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Senior Math Circles
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Trigonometry I

Trigonometry

The word *trigonometry* is derived in part from the Greek *trigonos*, meaning three-angled (think 3-gon).

Relation Identities

$$\cos(-\theta) = \cos\theta \qquad \cos\theta = \sin(90^\circ - \theta) \qquad \tan\theta = \tfrac{\sin\theta}{\cos\theta}$$
$$\sin(-\theta) = -\sin\theta \qquad \sin\theta = \cos(90^\circ - \theta)$$

Pythagorean Identities

$$\cos^2\theta + \sin^2\theta = 1 \qquad 1 + \tan^2\theta = \tfrac{1}{\cos^2\theta} \qquad 1 + \tfrac{1}{\tan^2\theta} = \tfrac{1}{\sin^2\theta}$$

Double-Angle Identities

$$\cos 2\theta \begin{array}{l} = 2\cos^2\theta - 1 \\ = 1 - 2\sin^2\theta \end{array} \qquad \sin 2\theta = 2\sin\theta\cos\theta \qquad \tan 2\theta = \tfrac{2\tan\theta}{1-\tan^2\theta}$$

Addition Identities

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$\cos A \cos B \cos C - \cos(A+B+C)$
$= \cos A \sin B \sin C + \sin A \cos B \sin C + \sin A \sin B \cos C$
$\sin A \sin B \sin C + \sin(A+B+C)$
$= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C$

We can now look at some trigonometry problems that may be solved by the identities above, and some neat tricks.

Example: Determine the maximum value of $\cos\theta + \sin\theta$, where $0^\circ \le \theta \le 90^\circ$.

Solution 1:

One trick is to square the expression. Since $\cos\theta + \sin\theta \ge 0$, maximizing $\cos\theta + \sin\theta$ is equivalent to maximizing $(\cos\theta + \sin\theta)^2$. We get

$$(\cos\theta + \sin\theta)^2 = \cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta = 1 + \sin 2\theta,$$

so we attain the maximum when $\theta = 45^\circ$.
The maximum is $\cos(45^\circ) + \sin(45^\circ) = \sqrt{2}$.

Solution 2:

$$\begin{aligned}\cos\theta + \sin\theta &= \cos\theta + \cos(90^\circ - \theta)\\ &= 2\cos\left(\frac{\theta + (90^\circ - \theta)}{2}\right)\cos\left(\frac{\theta - (90^\circ - \theta)}{2}\right)\\ &= 2\cos(45^\circ)\cos(\theta - 45^\circ)\\ &= \sqrt{2}\cos(\theta - 45^\circ)\end{aligned}$$

The maximum value is $\sqrt{2}$, attained when $\theta = 45^\circ$, as above.

Solution 3:

Theorem (Quadratic-Mean Arithmetic-Mean Geometric-Mean Harmonic-Mean Inequality):
For any positive integer n and numbers $x_1, x_2, \ldots, x_n \ge 0$,
$\sqrt{\frac{x_1^2+\cdots+x_n^2}{n}} \ge \frac{x_1+\cdots+x_n}{n} \ge \sqrt[n]{x_1 x_2 \cdots x_n} \ge \frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}$,
with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Using the first two terms in this inequality, with only two variables, we get that for all $x, y \geq 0$, $\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2}$ with equality if and only if $x = y$. Plugging in $x = \cos\theta$ and $y = \sin\theta$, we get $\frac{\cos\theta+\sin\theta}{2} \leq \sqrt{\frac{\cos^2\theta+\sin^2\theta}{2}} = \sqrt{\frac{1}{2}}$ with equality if and only if $\cos\theta = \sin\theta$. This agrees with our solutions above.

Example: Determine the least positive integer n such that
$\tan(19°n) = \dfrac{\cos 96° + \sin 96°}{\cos 96° - \sin 96°}$.

Solution 1:

$\frac{\cos 96°+\sin 96°}{\cos 96°-\sin 96°} < 0$ so we must have $\tan 19°n < 0$.

$$\begin{aligned} \tan^2(19°n) &= \frac{(\cos 96° + \sin 96°)^2}{(\cos 96° - \sin 96°)^2} \\ &= \frac{1 + 2\sin 96° \cos 96°}{1 - 2\sin 96° \cos 96°} \\ &= \frac{1 + \sin 192°}{1 - \sin 192°} \\ &= \frac{1 - \sin 12°}{1 + \sin 12°} \end{aligned}$$

We know $1 + \tan^2\theta = \frac{1}{\cos^2\theta}$, so

$$\begin{aligned} 1 + \tan^2(19°n) &= 1 + \frac{1 - \sin 12°}{1 + \sin 12°} \\ \frac{1}{\cos^2(19°n)} &= \frac{2}{1+\sin 12°} \\ 1 + \sin 12° &= 2\cos^2(19°n) \end{aligned}$$

Since $\cos 2\theta = 2\cos^2\theta - 1$, then $\sin 12° = \cos(38°n)$,
and since $\sin\theta = \cos(90° - \theta)$, then $\cos 78° = \cos(38°n)$.
Therefore $38n = 78 + 360k$ or $38n = -78 + 360k$, where k is an integer.
This simplifies to $19n = 39 + 180k$ or $19n = -39 + 180k$.
If $19n = 39 + 180k$ then $tan(19°n) > 0$, therefore $19n = -39 + 180k$.

$$\begin{aligned} 180k &\equiv 39 \mod 19 \\ 9k &\equiv 1 \mod 19 \\ -k &\equiv 2 \mod 19 \\ k &\equiv 17 \mod 19 \end{aligned}$$

To get the smallest n such that $19n = -39 + 180k$, we should take $k = 17$.
Then $n = \frac{-39+180\cdot 17}{19} = 159$.

Solution 2:

$$
\begin{aligned}
\frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ} &= \frac{\cos 96^\circ + \cos 6^\circ}{-\sin 6^\circ - \sin 96^\circ} \\
&= \frac{2\cos\left(\frac{96^\circ + 6^\circ}{2}\right)\cos\left(\frac{96^\circ - 6^\circ}{2}\right)}{-2\sin\left(\frac{96^\circ + 6^\circ}{2}\right)\cos\left(\frac{96^\circ - 6^\circ}{2}\right)} \\
&= -\frac{1}{\tan 51^\circ} \\
&= \tan(-39^\circ)
\end{aligned}
$$

So $\tan(19^\circ n) = \tan(-39^\circ)$.
Therefore, $19n = 180k - 39$, where k is an integer, and so

$$
\begin{aligned}
180k &\equiv 39 \mod 19 \\
k &\equiv 17 \mod 19
\end{aligned}
$$

To derive the identites for $\cos A + \cos B$ and $\sin A + \sin B$, note that

$$
\begin{aligned}
\cos(x+y) + \cos(x-y) &= (\cos x \cos y - \sin x \sin y) + (\cos x \cos(-y) - \sin x \sin(-y)) \\
&= 2\cos x \cos y
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\sin(x+y) + \sin(x-y) &= (\sin x \cos y + \cos x \sin y) + (\sin x \cos(-y) + \cos x \sin(-y)) \\
&= 2\sin x \cos y.
\end{aligned}
$$

Let $x = \frac{A+B}{2}$ and $y = \frac{A-B}{2}$, and you get the addition identities!

Problem:
Use this method to develop an identity for $\cos A - \cos B$.

Example: Determine $\cos(20^\circ)\cos(40^\circ)\cos(80^\circ)$.

Solution:

The above identity states
$\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$.

This gives us

$$\begin{aligned}\cos(40^\circ)\cos(20^\circ) &= \tfrac{1}{2}(\cos(60^\circ)+\cos(20^\circ))\\ &= \tfrac{1}{4}+\tfrac{1}{2}\cos(20^\circ)\end{aligned}$$

.

Plugging this in, we get

$$\begin{aligned}\cos(20^\circ)\cos(40^\circ)\cos(80^\circ) &= \tfrac{1}{4}\cos(80^\circ)+\tfrac{1}{2}\cos(20^\circ)\cos(80^\circ)\\ &= \tfrac{1}{4}\cos(80^\circ)+\tfrac{1}{4}(\cos(60^\circ)+\cos(100^\circ))\\ &= \tfrac{1}{4}\cos(80^\circ)+\tfrac{1}{8}+\tfrac{1}{4}\cos(100^\circ)\end{aligned}$$

.

But $\cos(100^\circ) = -\cos(80^\circ)$, so we get
$\cos(20^\circ)\cos(40^\circ)\cos(80^\circ) = \frac{1}{8}$.

Problem Set

1. Given that $\cos\theta + \sin\theta = \sqrt{\frac{3}{2}}$ and $0° \le \theta \le 45°$, find θ.

2. Let ACB be a right-angle triangle with right angle C, and let D be on AC such that $AD = 2DC$. If $BD = 5$ and $DC = 3$, find $\tan(\angle ABD)$.

3. Determine an angle x such that $\sin 2x \sin 3x = \cos 2x \cos 3x$.

4. Given that $\tan(\theta) = \dfrac{\sin 10° + \sin 20°}{\cos 10° + \cos 20°}$ and $0° \le \theta \le 90°$, find θ.

5. Given that $0° \le A, B \le 180°$ and

$$\sin A + \sin B = \sqrt{\frac{3}{2}}, \quad \cos A + \cos B = \sqrt{\frac{1}{2}},$$

determine $A + B$.

6. Suppose that x and y are angles such that

$$\sin^2 x + \cos^2 y = \frac{3}{2}a \quad \text{and} \quad \cos^2 x + \sin^2 y = \frac{1}{2}a^2.$$

Determine all possible values of a.

7. Given that $2\sin 2\theta = 1 + 3\sin^2\theta$, determine $\tan\theta$.

8. Show that $\dfrac{1}{\cos\theta} + \dfrac{1}{\sin\theta} \ge 2\sqrt{2}$ for $0° \le \theta \le 90°$.

9. Define $f(x) = \sin^6 x + \cos^6 x + k(\sin^4 x + \cos^4 x)$ for some real number k.
   (a) Determine all real numbers k for which $f(x)$ is constant for all values of x.
   (b) If $k = -\frac{7}{10}$, find all solutions to the equation $f(x) = 0$, where $0° \le x \le 90°$.
   (c) Determine all real numbers k for which there exists a real number c such that $f(c) = 0$.

10. Determine all values of x, $0° \le x \le 180°$, such that

$$\cos^3 3x + \cos^3 5x = 8\cos^3 4x \cos^3 x.$$

11. Determine the minimum value of

$$\left|\sin\theta + \frac{1}{\sin\theta} + \cos\theta + \frac{1}{\cos\theta} + \tan\theta + \frac{1}{\tan\theta}\right|,$$

where θ ranges over all real numbers such that $\sin\theta \ne 0$, $\cos\theta \ne 0$.