time. Consider the problem of the NHL salary cap, and the following restrictions it places on players 1 Holi Goalie, 2 Smitty Smirnov and 3 Roy Roy:

The total value of the players must be 20 million, twice the value of Holi plus 3 times the value of Smitty must equal 1.5 times the value of Roy, and the value of Holi must be equal to 4 times the value of Smitty plus 2.5 the value of Roy.

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We represent this trade attempt as equations with $x_1 =$ value of Holi, $x_2 =$ value of Smitty and $x_3 =$ value of Roy

$x_1 + x_2 + x_3 = 20$
$2x_1 + 3x_2 - 1.5x_3 = 0$
$x_1 - 4x_2 - 2.5x_3 = 0$

Are YOU in the Matrix?

We could solve this with substitution and elimination but this would take a very long time, and mistakes could easily be made. We place the coefficients in a matrix to preserve the information but make it simpler to work with:

$$\begin{bmatrix} 1 & 1 & 1 & 20 \\ 2 & 3 & -1.5 & 0 \\ 1 & -4 & -2.5 & 0 \end{bmatrix}$$

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Matrix Algebra Exercise

12. Using only the 3 rules below, turn $A = \begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix}$ into $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

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$$\begin{aligned} &\begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix} \\ \sim &\begin{bmatrix} 1 & 6 \\ 2 & 3 \end{bmatrix} \text{ by a)} \\ \sim &\begin{bmatrix} 1 & 6 \\ 0 & -9 \end{bmatrix} \text{ by c)} \\ \sim &\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \text{ by b)} \\ \sim &\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ by c)} \end{aligned}$$

Matrix Algebra

With some arithmetic involving these three rules, we can make any matrix look different, but still preserve all the information given. We call this new matrix an *equivalent matrix.*

Here is an equivalent matrix from our trade problem: $\begin{bmatrix} 1 & 0 & 0 & 9.27 \\ 0 & 1 & 0 & 0.7 \\ 0 & 0 & 1 & 10.73 \end{bmatrix}$

Solution Sets

We can read off the solution to this system: the only way the trade can happen is if Holi is offered 9.27M, Smitty only gets 0.7M and 10.73M would go to Roy.

What if our matrix ended up looking like $\begin{bmatrix} 1 & 0 & 0 & 9.27 \\ 0 & 1 & 0 & 0.7 \\ 0 & 0 & 0 & 10.73 \end{bmatrix}$

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This means that we are not told anything about Smitty's value, only x_1 and x_3 are given.

Solution Sets

If the original system was set up so that we ended up with a missing 1, or a repeating row, this means that x_2 can be anything, and the system will still be solved as long as $x_1 = 9.27$ and $x_3 = 10.73$. That is, there would be an infinite amount of solutions for this trade. Of course this is not the case.

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Even with a simple system, like three non-linear equations, it could end up being difficult or even impossible. In some cases engineers approximate solutions to non-linear equations with solutions to a similar linear system.

Questions

If you have any questions, please let us know or bring them in for next week.

Have a good night!