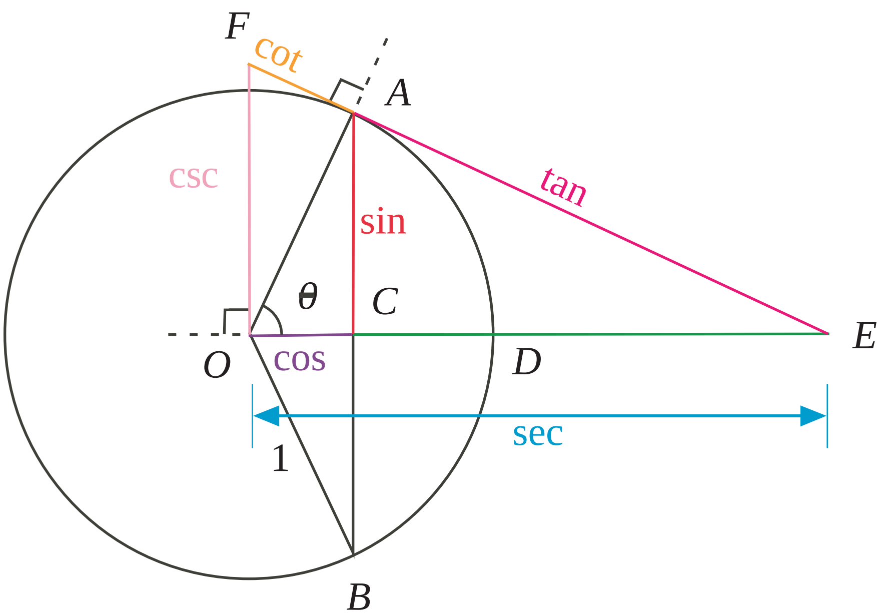
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Senior Math Circles
November 5, 2008
Trigonometry II

Trigonometry



In the diagram above, O is the centre of the circle, which has radius 1.
Therefore $OA = 1$. In $\triangle AOC$ $\sin\theta = \frac{AC}{OA}$, so $AC = \sin\theta$. Also, $\cos\theta = \frac{OC}{OA}$, so $OC = \cos\theta$.
Now, $\angle OAC = \angle FOA = 90° - \theta$, and $\angle ACO = \angle OAF = 90°$, so $\triangle AOC \sim \triangle OFA$.

$$\text{Therefore } \frac{FA}{OA} = \frac{OC}{AC}$$
$$\frac{FA}{1} = \frac{\cos\theta}{\sin\theta}$$
$$FA = \frac{1}{\tan\theta}$$
$$FA = \cot\theta$$

$$\text{Also } \frac{OF}{OA} = \frac{AO}{AC}$$
$$\frac{OF}{1} = \frac{1}{\sin\theta}$$
$$OF = \csc\theta$$

Now, $\angle AOC = \angle EOA = \theta$, and $\angle ACO = \angle EAO = 90^\circ$, so $\triangle AOC \sim \triangle EOA$.

$$\begin{aligned}
\text{Therefore } \frac{EO}{AO} &= \frac{OA}{OC} \\
\frac{EO}{1} &= \frac{1}{\cos\theta} \\
EO &= \sec\theta
\end{aligned}$$

$$\begin{aligned}
\text{Also } \frac{EA}{OA} &= \frac{AC}{OC} \\
\frac{EA}{1} &= \frac{\sin\theta}{\cos\theta} \\
EA &= \tan\theta
\end{aligned}$$

Notice that we can derive some of the Pythagorean trigonometry identities from the above diagram.
In $\triangle ACO, OC^2 + AC^2 = AO^2$, therefore $\cos^2\theta + \sin^2\theta = 1$.
In $\triangle OAE, OA^2 + AE^2 = OE^2$, therefore $1 + \tan^2\theta = \sec^2\theta$ or $1 + \tan^2\theta = \frac{1}{\cos^2\theta}$.
Also, in $\triangle AFO, OA^2 + AF^2 = OF^2$, therefore $1 + \cot^2\theta = \csc^2\theta$ or $1 + \frac{1}{\tan^2\theta} = \frac{1}{\sin^2\theta}$.

Radians

Definition:
A *radian* is defined as the measure of the angle at the centre of a circle that is subtended by an arc the length of the radius of the circle.

You can find the radian measure of an angle by taking the ratio of the arc it subtends, and the radius of the circle.
From this we may convert 360° into radians. If we let the radian angle equal θ, then $\theta = \frac{2\pi r}{r} = 2\pi$. Similarly 180° is equal to π radians.

Complex Numbers

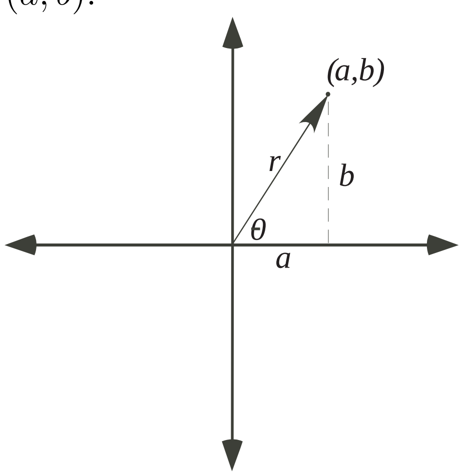
Definition:
We define $i = \sqrt{-1}$. That is, i is a solution to the equation $x^2 + 1 = 0$.

Definition:
Numbers of the form $a + bi$, where a and b are real, are called *complex numbers.*
The *imaginary part* of a complex number is b and the *real part* is a.

Example:
Solve $x^2 - 4x + 5 = 0$.

Solution:
Using the quadratic formula, we get
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$
$x = \dfrac{4 \pm \sqrt{-4}}{2}$
$x = \dfrac{4 \pm \sqrt{4}\sqrt{-1}}{2}$
$x = \dfrac{4 \pm 2i}{2}$
$x = 2 \pm i$
Therefore, $x = 2 + i$ or $x = 2 - i$.

We can represent complex numbers on a plane. The real part a is measured on the x-axis and the imaginary part b is measured on the y-axis. This means that the complex number $a + bi$ will be represented by the point (a, b).



Polar Coordinates:
Now, imagine a ray drawn from the origin to the point (a, b). Let the length of this ray be $r = \sqrt{a^2 + b^2}$, and the angle it makes with the positive x-axis be θ. If we draw a right triangle, we get that $\sin\theta = \frac{b}{r}$ or $b = r\sin\theta$, and $\cos\theta = \frac{a}{r}$ or $a = r\cos\theta$.
Hence, the complex number $a + bi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$.

The Euler Relationship states that $e^{i\theta} = \cos\theta + i\sin\theta$.
Therefore, the complex number $r(\cos\theta + i\sin\theta) = re^{i\theta}$.

Multiplication:

Now, consider multiplying a complex number $r_1(\cos\theta + i\sin\theta)$ by another $r_2(\cos\alpha + i\sin\alpha)$.
$[r_1(\cos\theta + i\sin\theta)][r_2(\cos\alpha + i\sin\alpha)]$
$= r_1r_2[\cos\theta\cos\alpha - \sin\theta\sin\alpha + i(\cos\theta\sin\alpha + \sin\theta\cos\alpha)]$
$= r_1r_2[\cos(\theta+\alpha) + i\sin(\theta+\alpha)]$

We can also multiply the Euler representation of these complex numbers.
$(r_1e^{i\theta})(r_2e^{i\alpha})$
$= r_1r_2e^{i\theta+i\alpha}$
$= r_1r_2e^{i(\theta+\alpha)}$

We can see that since the angle in the product is $\theta+\alpha$, this multiplication is equivalent to rotating the 2-D representation of the first complex number by the second angle, α, and multiplying its length by r_2.

Exponents:

Consider a complex number $z = re^{i\theta}$. If we raise this number to the nth power we have $z^n = (re^{i\theta})^n = r^ne^{in\theta}$.
If we take the nth root of this number, we have
$\sqrt[n]{z} = z^{\frac{1}{n}} = (re^{i\theta})^{\frac{1}{n}} = \sqrt[n]{r}e^{\frac{i\theta}{n}}$.

Roots of Unity:

We may represent the number 1 as $1 = \cos(2\pi) + i\sin(2\pi) = e^{i2\pi}$.
Therefore, an nth root of 1 would be $e^{\frac{2\pi i}{n}}$.
However 1 may also be represented as $e^{i4\pi}, e^{i6\pi}, \ldots$.
In general, we can write $1 = e^{i2k\pi}$ for $k = 0, 1, 2, \ldots$.

Therefore the n solutions to $z^n = 1$ are $z = e^{\frac{i2k\pi}{n}}$ for $k = 0, 1, \ldots, n-1$.
These numbers z are called the nth roots of unity.

Problem Set

1. Solve $x^2 + 2x + 2 = 0$.

2. Solve $x^4 + 5x^2 + 4 = 0$.

3. Solve $x^3 - 5x^2 + 17x - 13 = 0$.

4. Find the imaginary part of $(1 + i)(1 + 2i)(1 + 3i)$.

5. If $\sum_{n=0}^{\infty} \cos^{2n} \theta = 5$, what is the value of $\cos 2\theta$?

6. Determine $e^{i\pi}$.

7. Find $e^{\frac{i\pi}{6}} + e^{\frac{-i\pi}{6}}$.

8. Determine all solutions to $z^4 = 81$, and plot them in the plane. These points form a polygon. What are its area and perimeter?

9. Determine all solutions to $z^6 = 64$, and plot them in the plane. These points form a polygon. What are its area and perimeter?

10. Determine $(1 + i)^{100}$.

11. Determine $(i + \sqrt{3})^{41}$.

12. Given that z is a complex number such that $z + \frac{1}{z} = 2 \cos 3^\circ$, determine $z^{2000} + \frac{1}{z^{2000}}$.

13. The solutions to $z^4 + 4z^3 i - 6z^2 - 4zi - i = 0$, when plotted in the plane, form a quadrilateral. What is its area?

14. Find the product of the nonreal roots of $x^4 - 4x^3 + 6x^2 - 4x = 2008$.

15. Find the sum of the roots, real and non-real, of the equation $x^{2001} + (\frac{1}{2} - x)^{2001} = 0$, given that there are no multiple roots.

16. Simplify $\sin 10^\circ + \sin 20^\circ + \cdots + \sin 80^\circ$.