Math Circles: History of Equations II

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Last Week

Exercise: Given $P(x, y) = x^3 + y^3 - 4$, use the symmetric polynomials $\sigma_1 = x + y$ and $\sigma_2 = xy$ to express P as a function of σ_1 and σ_2. Think about how many **elementary** symmetric polynomials there are in two variables.

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Solution - Part 1: Here we use the fact that
$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
This gives us that $x^3 + y^3 = (x + y)^3 - 3x^2y - 3xy^2$
$= \sigma_1^3 - 3(x^2y + xy^2)$
$= \sigma_1^3 - 3(x + y)xy$
$= \sigma_1^3 - 3\sigma_1\sigma_2$.
Thus $P(x, y) = x^3 + y^3 - 4 = \sigma_1^3 - 3\sigma_1\sigma_2 - 4$, in terms of elementary symmetric polynomials.

Another Solution

We also found out that we could use the fact that $x^3 + y^3 = (x+y)(x^2 - xy + y^2) = (x+y)(x^2 + 2xy - 3xy + y^2)$. This gives us that $x^3 + y^3 - 4 = (x+y)(x^2 + 2xy + y^2 - 3xy) - 4$ $= (x+y)[(x+y)^2 - 3xy] - 4 = \sigma_1(\sigma_1^2 - 3\sigma_2) - 4$

This is the same expression as we got using the other method. Will we always get the same result no matter how we break down our polynomial?

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Yes, the building blocks σ_1 and σ_2 will always express a given symmetric polynomial in a unique way.

Fundamental Corollary of Symmetric Polynomials

Solution - Part 2: There are only two elementary symmetric polynomials (with real coefficients) in 2 variables. Here is the general rule:

Let $P(x_1, ...x_n)$ be a polynomial in the set of all polynomials in n variables $\{x_1, ...x_n\}$. Then P can be expressed in terms of the n symmetric polynomials $\sigma_1, ...\sigma_n$, where

$\sigma_1 = x_1 + ... + x_n$
$\sigma_2 = x_1x_2 + x_2x_3 + ... + x_{n-1}x_n$
$\sigma_3 = x_1x_2x_3 + x_2x_3x_4 + ... + x_{n-2}x_{n-1}x_n$
...all the way to
$\sigma_n = x_1x_2 \cdots x_{n-1}x_n$

Quadratic Formula Derivation

Now, we all know that given an equation $ax^2 + bx + c = 0$, there are two solutions, namely:

$x = (-b \pm \sqrt{b^2 - 4ac})/2a$. We can derive this result for the *monic* polynomial $x^2 + px + q$ using symmetric polynomials.

Exercise: We're given a quadratic equation $x^2 + px + q$. It has roots α and β. Write an expression linking the coefficients p and q with the roots.

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Solution: $x^2 + px + q = (x - \alpha)(x - \beta)$
Which gives: $x^2 + px + q = x^2 - \alpha x - \beta x + \alpha\beta$
$\Rightarrow x^2 + px + q = x^2 + x(-(\alpha + \beta)) + \alpha\beta$.

Thus we must have: $p = -(\alpha + \beta)$ and $q = \alpha\beta$.

Quadratic Formula Derivation

Let $t_1 = \alpha + \beta$ and $t_2 = \alpha - \beta$.
Then $\alpha = \frac{1}{2}(t_1 + t_2)$ and $\beta = \frac{1}{2}(t_1 - t_2)$.

Our goal is to express t_1 and t_2 in terms of p and q, which will then result in our finding α and β in terms of p and q. We can see that $t_1 = -p \star$.

Exercise: Express t_2 in terms of p and q. Begin with the equation:
$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$.

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Quadratic Formula Derivation

Solution: $t_2 = \alpha - \beta$, $-p = \alpha + \beta$ and $q = \alpha\beta$. We substitute these into the above equation to get:
$t_2^2 = (-p)^2 - 4q$
$\Rightarrow t_2 = \pm\sqrt{p^2 - 4q}$. We can take $t_2 = \sqrt{p^2 - 4q}$. $\ast$

This means that:
$\alpha = \frac{1}{2}(t_1 + t_2) = (-p + \sqrt{p^2 - 4q})/2$ by $\star$, and
$\beta = \frac{1}{2}(t_1 - t_2) = (-p - \sqrt{p^2 - 4q}/2$ by $\ast$.

These roots are consistent with the quadratic formula for a *monic* polynomial (why is this a general solution?). What happens if we take $t_2 = -\sqrt{p^2 - 4q}$ instead?

α and β are interchanged, but the overall solution remains the same.

Symmetric Polynomials and Roots

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There is only one way to permute α and β in p and in q - we switch them.

There is a connection between the number of ways to permute α and β and the ease of solving a degree 2 polynomial...but it is beyond the scope of our talks!

On Your Marks...

It wasn't until the 1000s that progress began toward solving equations of the form $ax^3 + bx^2 + cx + d = 0$.

In the early 1500s, an Italian mathematician named Nicolo Fontana a.k.a. Tartaglia made some solid headway in the problem.

The idea behind solving this equation was to start with special cases, for example $x^3 + ax + b = 0$, or $x^3 + ax^2 + b = 0$.

Tartaglia had a long, bitter debate with his once trusted colleague Cardan over rights to certain steps to solving cubic equations.

It turned out that Tartaglia's math combined with the work of Cardan and his assistant Ferrari resulted in the solution to the general cubic equation.

Rivalry

The story goes, Fontana heard that a collegue, Fior (a student of del Ferro), had declared he could solve cubic equations. Fontana became angered and challenged Fior to a competition.

Each mathematician was to submit 30 problems for the other to solve. Fior was only able to solve equations of the type $x^3 + ax + b = 0$. Fontana ended up winning.

When news of the competition came to Cardan, he contacted Fontana to ask him to share his cubic solutions. Fontana wanted it to be a secret, so he refused.

Cardan then tried to bait Fontana into visiting him for different reasons, until finally they ended up in each other's company...and Fontana gave his solution.

Cardan was instructed not to publish it, but eventually Cardan found out about del Ferro's work, including that which was not disclosed to Fior. He published everything he knew, including Fontana's method.

Cardano-Ferrari-Fontana

Fontana was furious. He and Cardan's math partner, Ferrari, wrote insulting letters back and forth for a long time. Eventually a new competition was organized between Fontana and Ferrari.

Fontana thought he'd win again, but he was wrong. In fact, Cardan and Ferrari had also solved the quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, so Ferrari was well prepared.

With the threat of defeat imminent, Fontana did the noble thing: he ran away. Ever since his career was not the same.

The cubic formula is credited to Fontana and Cardan. It's called the *Cardan-Tartaglia* formula.

Cubic Formula

Here is the formula:

For $x^3 + ax^2 + bx + c = 0$

$$x = \frac{-a}{3} - \sqrt[3]{\frac{-9q+\sqrt{12p^3+81p^2}}{18}} - \sqrt[3]{\frac{9q+\sqrt{12p^3+81q^2}}{18}}$$

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Where $p = b - \frac{a^2}{3}$

and $q = c - \frac{ab}{3} + \frac{2a^3}{27}$

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Derivation

We begin with $x^3 + ax^2 + bx + c = 0$.

Exercise: Let $x = y - \frac{a}{3}$. Substitute this change of variables into the cubic equation. It helps to calculate each term separately before putting it all together. Then simplify as much as possible. You should end up with only 6 terms on the left!

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Solution:

$$(y - \tfrac{a}{3})^3 = y^3 - \frac{3y^2a}{3} + \frac{3ya^2}{3^2} - \frac{a^3}{3^3}$$

$$= y^3 - y^2a + \frac{ya^2}{3} - \frac{a^3}{27}$$

$$a(y - \tfrac{a}{3})^2 = a(y^2 - \frac{2ya}{3} + \frac{a^2}{3^2})$$

$$= ay^2 - \frac{2ya^2}{3} + \frac{a^3}{9}$$

$$b(y - \tfrac{a}{3}) = by - \frac{ba}{3}$$

Derivation...

Subbing these expressions back into the cubic:

$$y^3 - y^2 a + \frac{ya^2}{3} - \frac{a^3}{27} + ay^2 - \frac{2ya^2}{3} + \frac{a^3}{9} + by - \frac{ba}{3} + c = 0$$

And to simplify:

a) $-y^2 a + ay^2 = 0$
b) $\frac{a^3}{9} - \frac{a^3}{27} = \frac{3a^3 - a^3}{27} = \frac{2a^3}{27}$
c) $\frac{ya^2}{3} - \frac{2ya^2}{3} = \frac{-ya^2}{3}$

Therefore the cubic becomes:

$$y^3 - \frac{ya^2}{3} + \frac{2a^3}{27} + by - \frac{ba}{3} + c = 0$$

Derivation...still

Exercise: So far we have: $y^3 - \frac{ya^2}{3} + \frac{2a^3}{27} + by - \frac{ba}{3} + c = 0$. Collect like terms and express the equation in the form $y^3 + py + q = 0$, where p and q are in terms of a, b, c. What have we accomplished so far?

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Exercise: So far we have: $y^3 - \frac{ya^2}{3} + \frac{2a^3}{27} + by - \frac{ba}{3} + c = 0$. Collect like terms and express the equation in the form $y^3 + py + q = 0$, where p and q are in terms of a, b, c. What have we accomplished so far?

Solution: We have eliminated the y^2 term. The method following was designed to solve only these types of equations, turns out it solves a general cubic now too!

We have $y^3 + y[b - \frac{a^2}{3}] + [c - \frac{ab}{3} + \frac{2a^3}{27}]$

Thus $p = b - \frac{a^2}{3}$ and $q = c - \frac{ab}{3} + \frac{2a^3}{27}$

More Derivation

Exercise: Let $y = u - v$ and substitute this into $y^3 + py + q = 0$. Simplify this to $q - (v^3 - u^3) + (3uv - p)(v - u) = 0$

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Solution: The cubic equation becomes $(u - v)^3 + p(u - v) + q = 0$. Expanding, we get:

$$
\begin{aligned}
& u^3 - 3u^2v + 3uv^2 - v^3 + pu - pv + q = 0 \\
& \Rightarrow q - v^3 + u^3 + 3uv^2 - 3u^2v + pu - pv = 0 \\
& \Rightarrow q - (v^3 - u^3) + 3uv^2 - pv - 3u^2v + pu = 0 \\
& \Rightarrow q - (v^3 - u^3) + v(3uv - p) - u(3uv - p) = 0 \\
& \Rightarrow q - (v^3 - u^3) + (3uv - p)(v - u) = 0
\end{aligned}
$$

Derivation, cont'd

Now we have $q - (v^3 - u^3) + (3uv - p)(v - u) = 0$. We can solve this equation by asking ourselves what conditions would make the LHS 0.

We need each *summand* to equal 0:

1) $q - (v^3 - u^3) = 0$
2) $3uv - p = 0$

This becomes:

1) $q = v^3 - u^3$
2) $p = 3uv$

How do we find u and v to make this work?

Derivation

Exercise: Find a single equation connecting p, q, u and v using:

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Solution: 2) gives $v = \frac{p}{3u}$. Subbing this into 1) gives
$q = (\frac{p}{3u})^3 - u^3 = \frac{p^3}{27u^3} - u^3$

Clearing the denominator, we have: $27qu^3 = p^3 - 27u^6$, or
$27u^6 - 27qu^3 - p^3 = 0$. This can be viewed as a quadratic equation in u^3:

$$27(u^3)^2 - 27q(u^3) - p^3 = 0$$

Derivation

To find a solution to this equation we may use the quadratic formula, but we only need the positive root.

$$u^3 = \frac{-(-27q)+\sqrt{(-27q)^2-4(27)(-p^3)}}{2(27)}$$

$$= \frac{27q+\sqrt{729q^2+108p^3}}{54}$$

$$= \frac{9q+\sqrt{12p^3+81q^2}}{18}$$ by dividing each term by 3

In the same way, we may find v^3: $v^3 = \frac{-9q+\sqrt{12p^3+81q^2}}{18}$.

Derivation

We have: $u^3 = \frac{9q+\sqrt{12p^3+81q^2}}{18}$
$v^3 = \frac{-9q+\sqrt{12p^3+81q^2}}{18}$

$x = y - \frac{a}{3}$ and
$y = u - v$.

Exercise: Using these 4 equations, find x in terms of p, q and a, thereby solving the cubic equation $x^3 + ax^2 + bx + c = 0$.

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where $p = b - \frac{a^2}{3}$ and $q = c - \frac{ab}{3} + \frac{2a^3}{27}$