University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Junior Math Circles
November 25, 2009
2D Geometry III

Warm Up Problem: Magic Squares

Magic squares are grids that are filled with numbers so that the sum of each column, row and main diagonal is the same. The sum is called the magic constant. Also magic squares must not contain duplicate numbers.

a.) This magic square contains each of the integers from 1 to 16, and has a magic constant of 34. Fill in the remaining numbers.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

b.) The magic square contains each of the integers from -3 to 5. Fill in the missing numbers.

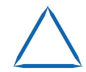
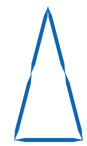
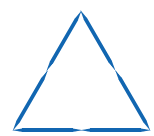
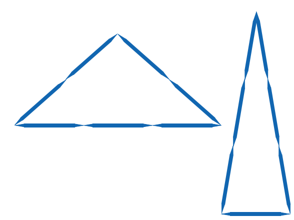
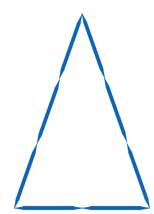
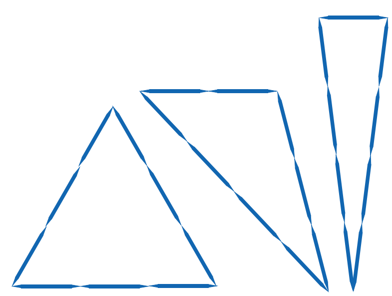
2	-3	4
3	1	-1
-2	5	0

Hints:

- We could use trial and error with the top right and middle square and since there are only 5 numbers not being used this approach is not too tedious.
- Another approach is to first find the magic constant. We are not given the magic constant, but we know that every row must add up to this value and the sum of all three rows are the summation of the integers from -3 to 5. So we know the summation of the integers from -3 to 5 is equal to 3 times the magic constant.

The Toothpick Challenge

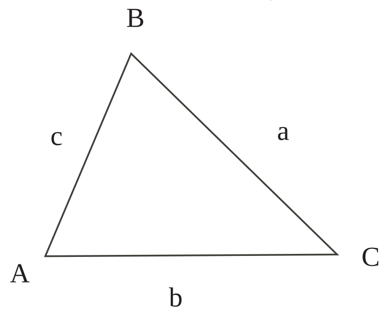
With three toothpicks arrange them so that they form a triangle. Do the same with 4, 5, 6, 7, 8 and 9 toothpicks.

Number of Toothpicks	Length of Sides	Sketch of the Triangle(s)
3	{1, 1, 1}	
4		
5	{2, 2, 1}	
6	{2, 2, 2}	
7	{3, 3, 1} {3, 2, 2}	
8	{3, 3, 2}	
9	{4, 4, 1} {4, 3, 2} {3, 3, 3}	

Why can't we create a triangle with 4 toothpicks? The reason has to do with the *Triangle Inequality*.

Triangle Inequality

The sum of the lengths of two sides of a triangle is always greater than the length of the third side.



$$a + b > c$$
$$b + c > a$$
$$a + c > b$$

Practice 1:
A triangle has one side of length 4 and another side of length 7.
a.) What is the largest possible integer length of the third side?
b.) List all possible integer values of the third side.

Solution:
Let x represent the length of the unknown side.
Using the Triangle Inequality we know that:

$$x + 4 > 7 \longrightarrow x > 3$$
$$x + 7 > 4 \longrightarrow x > -3$$
$$4 + 7 > x \longrightarrow 11 > x$$

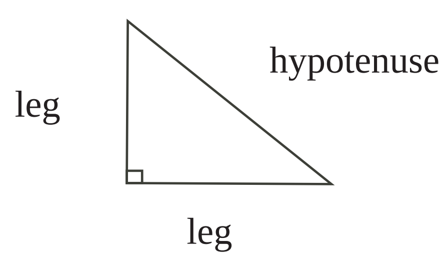
Now we know that x is less that 11 and greater then 3. The possible integer values for the third side length is 10, 9, 8, 7, 6, 5 and 4.

a.) Therefore, the largest possible integer length of the third side is 10.
b.) Therefore, the possible integer values for the third side length is 10, 9, 8, 7, 6, 5 and 4.

Pythagorean Theorem

The Pythagorean Theorem is a very famous and important theorem in mathematics and in particular geometry. It is used to find the length of one side of a right triangle when given the length of other two sides.

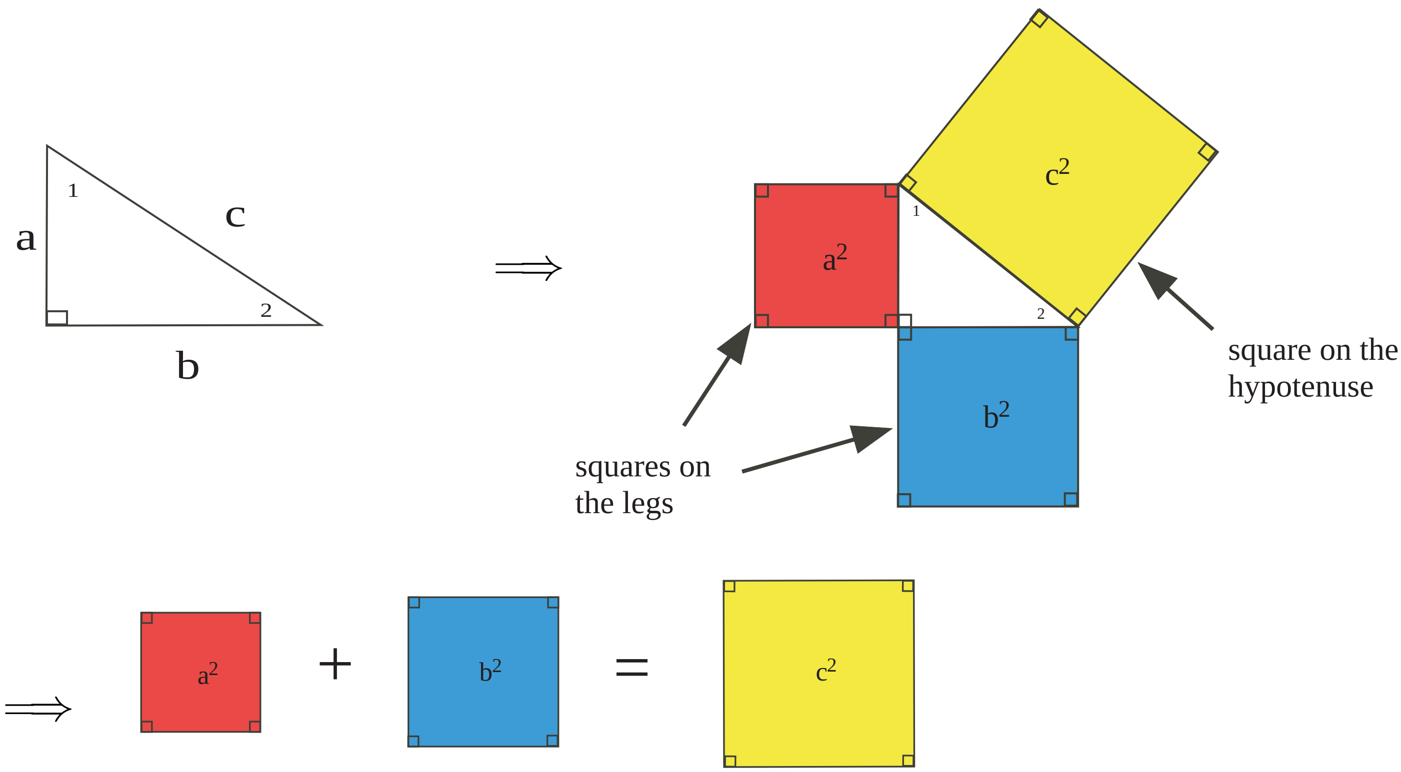
If we recall a right triangle is a triangle that has an interior angle with a measure of 90° . The longest side and the side that is opposite the 90° angle is called the hypotenuse. The other sides are called the legs.



Pythagorean Theorem
The sum of the areas of the squares on the legs of a right triangle is equal to the area of the square on the hypotenuse.

Let a and b represent the length of the legs of a right triangle.

Let c represent the length of the hypotenuse.



sum of the areas of the squares on the legs = area of the square on the hypotenuse

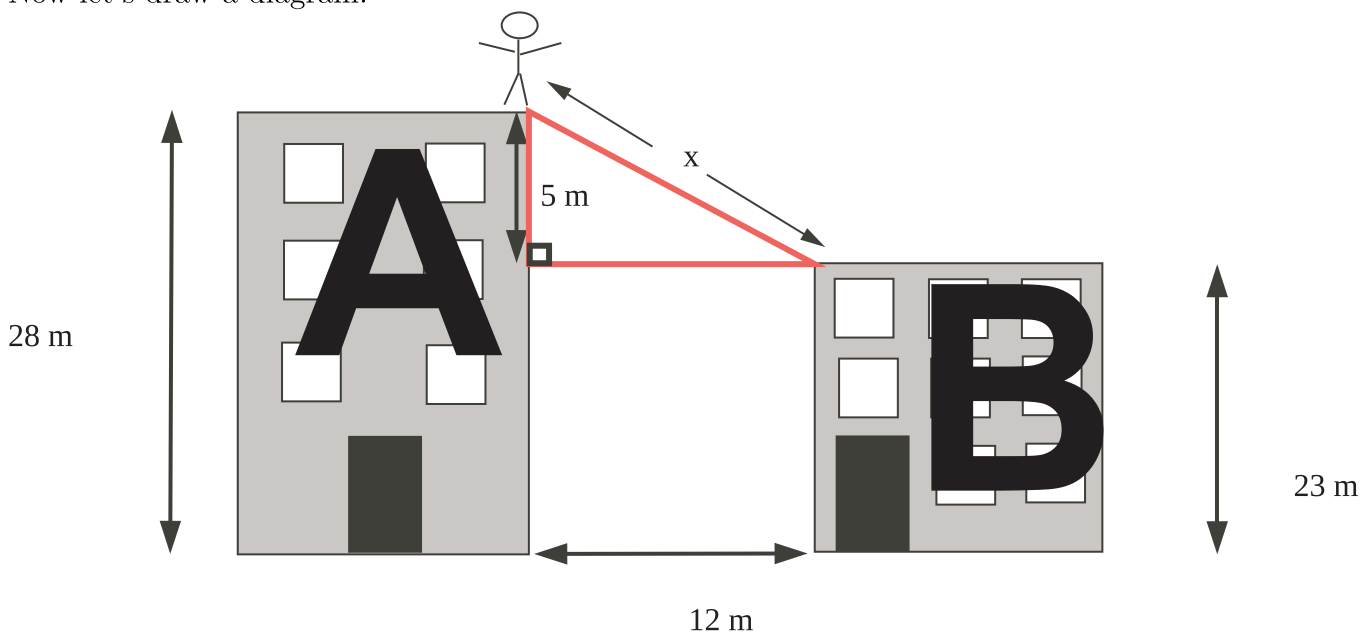
If we were to express this theorem with a equation it would look like this,

$$a^2 + b^2 = c^2$$

When asked "What is the Pythagorean Theorem?" the equation above is often what people will tell you, but the theorem above is a deeper explanation of the Pythagorean Theorem.

Practice 2:
Chuck Norris is finished with being a action star and has decided to become a tight rope walker. For his first daring feat, he plans to walk from building A which is 28m tall to building B which is 23m tall. At the point Chuck plans to cross the buildings are 12m apart. Assume the rope is perfectly straight and not curved, and that the building's are perpendicular to the ground and parallel to each each other. How long a rope does Chuck Norris need?

Solution:
Let x represent the length of the tight rope.
Now let's draw a diagram.



Since the building are parallel to each other and perpendicular to the ground we can use the Pythagorean Theorem to determine the length of the rope. We can draw in a right triangle where the hypotenuse is the tight rope and he difference in height of the buildings and the space between the buildings are the legs. Now we can apply the Pythagorean Theorem.

$$5^2 + 12^2 = x^2$$
$$25 + 144 = x^2$$
$$169 = x^2$$
$$\pm\sqrt{169} = x$$
$$x = \pm 13$$

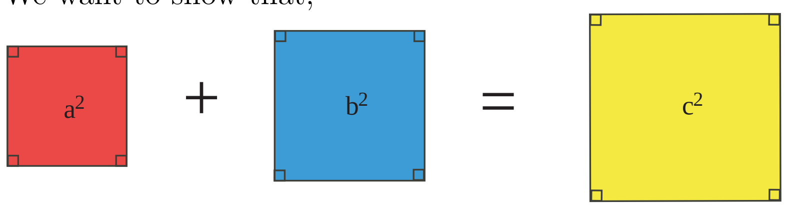
Since length is going to be a positive measurement we can ignore the negative solution.

Therefore, Chuck Norris needs a rope 13 m long.

Proof of the Pythagorean Theorem

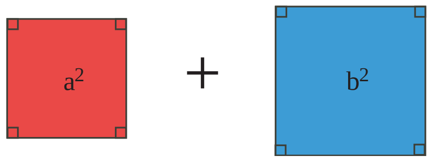
There are different ways to prove the Pythagorean Theorem. Leonardo Da Vinci developed a proof, so did James A. Garfield the 20th president of the United States of America. We will look at a proof called the Dissection Proof which uses the squares of each side of the right triangle directly.

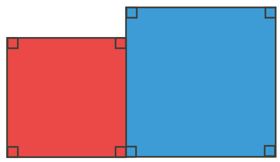
We want to show that,



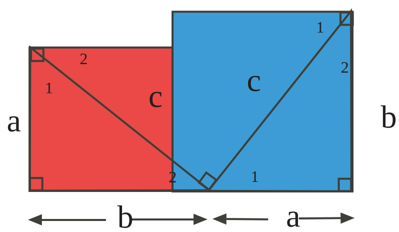
sum of the areas of the squares on the legs = area of the square on the hypotenuse

Let's start with the left side of this equation and try to show that those two squares can be rearranged to from the c^2 square.

$\Longrightarrow$ 

$\Longrightarrow$ 

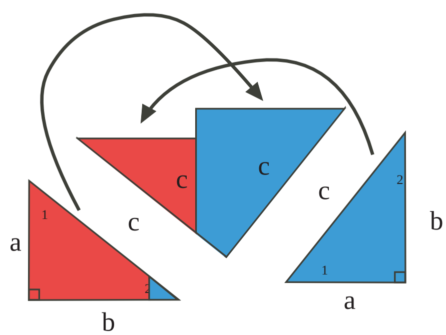
Now draw the original triangle twice within the two squares.

$\Longrightarrow$ 

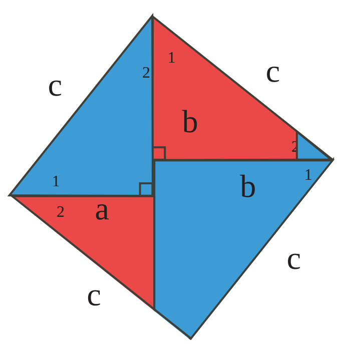
We know the sum of the interior angles of a triangle is 180°. So with the our right triangle:

$$\angle 1 + \angle 2 + 90^\circ = 180^\circ$$
$$\angle 1 + \angle 2 + 90^\circ - 90^\circ = 180^\circ - 90^\circ$$
$$\angle 1 + \angle 2 = 90^\circ$$

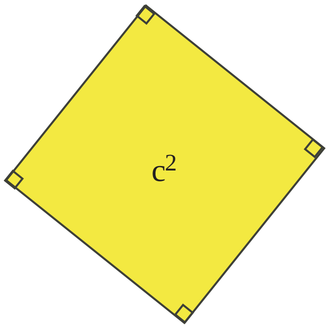
Using this information we can discover other angles within our shape.

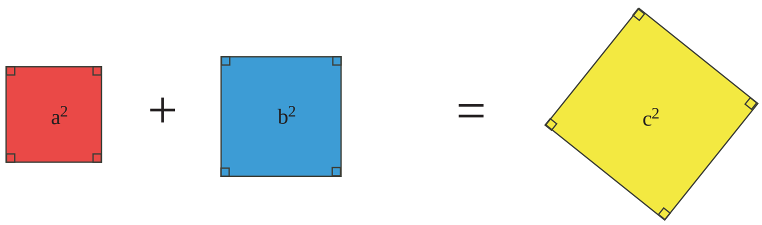
$\Longrightarrow$ 

Now if we take these triangles and remove them from our shape and reattach them in different positions

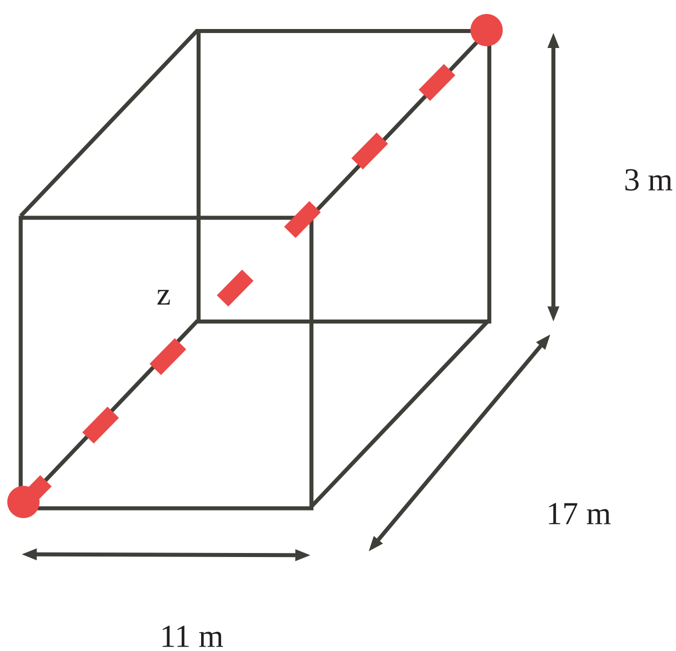
$\Longrightarrow$ 

Now each side of the quadrilateral has a length of c, but we don't can't assume it to be a square. We need to check if each corner is a right angle to confirm that this is a square. Each corner is made up of $\angle 1 + \angle 2$, which we know equals 90°. So we know it is a square with side length c.

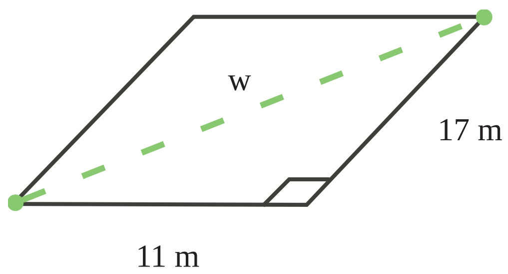
$\Longrightarrow$ 

Therefore, 

Practice 3:
A classroom has the following dimensions 11m×17m×3m. Assume the room is a rectangular prism. What is the distance from one corner on the floor to the corner on the ceiling that is diagonal the starting corner? Round your answer to two decimal places.



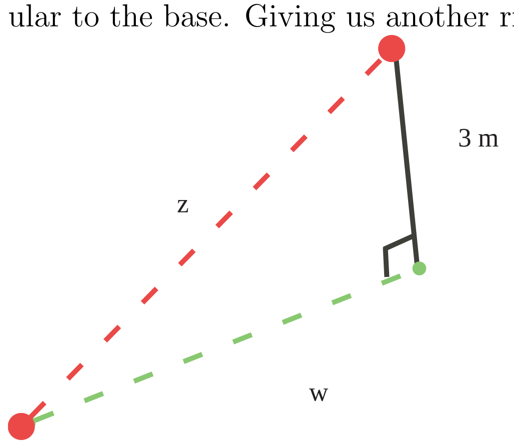
Solution:
Let z represent our desired distance.
Now we need to break this problem up into two steps. First, lets find the distance to the corner that is diagonal to our starting corner.



Let w represent our diagonal in the base of the classroom.

$$(11m)^2 + (17m)^2 = w^2$$
$$121m^2 + 289m^2 = w^2$$
$$410m^2 = w^2$$

Since the room is a rectangular prism, the line representing the height of the room is perpendicular to the base. Giving us another right triangle to which we can apply the Pythagorean Theorem.



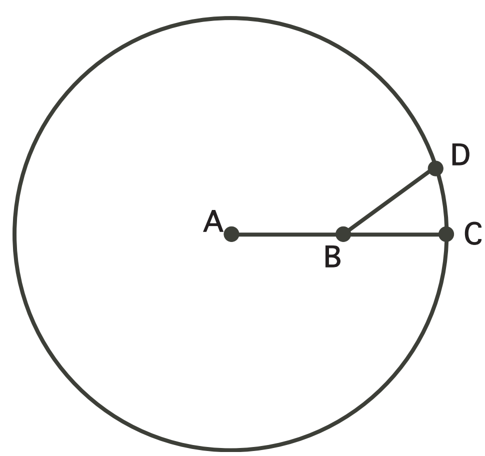
Let z represent the distance from corner to corner in the classroom.

$$w^2 + (3m)^2 = z^2$$
$$410m^2 + 9m^2 = w^2$$
$$419m^2 = w^2$$
$$w = \pm\sqrt{419m^2}$$
$$w \approx \pm 20.47m$$

Therefore, the distance from corner to corner is approximately 20.47m

Problem Set

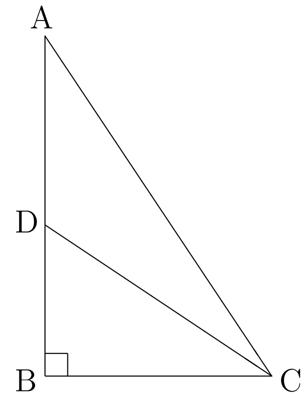
1. A triangle can be formed having side lengths 4, 5, 8. It is impossible, however, to construct a triangle with side lengths 4, 5 and 10. Using the side lengths 2, 3, 5, 7 and 11, how many different triangles **with exactly two equal sides** can be formed?



2. In the diagram, A is the center of the circle, and AC is a radius of the circle. B lies on the line segment AC and D is a point on the circle. Which of the following is always true, and why?

   (**A**) $BC \leq BD$ (**B**) $BC = BD$
   (**C**) $BC > BD$ (**D**) None of these

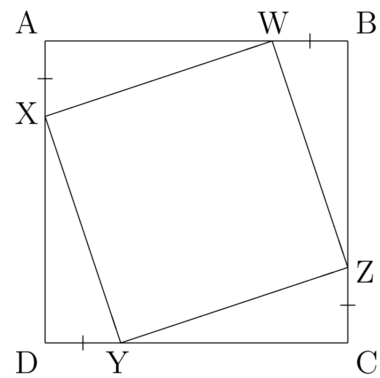
3. In the diagram, AD = 5, BD = 4, and the area of the triangle ACD is 15. What is the area of △ABC?



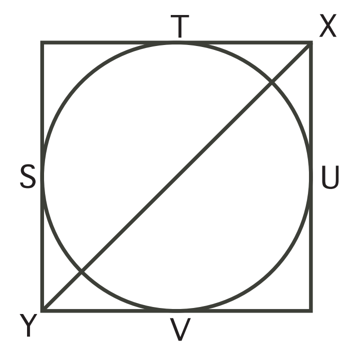
4. (20 G7 2004) The area of square $ABCD$ is 64 and

$$AX = BW = CZ = DY = 2.$$

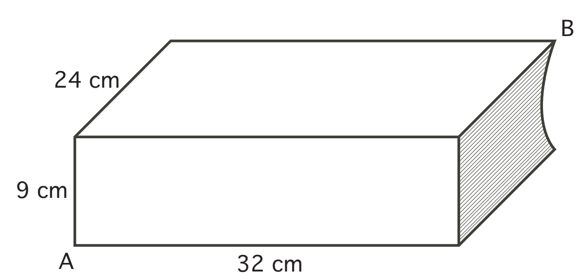
What is the area of square WXYZ?



5. (23 G8 2009) In the diagram, the circle is inscribed in the square. This means that the circle and the square share points S, T, U, and V, and the width of the square is exactly equal to the diameter of the circle. Rounded to the nearest tenth, what percentage of line segment XY is outside the circle?



6. (23 G8 2006) Tom and James go to the school library and try to find the biggest book possible. They find a math textbook that has dimensions 32 cm × 24 cm × 9 cm as shown. What is the length of the diagonal of this book AB?

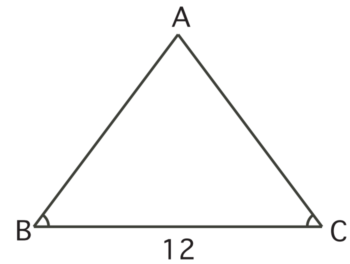


7. The sides of a triangle are 6, 8, and x.

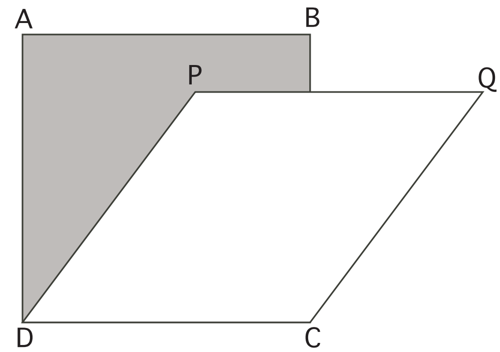
   a) How many possible integer values for x are there?

   b) How many of these values give an acute triangle? A right triangle? An obtuse triangle?

8. The perimeter of $\triangle ABC$ is 32. If $\angle ABC = \angle ACB$ and $BC = 12$, what is the area of $\triangle ABC$?



9. (24 G8 2003) In the diagram, ABCD is a square with area 25 cm^2. If PQCD is a rhombus with area 20 cm^2, what is the area of the shaded region?

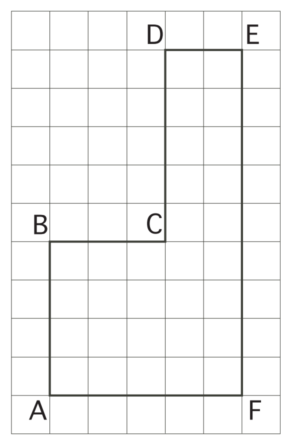


10. (20 G8 2005) Consider the 5 lengths listed below.

(**A**) AE (**B**) CD + CF (**C**) AC + CF
(**D**) FD (**E**) AC + CE

a) Which of these is the largest?

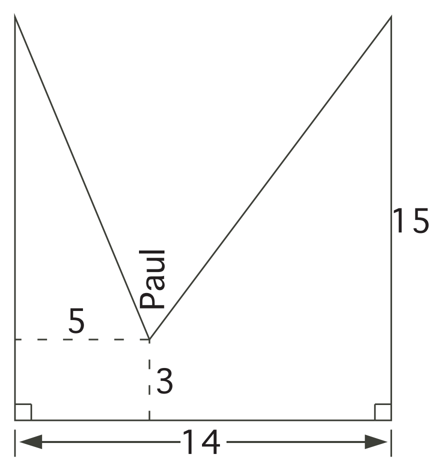
b) Which two are the same length?



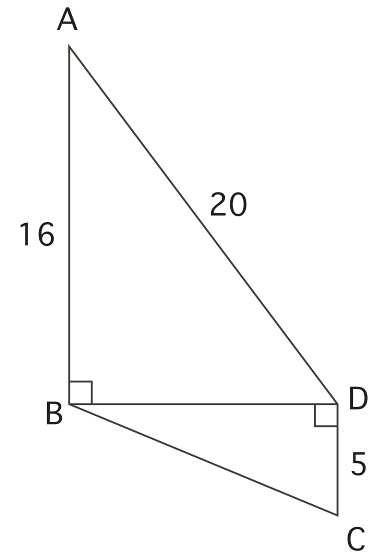
11. (24 G8 2003) A "slackrope walker" is much like a tightrope walker except that the rope on which he performs is not pulled tight. Paul, a slackrope walker, has a rope tied to two 15 m high poles which are 14 m apart. When he is standing on the rope 5 m away from one of the poles, he is 3 m above the ground.

a) How long is the rope? (round to two decimals)

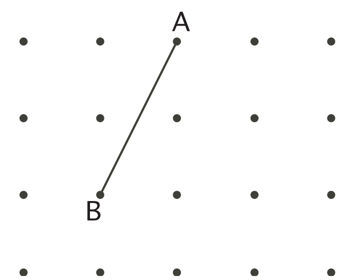
b) How close does Paul get to the ground? (round to two decimals)



12. (16 G8 2006) In the diagram, what is the length of BC?



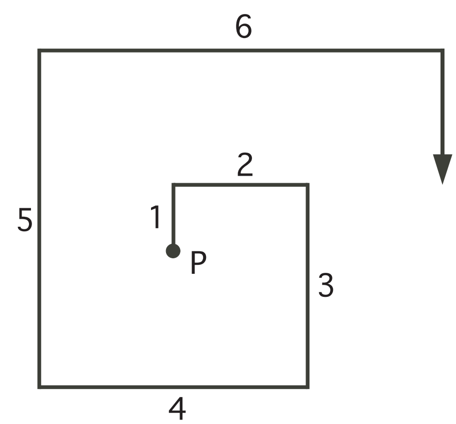
13. (23 G8 2006) In the diagram, the points are evenly spaced vertically and horizontally. A segment AB is drawn using two of the points, as shown. Point C is chosen to be one of the remaining 18 points. For how many of these 18 possible points is triangle ABC isosceles?



14. (24 G8 2009) Starting at point P, Breenah constructs a straight sided spiral so that:

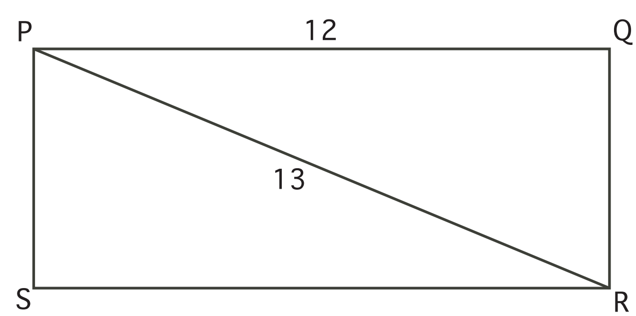
   - all angles are 90°
   - after starting with a line segment of length 1, each side is 1 longer than the previous side.

   After completing the side with length 21 Breenah's distance be from her original starting point P will be between:

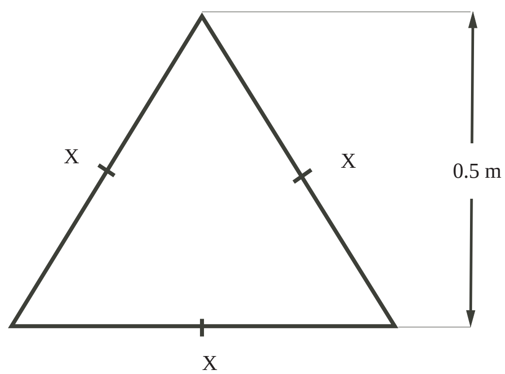


(**A**) 13 and 14 (**B**) 14 and 15
(**C**) 15 and 16 (**D**) 16 and 17

15. (15 G8 2009) In rectangle PQRS, PQ = 12 and PR = 13. What is the area of rectangle PQRS?



16. Find the length of the sides(labeled x) in the equilateral triangle below. The height of the triangle is 0.5m.



Answers

1. 14
2. (**A**)
   We have AB + BC = AC = AD $\leq$ AB + BD
   AB + BD = AD only if D = C, otherwise AB + BD > AD
3. 27
4. 40
5. 29.3%
6. 41 cm
7. a) 11
   b) 5 give an acute triangle, 1 gives a right triangle, and 5 give an obtuse triangle.
8. 48
9. 11 cm^2
10. a) (**E**)
    b) (**B**) and (**C**)
11. a) 30.49 m
    b) 1.46 m
12. 13
13. 6
14. (**B**)
15. 60
16. x = 0.58 m