University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Senior Math Circles
October 21, 2009
Infinite Series III

Last week: Harmonic series

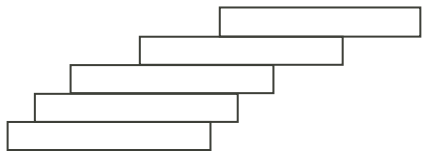
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \dots$$

Surprisingly, it diverges!

$$\sum_{n=1}^{N} \frac{1}{n} \approx \ln(N)$$

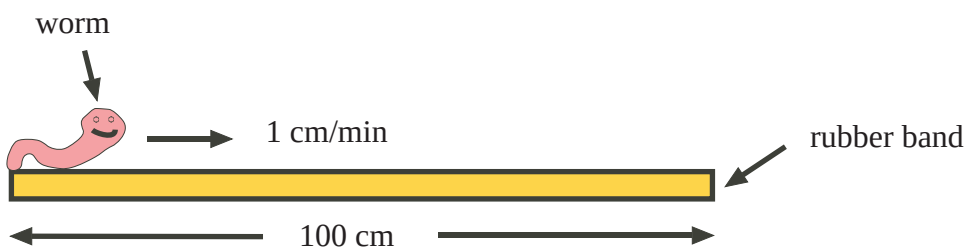
Remark: This is a result from calculus, but to see why you get logarithmic behaviour, look at our 1st proof, where we grouped terms to get the sum $> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$. Since we used 1, 2, 4, 8, 16, ... terms to get each $\frac{1}{2}$, i.e. an exponentially increasing *number* of terms, it follows that the series grows *logarithmically* as you add one term at a time.

Last week, we stacked a number of CD cases in the 'overhang' problem:



The overhang turned out to be a multiple of the harmonic series, but this diverges to ∞, so if you have infinitely many CD cases, you can make the overhang as big as you like. Remember 1 metre overhang needs a pile 8 km high!

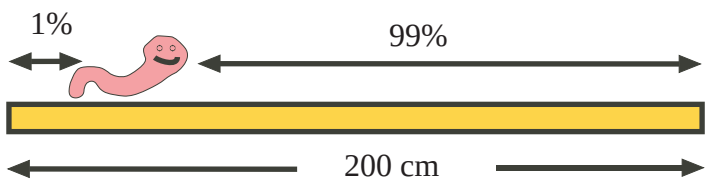
Now consider the 'worm problem':



A worm, at one end of a 1 m long rubber bad, crawls at a rate of $1\frac{cm}{min}$ toward the other end.

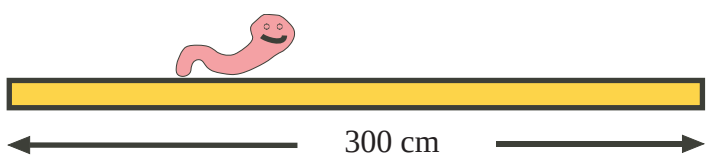
Question: How long will it take to reach the end?
Answer: 100 minutes, unless someone stretches the band!

Now let's suppose that, one minute after the worm begins its journey, you stretch the band so it's 2m in length.



The worm, which was 1% of the way before the stretch, will still be 1% of the way there, but now there's 198 cm left.

One minute later you stretch the band another metre:



Question: What percentage of the way is he now?
Answer: $(1 + \frac{1}{2})\%$, since he just crawled 1cm along the 200 cm band (so $1\frac{1}{2}$ of the band), and he maintains relative position after the stretch.

<u>**Exercise 1:**</u> If you lengthen the bad by 1 m each minute, will the worm ever reach the end?

<u>**Answer:**</u> Yes! After N steps, the worm will have travelled $(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{N})\%$ of the way. This is the harmonic series, which diverges to ∞, so we can make the percentage as big as we like!

Exercise 2: How long will it take (roughly) to reach the end, and how long will the rubber band be? (if you don't have a calculator, just simplify as much as you can)

Answer: (I don't have a calculator.)
To reach the end, we want $(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{N})\% \geq 100\%$
Using the $\ln N$ (rough) approximation, we set

$$\ln N \geq 100$$
$$\text{which implies } N \geq e^{100} \approx 10^{100(\log_{10} e)} \approx 10^{30} \text{ ish} \qquad \log_{10} e \approx 0.3$$
$$\text{So it travels } \approx \tfrac{1cm}{step} \times 10^{30}\text{steps} = 10^{30}cm \approx 10^{30}m$$

Alternating Series

Consider the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots \qquad \text{(signs alternate } + - + - + - \dots)$$

Does it converge or diverge?

Caution: You may be tempted to write this as

$$\left(1 + \frac{1}{3} + \frac{1}{5} + \dots\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots\right)$$

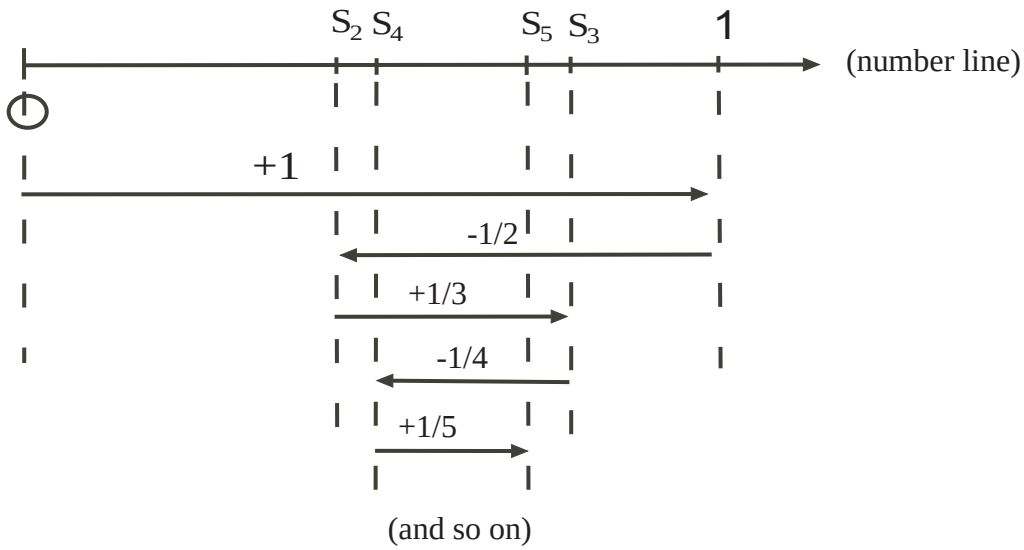
but each of these series diverges, as we saw last week. The only time you are not allowed to 'split up' an infinite sum is when each 'piece' converges. So, the expressions above are not the same thing.

Another way to look at this is by saying it 'looks like' $\infty - \infty$. This tells you nothing about the sum, since $\infty - \infty$ could mean many different things, depending on how 'big' the first ∞ is compared to the second.

Let's look at the partial sums:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots$$
$$S_2 = 1 - \frac{1}{2}$$
$$S_3 = 1 - \frac{1}{2} + \frac{1}{3}$$
$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$
$$\dots$$

Graphically:



Continuing this process, we suspect that the partial sum S_n converges to a certain number as we take more and more terms. So, the series appears to converge. This is proven in many calculus texts.

It turns out that

1. Any alternating series converges if its terms are decreasing in size (e.g. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$). Get 'close as you like' to 0 eventually when truncating the sum.
2. The 'error' is less than the first term thrown away. An example will make this more clear.

Example:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} \qquad \text{with error} \leq \frac{1}{10}$$

i.e. $$1 - \frac{1}{2} + \frac{1}{3} - \cdots = \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{9}\right) \pm 0.1$$

Exercises:

1. Decide if the series converges or diverges.

   a.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ b.) $\displaystyle\sum_{n=0}^{\infty} (-1)^n$ c.) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n$

2. Estimate the sum of each convergent series to within 0.01.

3. Joe and Olga, $1km$ apart, ride their bikes toward each other at $10\frac{m}{s}$. A fly starts on Joe's front tire, flies a $30\frac{m}{s}$ to Olga's front tire, then back to Joe's, the back to Olga's, and so on until the bikes collide. How far has the fly flown?

Answers:

1. a.) Converges b.) Diverges(see week 1) c.) Converges

2. a.) $\sum_{n=1}^{10} \frac{(-1)^{n-1}}{n^2}$ since error $< \frac{1}{10^2}$ c.) $\frac{1}{1 - \frac{-1}{2} = \frac{2}{3}}$

3. Hard way: Use an infinite sum.

   Smart way: Bikes collide in $\frac{500m}{10\frac{m}{s}} = 50s$. So fly travels $50s \times 30\frac{m}{s} = 1500m$.

Closing remark

Infinite series is a very counter-intuitive concept at times.

Why does? $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$ (geometric),

but $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \to \infty$ (harmonic)?

It is because the terms in the first series approach 0 quickly enough that they eventually don't add up to much. The terms in the second series don't go to 0 quickly enough, and that is why it "adds to ∞". (If the series alternates, e.q. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \ldots$, you get some cancellation which helps it converge.) If you study calculus, you will further explore this dance between zero and infinity. Enjoy!