University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Senior Math Circles
March 4, 2009
Number Systems

Number Systems

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \bar{\mathbb{Q}} \subseteq \mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H}$

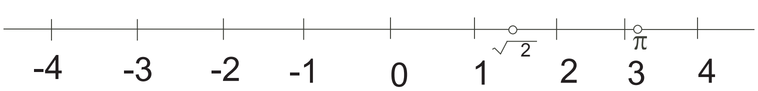
As we come across different situations, we need to extend our number systems.

$\mathbb{N}$: The natural numbers are the set of all non-negative integers $0, 1, 2, 3\ldots$
Consider $x + 2 = 1$.
x cannot be a number in $\mathbb{N}$, thus we need to extend our number system to include negative integers.

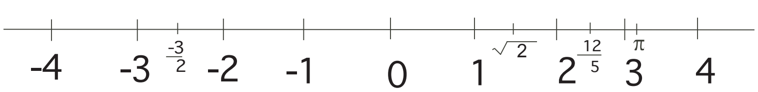
$\mathbb{Z}$: The set of all integers $0, \pm 1, \pm 2, \pm 3\ldots$
However, again our system is not sufficient. If we consider $2x = 1$, x must be $\frac{1}{2}$, thus we need to extend our number system to include fractions.

$\mathbb{Q}$: The set of all rational numbers is denoted by $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$.
However, there are numbers on the number line which do not correspond to a rational number such as π and $\sqrt{2}$.

$\bar{\mathbb{Q}}$: The set of all irrational numbers are the real numbers which are not rational.



$\mathbb{R}$: The system of numbers that correspond to all the points of the number line is called the set of real numbers.

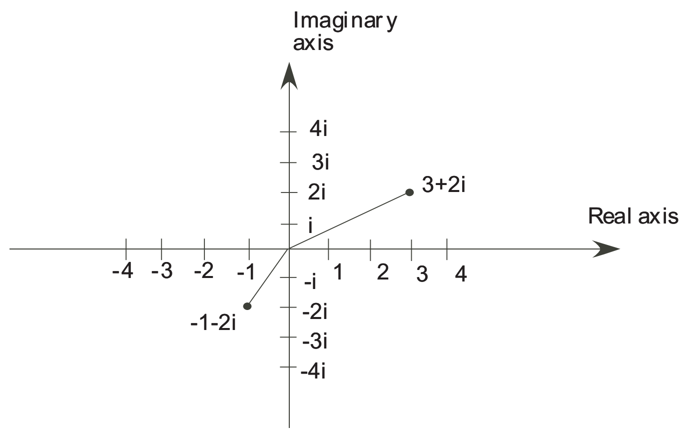


What if we wish to find the root of a negative number, such as $x^2 = -1$. We will define the complex number system to handle this.

$\mathbb{C}$: The set of all complex numbers is denoted by $\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$.
For $z = x + yi$, x is called the real part and y is called the imaginary part. i is a new symbol with the property that $i^2 = -1$.

Addition and Multiplication are defined by:
$(a + bi) + (c + di) = (a + c) + (b + d)i$.
$(a + bi)(c + di) = (ac - bd) + (ad + cb)i$.

The Complex Plane:
The real numbers have a geometric representation of points on a number line, the complex numbers are associated with points on a plane.We label the horizontal axis the real axis and the vertical the imaginary axis. The geometric representation of the complex number $z = x + yi$ is the point (x, y) in the plane.



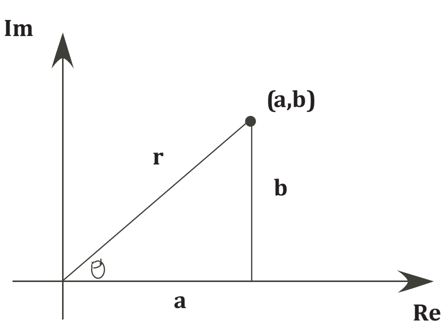
Polar form:
For any point in cartesian coordinates, we can also use polar coordinates.
$(x, y) \Longleftrightarrow (r, \theta)$
by $(x, y) \longmapsto (\sqrt{x^2 + y^2}, arctan(\frac{y}{x}))$
or backwards $(r, \theta) \longmapsto (rcos\theta, rsin\theta)$

We define the *modulus* of the complex number $z = x + yi$ to be:

$\|z\| = \sqrt{x^2 + y^2}$. This is just the distance from the point z to the origin.
Furthermore, the angle θ is called the *argument* of z.

Find the polar form of any complex number $z = a + bi$.

From the diagram, $sin\theta = \frac{b}{r}$ and $cos\theta = \frac{a}{r}$.
Rearranging we get $rsin\theta = b$ and $rcos\theta = a$, and thus we have $z = rcos\theta + irsin\theta$. This gives us the polar form for any complex number z.
$z = r(cos\theta + isin\theta)$. Which we denote as $rcis\theta$



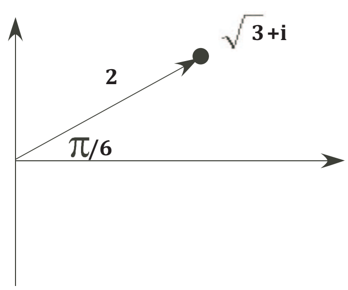
One reason for converting complex numbers into polar form is that multiplication is made easier.
If $z_1 = r_1(cos\theta_1 + isin\theta_1)$ and $z_2 = r_2(cos\theta_2 + isin\theta_2)$
Then $z_1 z_2 = r_1 r_2[cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)]$.

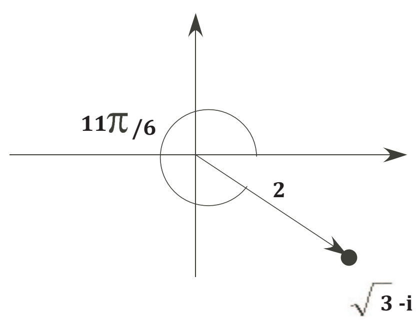
Exercise: Prove this.

De Moivre's Theorem:

Theorem
For any real number θ and integer n,
$(cos\theta + isin\theta)^n = cosn\theta + isinn\theta$.
Corollary Following immediately from the theorem we get:
If $z = r(cos\theta + isin\theta)$ then, for any integer n,
$z^n = r^n(cosn\theta + isinn\theta)$

Exercise: Calculate $(\sqrt{3} + i)^{11}$. We could expand using the Binomial Theorem, but it is easiest to convert to polar coordinates and use De Moivre's Theorem.





We have $\sqrt{3} + i = 2(cos\frac{\pi}{6} + isin\frac{\pi}{6})$
and so using the corollary we get:
$(\sqrt{3} + i)^{11} = 2^{11}(cos\frac{11\pi}{6} + isin\frac{11\pi}{6})$
$= 2^{11}(\frac{\sqrt{3}-i}{2})$
$= 1024\sqrt{3} - 1024i$.

We can also define the complex exponential function by
$e^{i\theta} = cos\theta + isin\theta$
This complex exponential obeys the usual laws of exponents because
$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$(by multiplication of complex numbers in polar form)
$(e^{i\theta})^n = e^{in\theta}$ (by De Moivre's Theorem)

The polar form of a complex number z can now be written as
$z = re^{i\theta}$,
where $r = |z|$ and θ is the argument of z.

Since $e^{i\pi} = cos\pi + isin\pi = -1 + 0i$, we obtain the famous equation, due to Euler, that connects the numbers $\pi, e, i, 1$ and 0, namely
$e^{i\pi} + 1 = 0$.

However, the complex numbers are not the end of the number system, we will now take a look at a set of numbers called the Quaternions.

$\mathbb{H} = \{a + bi + cj + dk\}$ with $a, b, c, d \in \mathbb{R}$ are called the Quaternions. The quaternions are an extension of the complex number system. As a set, the quaternions $\mathbb{H}$ are equal to R^4, a four-dimensional vector space over the real numbers.
Multiplication is defined by:
$i^2 = j^2 = k^2 = -1$
Thus it can be shown that $ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$

Hamilton Product
For two elements $a_1 + b_1i + c_1j + d_1k$ and $a_2 + b_2i + c_2j + d_2k$, their Hamilton Product $(a_1 + b_1i + c_1j + d_1k)(a_2 + b_2i + c_2j + d_2k)$ is determined by the product of the basis elements and the distributive law: $a_1a_2 + a_1b_2i + a_1c_2j + a_1d_2k + b_1a_2i + b_1b_2i^2 + b_1c_2ij + b_1d_2ik + c_1a_2j + c_1b_2ji + c_1c_2j^2 + c_1d_2jk + d_1a_2k + d_1b_2ki + d_1c_2kj + d_1d_2k^2$.
Now basis elements can be multiplied using our above rules.
$(a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)j + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)k$.

Problems

1. Let $z = 4 + i$ and $w = -3 + 2i$ be complex numbers. find (i)$z + w$, (ii) $z - w$, (iii)$z^2 w$, (iv) the real and imaginary parts of w^4.

2. Convert the complex number $i, -1 + i, \sqrt{3} - 3i$, and -4 to polar form.

3. Multiply the complex number $z_1 = 2(cos\frac{\pi}{4} + isin\frac{\pi}{4})$ by the complex number $z_2 = \frac{1}{\sqrt{2}}cos(\frac{7\pi}{6} + isin\frac{7\pi}{6})$.

4. Express each of the following number $rcis\theta = r(cos\theta + isin\theta)$ in the standard form $x + iy$ where $x, y \in \mathbb{R}$
   i) $4cis2\pi$ ii) $\sqrt{3}cis(\frac{4\pi}{3})$ iii) $2cis(\frac{-3\pi}{4})$

5. Calculate $(2 + i)^6$ using De Moivre's Theorem. Check your answer by using the Binomial Theorem.

6. Use De Moivre's Theorem to show that
   $cos2\theta = 2cos^2\theta - 1$
   $sin2\theta = 2sin\theta cos\theta$

7. Calculate the Hamilton Product $z_1 z_2$ where:
   $z_1 = \sqrt{2} - 3i + j - 4k$ and $z_2 = 3 + i - j + 2k$

Works Cited:
An Introduction to Mathematical Thinking: Algebra and Number Systems by William J. Gilbert and Scott A. Vanstone.

"Quaternions" from Wikipedia.org.