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Intermediate Math Circles
November 24, 2010
Equations and Inequalities with Two Variables

Linear Equations in two variables

A linear equation, such as $x + y = 3$ represents the set of points lying on a straight line.

To construct the graph of a line, we can find any two points that satisfy the equation and then draw the line through the two points.

If we graph two lines, there are 3 possibilities for their intersections:

1. The two lines could intersect at exactly one point
   (for example, $x + y = 3$ and $3x - y = 6$ intersect at a point)
2. The two lines could intersect at 0 points
   (for example, $x + y = 3$ and $x + y = 4$)
3. The two lines could intersect at infinitely many points
   (for example, $x + y = 3$ and $2x + 2y = 6$)

How do we find the point of intersection?

Linear Inequalities in two variables

Now consider a linear inequality, such as $x + y \leq 3$.

First consider the line $x + y = 3$. It divides the xy-plane into three regions - points that lie on the line, points that lie below the line, and points that lie above the line.

To graph an inequality, start by graphing the corresponding line. Then choose a test point that is not on the line to decide which region corresponds to points that satisfy the inequality.

In problems involving a system of inequalities, the solution of the system is the set of points that satisfy **all** the inequalities simultaneously.

Linear Programming

A linear programming problem is a problem of maximizing or minimizing some expression (called the **objective function**) subject to **limits** or **constraints**.

Consider the following problem:

A company carries two types of cell phones. The company makes a profit of $75 on each sale of a cell phone Model A, and $100 on each sale of a cell phone Model B. On each order, the company must purchase at least 50 cell phones, but no more than 80. The company also finds that sales of Model A are at least 4 times more than sales of Model B. How many cell phones of each model should the company order to maximize revenue?

Steps for solving Linear Programming Problems:

1. Sketch the feasible region.
2. Find all "corner points" of the feasible region.
3. Substitute all corner points into the objective function.
4. Select the largest (or smallest) of these values as the maximum (or minimum).

Problem Set 1

1. Sketch the inequality $x + 2y < 4$

2. Sketch the inequality $5x + 2y \geq -10$

3. John and Mary wrote a Math test. Two times John's score was 60 more than Mary's score. Two times Mary's score was 90 more than John's score. Determine their two scores.

4. Sketch the equations $4x - y = 8$ and $2x - 3y = 2$ on the same graph. Find the point of intersection of the two lines.

5. Sketch the points that satisfy:
   $3x + y \geq 12$
   $x - y \leq 4$
   $2x - 3y \leq 6$

6. Sketch the points that satisfy:
   $2x + y > 4$
   $x \geq 2$
   $x + y < 5$

7. A company makes cars and trucks. In any given week, a total of up to 400 vehicles can be made. Draw a graph showing the number of cars and trucks that could be made in one week.

8. Michael plans to spend up to 12 hours reviewing Science and Math in preparation for examinations. Michael is not as good in Science as he is in Math, so he wants to study at least two times more for science then he does for math. Draw a graph showing how much time she could spend studying each subject.

9. Find the area of the triangle enclosed by the lines $-4x+3y = 5$, $3x+2y = 43$ and $-5x+8y = 19$.

10. If the system of equations

$$\begin{aligned} px + qy &= 8 \\ 3x - qy &= 38 \end{aligned}$$

    has the solution $(x, y) = (2, -4)$. Determine p and q.

Problem Set 2

1. Maximize:

$$P = 3x + 4y$$

Subject to:

$$\begin{gathered} 2x + y \leq 6 \\ x + y \leq 4 \\ x \geq 0 \\ y \geq 0 \end{gathered}$$

2. Minimize:

$$C = 2x - 3y$$

Subject to:

$$\begin{gathered} 4x + 5y \leq 40 \\ 2x - y \geq 0 \\ x \leq 6 \\ x \geq 0 \\ y \geq 0 \end{gathered}$$

3. A tailor has 80 square metres of cotton material and 120 square metres of wool. A suit requires 1 square metre of cotton and 3 square metres of wool. A dress requires 2 square metre of cotton and 1 square metre of wool. How many of each should the tailor make to maximize revenue, if a suit sells for \$110 and a dress sells for \$80?

4. A company makes two types of calculators, Calculator A and Calculator B. Each calculator must be tested after it is assembled. The amount of time required for assembling Calculator A is 4 hours and the amount of time required for assembling Calculator B is also 4 hours. The amount of time for testing Calculator A is 2.5 hours, and the amount of time for testing Calculator B is 1.5 hours. Each week there are 104 working hours for assembling and 60 working hours for testing. If the company makes a profit of \$4 on each Calculator A and \$2.50 on each Calculator B, how many of each should it produce to maximize its weekly profits?

5. Minimize:

$$C = x - 4y$$

Subject to:

$$\begin{gathered} 2x + 3y \geq 6 \\ x \leq 8 \\ y \leq 12 \\ x \geq 0 \\ y \geq 0 \end{gathered}$$