University of Waterloo
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# Senior Math Circles
# February 24, 2010
# Probability and Expectation

**Some Preliminary Terminology**

An *experiment* is a theoretically repeatable process or phenomenon.

Each repeat of an experiment is called a *trial* and the result of each trial is called an *outcome* (or simple event).

The collection of outcomes is called a *sample space*, $S$.

If we combine simple events we create a compound event (otherwise called "event"). Events are denoted by capital letters from the start of the alphabet like $A, B, C, D,$ or $E$.

Eg. List all outcomes of rolling a die twice.
Eg. Can you think of a compound event?

Notes: Dice (singular die) are 6 sided and fair. There are 52 cards in a deck (unless stated otherwise) and the deck is well shuffled.

**Definition of Probability (Relative Frequency)**

The relative frequency definition of probability states that the probability of event $A$ is the number of times the event $A$ occured, denoted by $|A|$, in $n$ trials. Hence $Pr(A) = \dfrac{|A|}{n}$.

Eg. I flip a coin 50 times and get 35 heads. What is the probability of getting a head on a coin?

**Definition of Probability(Mathematical)**

Let $|E|$ be the number of outcomes in event $E$.
Let $|S|$ be the number of outcomes in sample space $S$.
Then the probability is $Pr(E) = \dfrac{|E|}{|S|}$.

**Exercise**

1. What is the probability I roll a 5 on a fair die?
2. What is the probability I draw a diamond from a shuffled deck of cards?

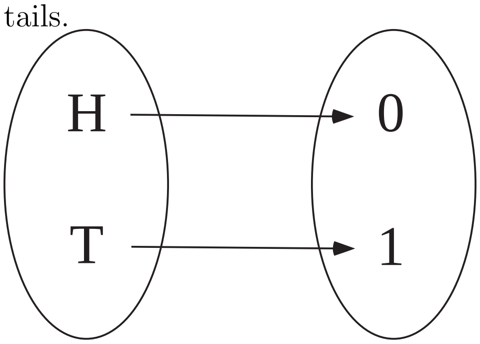
**Probability Properties**

- Let $E_i$ be an event. Then, $0 \leq Pr(E_i) \leq 1$.
- Let $|S| = n$ (ie., there are $n$ simple events). Let $E_i$ be a simple event (there are $n$ of them). Then $\sum_{i=1}^{n} Pr(E_i) = 1$.

**Random Variables**

A *random variable* is a function that assigns a number to each outcome. Random varaibles ($RV$s) are denoted by capital letters.

Eg. I flip a coin. The random variable illustrated below is a function assigning a 0 to heads and 1 to tails.




The random variable is denoted by a capital letter $X$. The *realization* is denoted by the lower case letter $x$ and is the output of a random variable, $X$.

We write:

$$Pr(X = x) = f(x)$$

$f(x) = Pr(X = x)$ is called a *probability function*.

It has properties:

- $0 \leq f(x) \leq 1$
- $\sum_{\text{all } x} f(x) = 1$

**Exercise**

1. Let $X = 1$ if we flip a head on a coin.What is $P(X = 1)$?

2. Let $f(x) = a2^{-x}$ for $x = 1, 2, 3, 4$, where $a$ is a constant. Find $a$.

**Expectation**

The sample average is defined to be $\sum_{\text{all } x} x \frac{\text{frequency } x}{n}$ where $n$ is the total number of trials in the sample.

Suppose I roll a die 6 times with results:

| **Value** | **Frequency** |
|---|---|
| 1 | 1 |
| 4 | 2 |
| 5 | 1 |
| 6 | 2 |

The average (same mean) value of the rolls is $1(\frac{1}{6}) + 4(\frac{2}{6}) + 5(\frac{1}{6}) + 6(\frac{2}{6}) = \frac{26}{6} = \frac{13}{3}$.

Now, imagine if you will, rolling the die an infinite number of times. What would the population average value of the rolls be?

The long run (population mean) expectation is denoted by:

$$\mu = E(X) = \sum_{\text{all } x} x(f(x))$$

where $f(x)$ is a probability function.

You can think of this as:

1. A weighted average where the weights are probabilities.

2. The average of an infinite number of trials.

3. The average of a population.

**Exercises**

1. In a particular class we know that the age of a randomly selected student is given by the following proportions.

| **Age** | 15 | 16 | 17 |
|---|---|---|---|
| **Proportion** | 0.3 | 0.5 | 0.2 |

Find the expected age of a randomly selected student.

2. What value would you expect to roll on a die?
3. Common Sense. If I flip a coin 12 times, how many heads would you expect me to get?

**Properties**

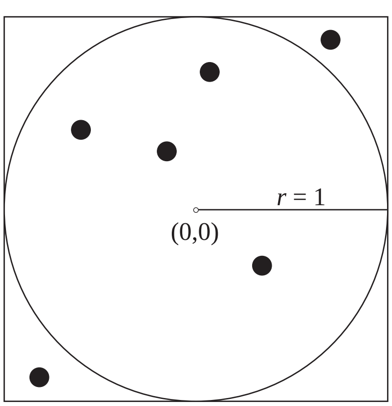
- $E(a) = a$
- $E(a + X) = a + E(X)$
- $E(aX) = aE(X)$

**Problem Set**

1. List a sample space for tossing a fair coin 3 times. Use this list to determine the probability of 2 consecutive tails (but not 3).
2. If we roll two die (a red and a green) what is the probability we get a total of 5?
3. What is the probability I draw a heart from a standard deck of cards?
4. The following probability function is given for a random variable $X$. Find $c$.

| $X$ | 1 | 2 | 3 |
|---|---|---|---|
| $Pr(X = x) = f(x)$ | $0.3c$ | $0.075c^2$ | $0.1$ |

5. What is the main disadvantage of the relative frequency definition of probability?
6. For Question 4, determine $E(X)$.
7. Suppose I flip a coin 2 times. Let $X$ be the number of heads obtained.
   (a) What is the probability function?
   (b) What is the expected number of heads?
8. A game lands on the values 1, 2, and 3 with probability 0.2, 0.6, and 0.2 respectively. Find $E(X)$.
9. Show that $E(aX) = aE(X)$.
10. We define the expectation of a function $g(x)$ to be $E(g(X)) = \sum_{\text{all } x} f(x)g(x)$.
    (a) For Question 8, you will $(-2)^x$ dollars on every roll. How much do you expect to win?
    (b) Show that $E(ag(X)) = aE(g(x))$.
    (c) Show that $E(a + g(X)) = a + E(g(x))$.
11. **Challenge**: Merlin the magician would like to determine the value of $\pi$. To do so he:
    i) Draws a circle inside a square on the ground with a stick.
    ii) Randomly drops marbles on the square
    Note: He always hits the square.
    (a) How many, of the 20 marbles he drops, would you expect to land in the circle?
    (b) Estimate the value of $\pi$, using the diagram below.



**Solutions**

1. The sample space is $S = \{$ HHH, HHT, HTH, THH, TTH, THT, HTT, TTT $\}$.
   $P(\text{2 consectuve tails}) = \dfrac{1}{4}$.

2.

| + | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

   So, $P(\text{a total of 5}) = \dfrac{4}{36} = \dfrac{1}{9}$.

3. $P(\text{drawing a heart from a deck}) = \dfrac{13}{52} = \dfrac{1}{4}$.

4.
$$\begin{aligned} 0.3c + 0.075c^2 + 0.1 &= 1 \\ 0.3c + 0.075c^2 - 0.9 &= 0 \\ c^2 + 4c - 12 &= 0 \\ (c-2)(c+6) &= 0 \end{aligned}$$
   If $c = -6$, then $f(1) = -1.8$ which is not possible.
   So $c = 2$, where $f(1), f(2), f(3)$ are between 0 and 1.

5. We should repeat the experiment an infinite number of trials, to obtain an accurate result.

6.

| $X$ | 1 | 2 | 3 |
|---|---|---|---|
| $Pr(X = x) = f(x)$ | 0.6 | 0.3 | 0.1 |

$$E(X) = \sum_{\text{all } x} x f(x) = 1(0.6) + 2(0.3) + 3(0.1) = 1.5$$

7. (a)

| $X$ | 0 | 1 | 2 |
|---|---|---|---|
| $Pr(X = x) = f(x)$ | 0.25 | 0.5 | 0.25 |

   (b) $E(X) = 0(0.25) + 1(0.5) + 2(0.25) = 1$

8. $E(X) = 1(0.2) + 2(0.6) + 3(0.2) = 2$

9.

$$\begin{aligned} E(X) &= \sum_{\text{all } x} x f(x) \\ E(aX) &= \sum_{\text{all } x} a x f(x) \\ &= a \sum_{\text{all } x} x f(x) \\ &= aE(X) \end{aligned}$$

10. (a)

| $(-2)^x$ | -2 | 4 | -8 |
|---|---|---|---|
| $Pr(X = x) = f(x)$ | 0.2 | 0.6 | 0.2 |

$E(X) = (-2)(0.2) + (4)(0.6) + (-8)(0.2) = 0.4$

(b)

$$\begin{aligned} E(ag(X)) &= \sum_{\text{all } x} a f(x) g(x) \\ &= a \sum_{\text{all } x} f(x) g(x) \\ &= aE(g(X)) \end{aligned}$$

(c)

$$\begin{aligned} E(a + g(X)) &= \sum_{\text{all } x} (a + g(x)) f(x) \\ &= \sum_{\text{all } x} a f(x) + \sum_{\text{all } x} f(x) g(x) \\ &= E(a) + E(g(X)) \\ &= a + E(g(X)) \end{aligned}$$

11. (a) The probability that a marble lands in the circle is: $\dfrac{\text{Area of Circle}}{\text{Area of Square}} = \dfrac{\pi}{4}$.

So we expect $20(\dfrac{\pi}{4}) \approx 16$ marbles to land in the circle.

(b) The probability we hit the circle is $\dfrac{\pi}{4}$.

We have 4 out of 6 marbles that hit the circle in the diagram.

Using these experimental results to approximate $\pi$, we have, $\dfrac{4}{6} \approx \dfrac{\pi}{4}$.

So, $\pi \approx \dfrac{8}{3}$.