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Senior Math Circles
November 11, 2009
History of Solving Polynomial Equations

After 3rd and 4th degree polynomials were solved in generality, people tried to find a 5th degree polynomial formula.

Expected it to look like: $x = \sqrt[5]{\ldots \sqrt[4]{\ldots \sqrt[3]{\qquad}}} \ldots$

No one made any headway for at least a century.

In the early 1800's, the problem was solved by a young French mathematician named Évariste Galois.

Évariste Galois: 1811-1821, France

In order to solve this, he had to invent a whole branch of mathematics: "Galois Theory". The advancement of group theory is largely due to his work.

A bit about Galois:

- was always a great student, became seriously interested in math when he was 14 years old.
- read work by Lagrange, Abel.
- was often frustrated, grieving the loss of his father to suicide; dealt with lots of rejection (publicly).
- was a "global learner" - had trouble expressing details and connections behind his discoveries.
- expelled from university for political activism, which was more important to him than math.
- died in an infamous duel by a shot to the stomach
- before the fight, he collected his mathematical ideas and sent them to August Chevalier, who wrote:

> "This letter, if judged by the novelty and profundity of ideas it contains, is perhaps the most substantial piece of writing in the whole literature of mankind."

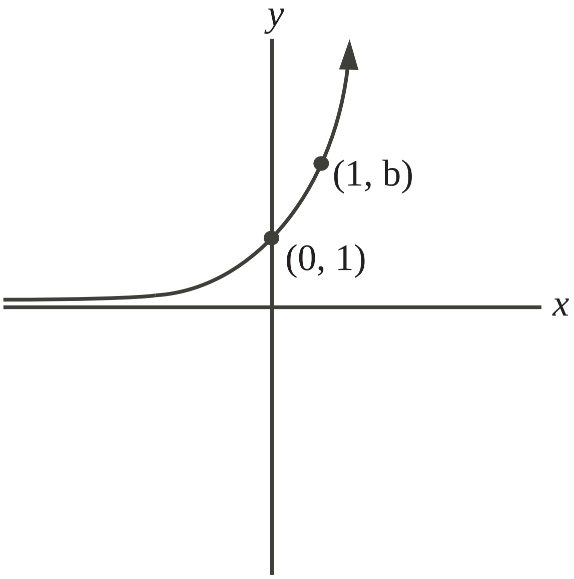
There is <u>no</u> formula using $+$, $-$, $\times$, $\div$, or $\sqrt[n]{\quad}$ for a general 5th degree polynomial . . . or higher.

Why? It has to do with the groups S_5, S_6, . . . and how they break down.

In other words, we can still solve some higher-degree polynomials, just no general formula exists.

History of $y = e^x$

Given a positive base b, the equation $y = b^x$ can be graphed:



e was first used within a compound interest setting:

$A = P(1 + \frac{1}{n})^n$ is a formula used to calculate compound interest growth.

Exercise:

Let $\{a_n\}_{n=1}^{\infty} = \{(1 + \frac{1}{n})^n\}$. Is this sequence increasing or decreasing? Why? Find a lower bound for $\{a_n\}$.

Solution:

Let's test some numbers:
$a_1 = (1 + 1)^1 = 2$
$a_2 = (1 + \frac{1}{2})^2 = (\frac{3}{2})^2 = \frac{9}{4} = 2\frac{1}{4}$
$a_3 = (1 + \frac{1}{3})^3 = (\frac{4}{3})^3 = \frac{64}{27} = 2\frac{10}{27}$

It appears a_n increases as n increases.

Claim: $a_{n+1} > a_n$ for a general $n \in \mathbb{N}$.

Proof: $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ (Binomial Theorem)

$$
\begin{aligned}
\therefore \quad (1+\tfrac{1}{n})^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\
&= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \frac{1}{n^k} \\
&= \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{n(n-1)(n-2)\cdots(n-(k-1))\cancel{(n-k)!}}{\cancel{(n-k)!}n^k} \\
&= \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{\cancel{n}}{\cancel{n}} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{n-(k-1)}{n} \\
&= \sum_{k=0}^{n} \frac{1}{k!} \cdot \left(1-\frac{1}{n}\right) \cdot \left(1-\frac{2}{n}\right) \cdot \ldots \cdot \left(1-\frac{k-1}{n}\right)
\end{aligned}
$$

Thus, $a_{n+1} = \sum_{k=0}^{n+1} \frac{1}{k!} \cdot \left(1-\frac{1}{n+1}\right) \cdot \left(1-\frac{2}{n+1}\right) \cdot \ldots \cdot \left(1-\frac{k-1}{n+1}\right)$.

It is important to note

$$\left(1-\frac{1}{n}\right) < \left(1-\frac{1}{n+1}\right), \quad \left(1-\frac{2}{n}\right) < \left(1-\frac{2}{n+1}\right), \quad \ldots, \quad \left(1-\frac{n}{n}\right) < \left(1-\frac{n}{n+1}\right),$$

Therefore:

$$
\begin{aligned}
&\left[\frac{1}{k!} \cdot \left(1-\frac{1}{n}\right) \cdot \left(1-\frac{2}{n}\right) \cdot \ldots \cdot \left(1-\frac{k-1}{n}\right)\right] \\
&\qquad\qquad \le \left[\frac{1}{k!} \cdot \left(1-\frac{1}{n+1}\right) \cdot \left(1-\frac{2}{n+2}\right) \cdot \ldots \cdot \left(1-\frac{k-1}{n+1}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad &\left[\sum_{k=0}^{n} \frac{1}{k!} \cdot \left(1-\frac{1}{n}\right) \cdot \left(1-\frac{2}{n}\right) \cdot \ldots \cdot \left(1-\frac{k-1}{n}\right)\right] \\
&\le \quad \left[\sum_{k=0}^{n} \frac{1}{k!} \cdot \left(1-\frac{1}{n+1}\right) \cdot \left(1-\frac{2}{n+1}\right) \cdot \ldots \cdot \left(1-\frac{k-1}{n+1}\right)\right] \\
&\le \quad \left[\sum_{k=0}^{n+1} \frac{1}{k!} \cdot \left(1-\frac{1}{n+1}\right) \cdot \left(1-\frac{2}{n+1}\right) \cdot \ldots \cdot \left(1-\frac{k-1}{n+1}\right)\right]
\end{aligned}
$$

Therefore, $a_{n+1} \ge a_n$ for any n. (proof)□

Thus $\{a_n\}$ is increasing.

Since $a_1 = 2$ is the leading value and $a_{n+1} \ge a_n \quad \forall n$, we know 2 is the "best" lower bound for $\{a_n\}$. Note that 0, -12, -10^6 or any number less than 2 could also be a lower bound. □

This sequence converges:

$$\lim_{n\to\infty} a_n = e$$

e was originally named "b" by Leibniz. It was Euler who re-named it to e in 1731.

Euler discovered that $e = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots$
Along with other formulae, Euler decided that e, like π, was irrational. We still aren't sure if e^e is rational!

Peirce would show this equation to his student:

$$i^{-i} = \sqrt{e^{\pi}}$$

Exercise:

Let $f(x)$ be a function, $f'(x)$ be its derivative. What function satisfies the differential equation

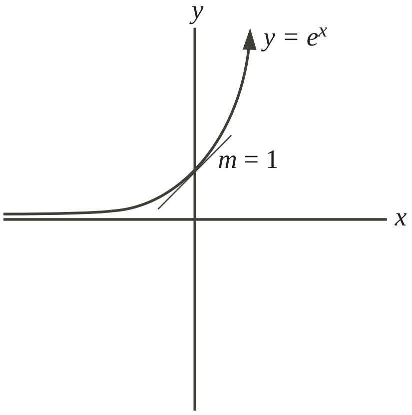
$$f'(x) = f(x)$$
$$f(0) = 1$$

Solution:

$f(x) = e^x$ is the only such function! $\square$

In Calculus, e is defined to be the base of the exponential function $f(x)$ such that the slope of the tangent line at $x = 0$ to f(x) is 1:

$f(x) = 2^x \Rightarrow f'(x) = 2^x \ln(2) => f'(0) = \ln(2) < 1$
$f(x) = 3^x \Rightarrow f'(x) = 3^x \ln(3) => f'(0) = \ln(3) > 1$



Exercise:

Below are the Taylor series expansions for e^z, $\sin z$, and $\cos z$, where $z \in \mathbb{C}$

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

Use these to derive the famous Euler Identity

$$e^{i\pi} + 1 = 0$$

Solution:

We notice that:

$$\begin{aligned} e^{iz} &= 1 + \frac{iz}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \dots \\ &= 1 + \frac{iz}{1!} + \frac{(-1)z^2}{2!} + \frac{(-i)z^3}{3!} + \frac{z^4}{4!} + \dots \end{aligned}$$

Since $i^1 = i$ $\quad i^2 = -1$
$i^3 = -i$ $\quad i^4 = 1$

re-arranging terms:

$$\begin{aligned} e^{iz} &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right) + \left(\frac{iz}{1!} - \frac{iz^3}{3!} + \frac{iz^5}{5!} - \dots\right) \\ &= \cos z + i\left(\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) \\ &= \cos z + i \sin z \end{aligned}$$

$$\therefore \quad e^{iz} = \cos z + i \sin z$$

$$\begin{aligned} \text{Let } z = \pi \Rightarrow e^{i\pi} &= \cos\pi + i\sin\pi \\ \Rightarrow e^{i\pi} &= -1 + 0 \\ \Rightarrow e^{i\pi} + 1 &= 0 \end{aligned}$$

A bit about Leonhard Euler: (c. 1750)

- Was expected to follow his father's footsteps in Christian ministry.
- Taught himself mathematics throughout grade/high school.
- Was a student of Johann Bernoulli.
- Worked on many government projects and even served as a medical officer in the Russian Navy.
- Studied at St. Detersburg Academy of Sciences, leaving Switzerland for Russia in 1727 . . . his big career move occurred the year Newton died!
- Had some health issues including eye sight loss; he had a photographic memory, so continued work after becoming blind.
- His last words: "I am dying. . . " 1783.

History of Diophantine Equations

Diophontus was a 3rd century Alexandrian mathematician, who first introduced equations solvable by integer values only.

Examples:

$$ax + by = 1 \qquad ; \text{a, b constants}$$

$$x^n + y^n = z^n \qquad ; n \geq 2$$

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \qquad ; x, y, z > 2$$

We study Diophantine equations by asking:

$\hookrightarrow$ are there solutions?

$\hookrightarrow$ how many solutions?

$\hookrightarrow$ how can we express all solutions?

India was a hot spot for Diophantine theory, beginning even before 3rd century: 800 BC. They continued to be popular there in medieval times.

Fermat's Last Theorem was introduced in 1637:
$x^n + y^n = z^n$ has no solutions for integers $x, y, z, n > 2$.

Euler solved many Diophantine equations.

Hilbert's 10th Problem is to solve all Diophantine equations.

Fact: A Diophantine equation of the form $d = ax + by$ has a solution (x, y) if and only if $d = \gcd(a, b)$ or d is a multiple of $\gcd(a, b)$.

Exercise:

For each pair a, b, find $\gcd(a, b)$.

- a = 4, b = 6
- a = 21, b = 56
- a = 3, b = 7
- a = 1350, b = 255

Solution:

$\gcd(4, 6) = 2$
$\gcd(21, 56) = 7$
$\gcd(3, 7) = 1$ (called "co-prime")
$\gcd(1350, 255) = \quad ?$

To find $\gcd(1350, 255)$, we could write the prime factorization of each number. Another way is to use the Euclidean Algorithm:

Write $\qquad a = qb + r \qquad q =$ quotient of $\frac{a}{b}$, $r =$ remainder of $\frac{a}{b}$.

1. $1350 = (5)(255) + 75$ Change a to 255 and b to 75 and repeat.
2. $255 = (3)(75) + 30$ Change a to 75 and b to 30 and repeat.
3. $75 = (2)(30) + 15$
4. $30 = (2)(15) + 0 \quad \longleftarrow$ at this point, we stop.

The last non-zero remainder is 15.

$\therefore \quad 15 = \gcd(1350, 255)$

Exercise:

Use 1.) to 3.) above to solve the Diophantine equation.

$$15 = 1350x + 255y$$

Hint: we start by rearranging 3.)

$$15 = 75 + (2)(30)$$

Solution:

$$\begin{aligned}
15 &= 75 - 2(30)\\
\text{2.) gives: } 30 &= 255 - 3(75)\\
\therefore\ 15 &= 75 - 2\left(255 - 3(75)\right)\\
&= 75 - 2(255) + 6(75)\\
&= -2(255) + 7(75)\\
\text{1.) gives: } 75 &= 1350 - 5(255)\\
\therefore\ 15 &= -2(255) + 7\left(1350 - 2(255)\right)\\
&= -2(255) + 7(1350) - 35(255)\\
\Rightarrow\ 15 &= 7(1350) - 37(255)\\
\therefore\ 15 &= 7a - 37b\\
\therefore\ (x, y) &= (7, -37)
\end{aligned}$$

Exercise:

Consider $15 = 1350x + 255y$ Are there any other solutions? If so how many?

Solution:

$$\begin{aligned}
15 &= 1350x + 255y\\
\Rightarrow\ 1 &= 90x + 17y\\
x = -7,\ y &= -37 \text{ still works}\\
\text{Let } \bar{x} &= 7 + 17\\
\bar{y} &= -37 - 90\\
\text{Then } 90\bar{x} &+ 17\bar{y}\\
&= 90(7 + 17) + 17(-37 - 90)\\
&= 90(7) + 90(17) - 37(17) - 90(17)\\
&= 90(17) - 90(17) + 90(7) - 37(17)\\
&= 0 + 1\\
\text{So } \bar{x},\ \bar{y} &\text{ works too!}
\end{aligned}$$

For the same reason, $\dot{x} = 7 + 2(17)$ and $\dot{y} = -37 - 2(90)$ works.

There are actually infinitely many solutions.

$$\begin{aligned}
x &= 7 + 17n\\
y &= -37 - 90n
\end{aligned}$$
$$\text{where } n \in Z, \text{ is the solution set.}$$

Game:

Match the following mathematician with his/her accomplishments

Cardon

Euler

Recorde

Al-Khwarizmi

Lagrange

Tartaglia

Galois

Diophantus

1.Challenged two prominent mathematicians to math contests.

2. Coined “=” as our equals sign.

3. Proved there is no general quintic equation formula.

4. Married his cousin.

5. Solved cubic and quadratic equations with integers only.

6. Popularized solving equations with integers only.

7. Coined the term “algebra”.

8. Found many expressions for the constant e.