Math Circles:

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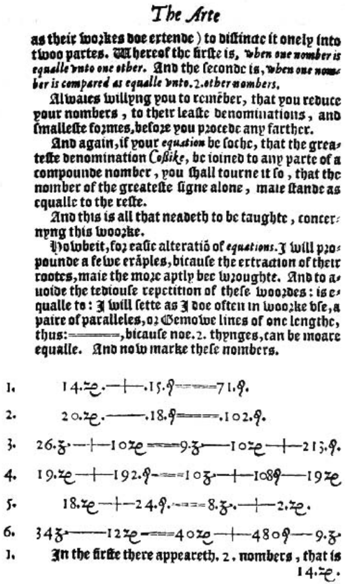
October 30, 2009

First Equation Ever by Robert Recorde c. 1557



The Arte

as their workes doe extende) to distincte it onely into twoo partes. Whereof the firste is, *when one nomber is equalle unto one other*. And the seconde is, *when one nomber is compared as equalle unto.2.other nombers.*

Alwaies willyng you to remẽber, that you reduce your nombers, to their leaste denominations, and smalleste formes, before you procede any farther.

And again, if your *equation* be soche, that the greateste denomination *Cosike*, be ioined to any parte of a compounde nomber, you shall tourne it so, that the nomber of the greateste signe alone, maie stande as equalle to the reste.

And this is all that neadeth to be taughte, concernyng this woorke.

Howbeit, for easie alteratiõ of *equations*. I will propounde a fewe exãples, bicause the extraction of their rootes, maie the more aptly bee wroughte. And to auoide the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe lines of one lengthe, thus:=====, bicause noe.2. thynges, can be moare equalle. And now marke these nombers.

1. 14.ꝛꝫ.—+—.15.ϑ=====71.ϑ.
2. 20.ꝛꝫ.———.18.ϑ=====.102.ϑ.
3. 26.ℨ.—+—10ꝛꝫ=====9.ℨ——10ꝛꝫ—+—213.ϑ.
4. 19.ꝛꝫ—+—192.ϑ=====10ℨ—+—108ϑ——19ꝛꝫ
5. 18.ꝛꝫ—+—24.ϑ.=====8.ℨ.—+—2.ꝛꝫ.
6. 34ℨ———12ꝛꝫ=====40ꝛꝫ—+—480ϑ——9.ℨ.

1. In the firste there appeareth. 2. nombers, that is 14.ꝛꝫ.

Robert Recorde - Welsh

A bit about Robert

Attended Oxford University for medicine in c. 1525

Oversaw a large-scale mining project for the production of silver coins

Wrote many mathematics texts in English geared toward a general audience, not just the few educated scholars who knew Latin and Greek

Most of these texts had the form of a student diologue with a scholar

Had interest in the astronomical theory of Ptolemy and Copernicus, although careful to declare his agreement with Copernican theory

Muhammad Al-Khwarizmi - Persian c. 800 AD

A page from Al-Khwarizmi's Algebra Book

on Muhammad ibn Musa al-Khwarizmi

Considered the founder of algebra, he was the first mathematician to devise general solutions to linear and quadratic equations

Note that the Babylonians were solving such equations 4000 years ago

Contributed to the areas of astronomy, cartography and geometry as well as mathematics

Was thought to be brought up as a Zoroastrian but later an orthodox Muslim

The modern term **algebra** is derived from the term **al-gabr**, a term used by al-Khwarizmi in his calculations

Linear Equations

Linear equations were solved thousands of years ago, but the formality and generality of solving them was devised much later.

The idea is that a given equation in one variable can eventually be solved by inspection.

Example: $2x = 6$ and $4 + x = 9$ had been solved many times before al-Khwarizmi came up with the forms $ax = b$ and $a + x = b$ and their solutions.

Note also that "$=$" hadn't even been invented, so many equations were written in prose: *Twice the unknown quantity yields a value of six* could be a modern translation.

Babylonian Clay Tablet

Quadratic Equations

In a similar way, quadratic equations were solved as early as 2000 BC but the quadratic formula was later derived partly by al-Khwarizmi millenia later.

The full solution to a quadratic equation was first presented by the Jewish mathematician Abraham bar Hiyya Ha-Nasi c. 1100 AD.

There are many different derivations of the quadratic formula...

Symmetric Group

The **Symmetric Group of order** n is the set of all the different ways to permute n elements.

An example: say we have three elements, let's label them $1, 2, 3$. Then the symmetric group of order 3, written S_3 is the set of permutations $\{(123), (132), (213), (231), (312), (321)\}$.

Exercise: Find the symmetric group of order 2 and order 4.

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Exercise: Find the symmetric group of order 2 and order 4.

Solution: Let $1, 2$ be the two elements. Then $S_2 = \{(12), (21)\}$.

Similarly, $S_4 =$
$\{(1234), (1243), (1324), (1342), (1423), (1432), (2134), (2143), (2314),$
$(2341), (2413), (2431), (3124), (3142), (3214), (3241), (3412), (3421), (4123),$
$(4132), (4213), (4231), (4312), (4321)\}$.

Induction

Exercise: Come up with an idea about the size of S_n. Think of a way to explain or "defend" your idea.

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Solution: The size of S_n is $n!$. An idea to convince yourself of this is something called *Induction.*

We know that there are 2 permutations of two elements, 6 of three elements and 24 of four elements.

Say we take $1, 2, 3, 4, 5$ and begin to arrange them, by listing first all the permutations that begin with 1.

The possibilities for these look like: $(1 * * * *)$ where $*$ is any of $2, 3, 4$ or 5.

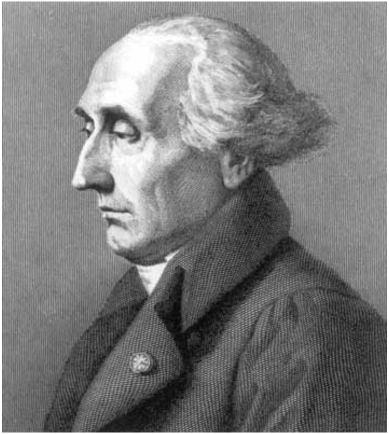
Induction

Thus we can turn to our knowledge of S_4 to determine that there are exactly 24 of these.
Should we change the first element to 2, there will again be 24 permutations of this type.
We see then that there are 5 different sets of 24 permutations, giving us $5 \cdot 24 = 5!$ permutations in all.

The same logic can now be used to show S_6 has 6! elements, and so on.
The use of a previous case to prove all subsequent cases when counting numbers are involved is called the **Principle of Mathematical Induction.**

Part of the Group

These sets S_n are called **groups.** Groups had many inventors. It is thought that the earliest group theorist was Joseph-Lois Lagrange, born in 1736:



Lagrange

Lagrange started out his mathematical career alone, as a self-taught academic. At first, it was hard for him to get noticed.

Calculus was in its infancy at this time, and it was Lagrange's work in this area that got him noticed by Euler. In 1755, Lagrange landed tenure! He was only 19 years old.

We might think of Lagrange as an applied mathematician, because he did math with application to physics, in particular fluid dynamics and cosmology.

His bride, Vittoria Conti, was also his cousin. This was common back then.

There is a lunar crater named after him.

Lagrange once said: "If I had not inherited a fortune I should probably not have cast my lot with mathematics."

Group Theory

A **group** is a set with the following 3 properties:

1. You can combine any two elements in the set *somehow* to get a third element.
2. There is just 1 element that doesn't change anything when combined with any other element. It's called an *identity.*
3. For every element, there is another that combines with it to make the identity. It's called an *inverse.*

Exercise: How can we translate these 3 properties to sound more mathy? Hint: Call your group G and your elements a, b and if needed c; call your identity e and your inverse for a can be called a^{-1} if you want. You can combine elements using any symbol you want.

Group Theory

Solution: Let's call our combining operation $*$. We could also use $\cdot$ like multiplication or $+$ like adding, because these are ways of combining elements.

1. For all $a, b \in G$ we have $a * b = c$ where $c \in G$.
2. There exists one $e \in G$ such that $a * e = a$ for all $a \in G$.
3. For all $a \in G$ there exists one a^{-1} such that $a * a^{-1} = e$.

Exercise: Prove that the integers $\mathbf{Z}$ are a group if combined by addition/subtraction. What is the identity? What are the inverses?

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Solution: Let's go through the 3 requirements:

1. Let a and b be an integer. Then it is easy to see that $a + b$ is another integer.
2. Consider 0. If $a \in \mathbf{Z}$ then $a + 0 = a$.
3. Given $a \in \mathbf{Z}$, consider $-a$. Then $a + -a = 0$.

The Point Being?

Groups are very useful. They are sets with just the right amount of structure (properties), and they pop up in everything:

Solving polynomials

Modern geometry, called topology

Cryptography

Quantum unification theory

Computer techniques such as pattern recognition

Classification of crystal structures

Spectroscopy

Music theory

They even helped mathematicians make progress on the famous Fermat's Last Theorem.

Symmetric Polynomials

A **Symmetric Polynomial** is a polynomial for which you can move the variables about and end up with the same polynomial. "The polynomial is invariant under any permutation of the variables," is one way to say the definition.

An example of a symmetric polynomial in 3 variables is $P(x, y, z) = ax + ay + az$ for any coefficients $a \in \mathbf{R}$.

Exercise: Name two or three of the symmetric polynomials in 2 variables. Can you decide how many there are?

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Solution:
$ax + ay$ for any $a \in \mathbf{R}$
axy for any $a \in \mathbf{R}$
or $ax^2y^2 + x^3y + xy^3 + (x + y)^4$

There are an infinte amount of these. Consider the set $\sum_{i=1}^{n} x^i y^i$. These polynomials are symmetric because multiplication is commutative.

Elementary Symmetric Polynomials

An **Elementary Symmetric Polynomial** is a type of symmetric polynomial that can be used to generate *all* symmetric polynomials of its kind. Written formally:

Let $P(x, y)$ be a symmetric polynomial (in two variables). Then $P(x, y) = P(\sigma_1(x, y), \sigma_2(x, y))$, where σ_1 and σ_2 are elementary symmetric polynomials.

That is, we can make symmetric polynomials in x and y by using the elementary symmetric polynomials as variables.

"Elementary" Example

Exercise: Given $P(x, y) = x^3 + y^3 - 4$, use the symmetric polynomials $\sigma_1 = x + y$ and $\sigma_2 = xy$ to express P as a function of σ_1 and σ_2. Think about how many **elementary** symmetric polynomials there are in two variables.

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Solution - Part 1: Here we use the fact that
$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
This gives us that $x^3 + y^3 = (x + y)^3 - 3x^2y - 3xy^2$
$= \sigma_1^3 - 3(x^2y + xy^2)$
$= \sigma_1^3 - 3(x + y)xy$
$= \sigma_1^3 - 3\sigma_1\sigma_2$.
Thus $P(x, y) = x^3 + y^3 - 4 = \sigma_1^3 - 3\sigma_1\sigma_2 - 4$, in terms of elementary symmetric polynomials.