



University of Waterloo
Faculty of Mathematics



Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

Fall 2010

Exponents and Binary Numbers

Powers

What is the product of three 2s?

$$2 \times 2 \times 2 =$$

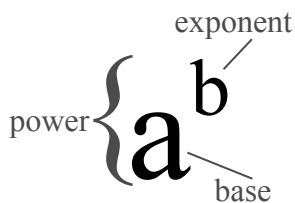
What is the product of five 2s?

$$2 \times 2 \times 2 \times 2 \times 2 =$$

What is the product of twelve 2s?

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 =$$

An easier way to write this is as the power 2^{12} .



The **base** is the number that you are multiplying over and over again, and the **exponent** is the number of times that you are multiplying the base. The entire expression is called a **power**.

What is the product of fifty 2s? How long would it take you to multiply it out?
Can you write it as a power instead?

Exercise 1. Write the following expressions as powers.

a) $2 \times 2 \times 2 \times 2 \times 2 =$

b) $4 \times 4 \times 4 =$

c) $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 =$

d) $5 =$

Excercise 2. Complete the following chart.

100 000	$10 \times 10 \times 10 \times 10 \times 10$	10^5
10 000	$10 \times 10 \times 10 \times 10$	10^4
1000		
100		
10		
1		
$\frac{1}{10}$		
$\frac{1}{100}$		

Summary:

$$a^0 = 1$$

$$\text{eg. } 2^0 = 1 \quad 3^0 = 1 \quad 10^0 = 1$$

$$a^{-b} = \frac{1}{a^b}$$

$$\text{eg. } 2^{-4} = \frac{1}{2^4} \quad 3^{-2} = \frac{1}{3^2} \quad 10^{-3} = \frac{1}{10^3}$$

Base-10 System

A number can be expressed as a sum of each digit times a power of 10.

$$100 = 10^2$$

$$200 = 2 \times 10^2$$

$$540 = (5 \times 10^2) + (4 \times 10^1)$$

$$407 = (4 \times 10^2) + (7 \times 10^0)$$

Exercise 3. Write each number as an expression as in the example above.

a) $50 =$

b) $901 =$

c) $40320 =$

Exercise 4. Write each expression as a number.

a) $(5 \times 10^2) + (7 \times 10^1) + (1 \times 10^0) =$

b) $(2 \times 10^3) + (2 \times 10^2) =$

c) $(3 \times 10^5) + (1 \times 10^3) + (8 \times 10^2) + (3 \times 10^0) =$

Did you know...

that our base-10 number system, also known as the decimal system, is only one of many number systems? The Babylonians used the sexagesimal (base-60) system, the Romans used the duodecimal (base-12) system, and the Mayans used the vegesimal (base-20) system. Our computers today store information using the **binary number system**, which is based on powers of 2.

The Binary Number System (Base-2)

Write out the powers of 2.

$$2^0 = 1$$

$$2^1 = 2$$

$$2^2 = 4$$

$$2^3 = 8$$

$$2^4 = 16$$

$$2^5 = 32$$

Note: Numbers in base-10 use ten different digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. It then makes sense that numbers in base-2 use only two different digits, 0 and 1.

Converting binary numbers to base-10.

The binary number 1101 can be represented as $(1 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$, which equals $8 + 4 + 1 = 13$. So changing 1101 from base-2 to base-10 would give us 13. Similarly, changing 13 from base-10 to base-2 would give us 1101.

Exercise 5. Convert the following binary numbers to numbers in base-10.

Base-2	$2^5 = 32$	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$	Base-10
1101	0	0	1	1	0	1	13
111000							
1111							
10101							
110010							
1							

Converting numbers in base-10 to binary numbers.

To go backwards from base-10 to binary requires more thought, but it is not difficult to do. Think about how you would convert 13 back into binary form. You would first have to change it into an expression of the form

$$(a \times 2^4) + (b \times 2^3) + (c \times 2^2) + (d \times 2^1) + (e \times 2^0)$$

where a , b , c , d and e are either 0 or 1.

To do this, we need to find the largest power of 2 that is less than 13. 2^4 is too large, so we can put a 0 in that column. 2^3 is less than 13 and can divide into 13 one time, so we put a 1 in that column. Next, we subtract 2^3 from 13 and are left with 5. Again, we need to find the largest power of 2 that is less than 5, which is 2^2 . It divides into 5 one time. Subtracting 2^2 from 5 leaves us with 1, which is 2^0 . This gives us $13 = (1 \times 2^3) + (1 \times 2^2) + (1 \times 2^0)$, which is 1101 in binary form.

Exercise 6. Convert the following base-10 numbers to binary numbers.

Base-10	$2^5 = 32$	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$	Base-2
13	0	0	1	1	0	1	1101
31							
8							
35							
22							
1							

Binary Square Trick

1. Get a friend to choose a number from one of the squares.
2. Ask him/her to tell you which squares have their number in them.
3. Add up the first number in each square that contains your friend's number.
4. Your sum is the number!

A

1	3	5	7
9	11	13	15
17	19	21	23
25	27	29	31

B

2	3	6	7
10	11	14	15
18	19	22	23
26	27	30	31

C

4	5	6	7
12	13	14	15
20	21	22	23
28	29	30	31

D

8	9	10	11
12	13	14	15
24	25	26	27
28	29	30	31

E

16	17	18	19
20	21	22	23
24	25	26	27
28	29	30	31

Exercises

- Write $(2 \times 10^7) + (4 \times 10^5) + (4 \times 10^4) + (3 \times 10^1) + (9 \times 10^0)$ as a number in base-10.
- Write $(1 \times 2^6) + (1 \times 2^4) + (1 \times 2^3) + (1 \times 2^0)$ as a number in binary form.
- Convert the binary number 10011011 to a number in base-10.
- Convert 507 to binary form. Convert 49 381 to binary form.
- How many digits are needed for base-3 numbers? Base-7? Base-85? (Hint: See the earlier note)
- Write $(6 \times 7^3) + (5 \times 7^2) + (1 \times 7^1) + (3 \times 7^0)$ as a number in base-7.
- Write $(2 \times 3^4) + (1 \times 3^3) + (1 \times 3^2) + (2 \times 3^0)$ as a number in base-3. Write it as a number in base-10.
- Write 363 as a number in base-5.
- Convert the base-5 number, 43012, to a number in base-10.
- Why do you think we use the base-10 number system? Think about how we learn how to count.
- Can we have a number system based on powers of 1? Why or why not?
- The number 0 is written the same way in every number system. What about the number 1? How is 1 written as in binary form? In base-3? In base-7? In base-85?
- Do the following binary calculations (you may want to convert them to base-10 to check your answers):

a) $0 + 0 =$

b) $1 + 0 =$

c) $0 + 1 =$

d) $1 + 1 =$

Can you develop rules for adding in binary?

- Try the following binary calculations:

$$\begin{array}{r} 10 \\ +1 \\ \hline \end{array}$$

$$\begin{array}{r} 1001 \\ +110 \\ \hline \end{array}$$

$$\begin{array}{r} 10 \\ +10 \\ \hline \end{array}$$

$$\begin{array}{r} 110 \\ +10 \\ \hline \end{array}$$

$$\begin{array}{r} 101 \\ +11 \\ \hline \end{array}$$

- What does the binary equation $1 + 1 + 1 = ?$

16. Use your result from 15 to do the following binary calculation.

$$\begin{array}{r} 111 \\ +1011 \\ \hline \end{array}$$

17. Linda is converting a base-10 number to binary form and ends up with the number 100210. She knows that this can't be right because binary numbers are only made up of ones and zeroes. What should Linda's number actually be?

Solutions:

1. 20440039

2. 1011001

3. $2^7 + 2^4 + 2^3 + 2^1 + 2^0 = 155$

4. $507 - 2^8 = 251$ $49281 - 2^{15} = 16\ 613$

$251 - 2^7 = 123$ $16613 - 2^{14} = 229$

$123 - 2^6 = 59$ $229 - 2^7 = 101$

$59 - 2^5 = 27$ $101 - 2^6 = 37$

$27 - 2^4 = 11$ $37 - 2^5 = 5$

$11 - 2^3 = 3$ $5 - 2^2 = 1$

$3 - 2^1 = 1$ $1 - 2^0 = 0$

$1 - 2^0 = 0$

$\Rightarrow 111111011$

$\Rightarrow 1100000011100101$

5. 3; 7; 85

6. 6513

7. 21102; 200

8. $363 - 2 \times 5^3 = 113$

$113 - 4 \times 5^2 = 13$

$13 - 2 \times 5^1 = 3$

$3 - 3 \times 5^0 = 0$

Therefore 383 in base-5 is 2423

9. $(4 \times 5^4) + (3 \times 5^3) + (1 \times 5^1) + (2 \times 5^0) = 2882$

10. We have 10 fingers.

11. Any power of 1 is equal to 1.

12. It is always 1 in any base.

13. a) 0 b) 1 c) 1 d) 10

14. 11; 1111; 100; 1000; 1000

15. 11

$$\begin{array}{r} 16. \quad \quad 11 \\ \quad \quad 1111 \\ \quad \quad \underline{+1011} \\ \quad \quad 10010 \end{array}$$

$$17. \quad 101010$$