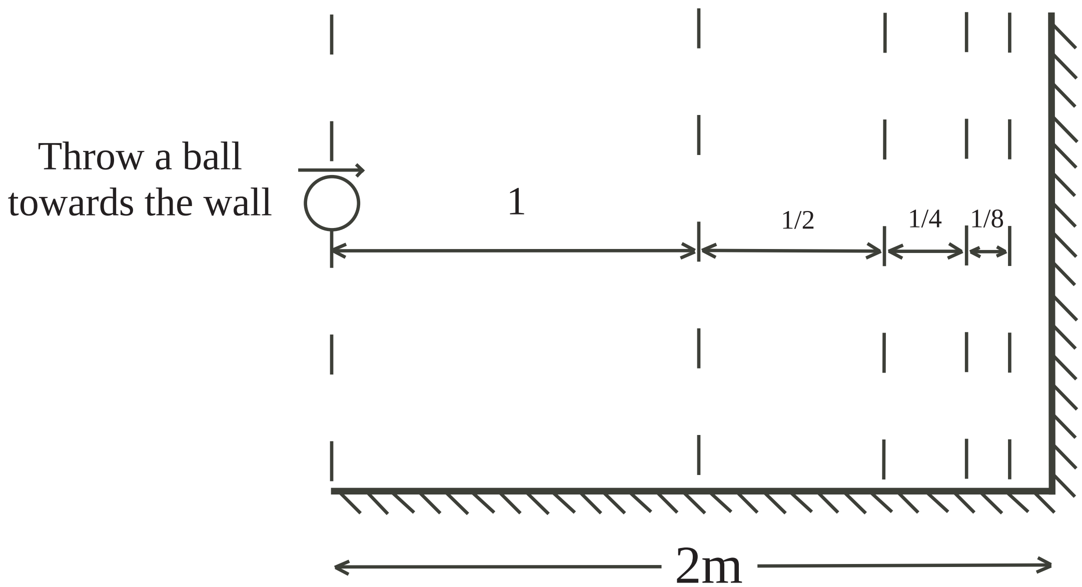
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Senior Math Circles
October 7, 2009
Infinite Series I

Zeno's Paradox (one version)



Zeno's arguement: the ball will never reach the wall, because there is always more distance to cover (half the remaining distance) before reaching the wall.

This seems logical, but is clearly not true!

Distance $= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$, an infinite series.

Variations of the paradox:

- Achilles and the tortoise
- Dichotomy
- *Time* taken (not distance)

Sigma notation (review, hopefully)

Examples:

$$\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^3 + 4^4 + 5^5$$

$$\sum_{i=2}^{4} \frac{i}{i+1} = \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1}$$
$$= \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$$

$$\sum_{n=0}^{N} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^N}$$

$$\sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_N$$

Exercise: Write $\frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \frac{6}{64} + \frac{7}{128}$ in $\sum$ notation.

Infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \ldots$$

Recall that

$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$
$\ldots$
$S_n = a_1 + a_2 + \cdots + a_n$

We say the series **converges** if the partial sum approaches a number as $n \to \infty$; otherwise, we say the series **diverges**.

Note: You'll learn more precise definitions of these terms in calculus class. Let's be like the engineers and learn by example.

Example 1: $\displaystyle\sum_{n=1}^{\infty} 10^{-n} = 0.1 + 0.01 + 0.001 + 0.0001 + \ldots$

$S_2 = 0.11, \quad S_3 = 0.111, \quad S_4 = 0.1111, \quad$ etc.

The series converges to $0.\overline{1} = \dfrac{1}{9}$.

Example 2: $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \ldots$

$S_2 = \ 1.17, \quad S_3 = \ 1.92, \quad S_4 = \ 2.72, \quad S_5 = 3.55$

The partial sum keeps increasing by approximately 1 each time; eventually the series looks like $\ldots 1 + 1 + 1 + 1 + \ldots$ which adds to infinity (not a number). So, the series *diverges*.

Example 3: $\displaystyle\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$

$S_2 = 0, \quad S_3 = 1, \quad S_4 = 0, \quad$ etc.

The partial sums do not approach a number (0, 1, 0, 1, 0, 1, . . .), so the series *diverges*

<u>Comment</u>:
Writing the series as $(1-1) + (1-1) + (1-1) + \cdots = 0 + 0 + 0 + \cdots = 0$ is *meaningless*, since we could write the series also as $1 + (-1+1) + (-1+1) + \cdots = 1$ or even $\frac{1}{2} + (\frac{1}{2} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{2}) + \cdots = \frac{1}{2}$. We need convergence in order to find an infinite sum.

Question:

Does $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ converge?

How about $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$? (covered next week)

Geometric Series

Recall $\displaystyle\sum_{n=0}^{N} ar^n = a + ar + ar^2 + ar^3 + \cdots + ar^N = \frac{a(1-r^{N+1})}{1-r}$ from high school.

Example: $\displaystyle\sum_{n=0}^{4} 2\left(\frac{1}{10}\right)^n = 2 + \frac{2}{10} + \frac{2}{10^2} + \frac{2}{10^3} + \frac{2}{10^4} = \frac{2(1-(\frac{1}{10})^5)}{1-\frac{1}{10}} = 2.2222$

Question:

What about $\displaystyle\sum_{n=0}^{100} 2\left(\frac{1}{10}\right)^n$?

Answer:

$$\sum_{n=0}^{100} 2\left(\frac{1}{10}\right)^n = \frac{2(1-(\frac{1}{10})^{101})}{1-\frac{1}{10}} \approx \frac{2}{(\frac{9}{10})} = \frac{20}{9}$$

Since $(\frac{1}{10})^{101}$ is so small (so small that a calculator gives the value 0), we can estimate the value as above.

Infinite Geometric Series:

$$\sum_{n=0}^{\infty} ar^n \begin{cases} = \dfrac{a}{1-r} & \text{if} |r| < 1 \\ diverges & \text{if} |r| \geq 1 \end{cases}$$

Exercises:

(i) Find $\displaystyle\sum_{n=0}^{\infty} \frac{2}{3^n}$

(ii) Show $\displaystyle\sum_{n=1}^{\infty} 10^{-n} = \frac{1}{9}$

Back to Zeno's paradox:

Distance to wall
$= 1 + \frac{1}{2} + \frac{1}{4} + \cdots = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2\text{m}$, as expected!

Aside: The time taken, if moving with velocity v:

$$t = \frac{1m}{v\frac{m}{s}} + \frac{\frac{1}{2}m}{v\frac{m}{s}} + \frac{1}{4v} + \frac{1}{8v} + \ldots \qquad \textit{seconds}$$

$$= \sum_{n=0}^{\infty} \frac{1}{v}\left(\frac{1}{2}\right)^n \qquad \textit{seconds}$$

$$= \frac{\frac{1}{v}}{1-\frac{1}{2}} \qquad \textit{seconds}$$

$$= \frac{2}{v} \qquad \textit{seconds}$$

eg. If $v = 1\frac{m}{s}$, then $t = 2$ seconds

Paradox solved: There are infinitely many steps, but they become infinitely small! (and there is no "pause" between steps, of course.)

Problem Set

1. Find the sum, or state that the series diverges and why.

   a.) $\sum_{n=0}^{\infty} \frac{3^n}{4^n}$

   b.) $\sum_{n=1}^{\infty} n^2$

   c.) $\sum_{n=2}^{\infty} \frac{e^n}{\pi^{n+1}}$

   d.) $\sum_{n=1}^{\infty} 2\cos(\pi n)$

   e.) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ (A "telescoping series")

2. Without using a calculator, estimate(to the nearest 10) how many "steps" the ball must take in the Zeno example to get within one nanometer(10^{-9}m) of the wall.
   (Hint: $\log_2 10 \approx 3.3$)

3. **The Overhang Problem**
<u>Note:</u> This problem will be taken up and continued next week.

Consider the problem of balancing two pieces of wood, each 2m long, on the edge of a table, so that they have maximum overhang. This happens when the centre of mass of the two boards is above the edge of the table, and the centre of mass of the top board is over the edge of the bottom board.

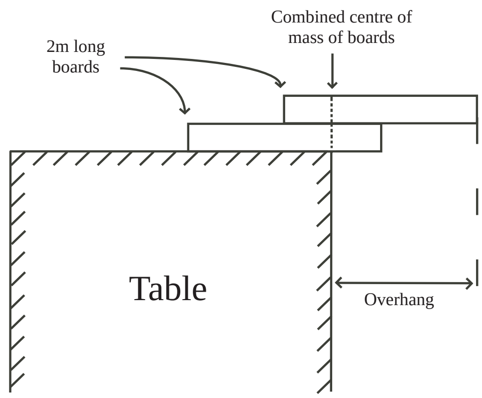


Figure 1.

a.) Find the "overhang", i.e. the horizontal distance from the edge of the topmost piece to the table.

***b.)** Contunuing this process using more boards how many boards are required so that the top board is completely to the right of the table?(i.e. overhang is 2m)

***c.)** To think about for next week: what do you think is the maximum possible overhang if you have infinitely many boards?