UNIVERSITY OF WATERLOO
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles
Fall 2010
Game Theory

Opening Question

Calvin and Hobbes are sent to the office for starting a big snowball fight. The principal sits them down in **separate rooms** and tells them each the same thing:

- If they both confess to starting the snowball fight, they will each be given a 1-hour detention
- If neither of them confess, they will each be given a 3-hour detention
- If one of them confesses and the other doesn't, the one who confesses will be given a full day (7-hour) detention, and the other will be given a warning (no detenion).

If you were Calvin or Hobbes, would you confess or not confess? Why or why not?

Calvin's Point of View

Hobbes \ Calvin	Confess	Don'tconfess
Confess	1	0
Don't confess	7	3

Hobbes's Point of View

Hobbes \ Calvin	Confess	Don'tconfess
Confess	1	7
Don't confess	0	3

Both

Hobbes \ Calvin	Confess	Don'tconfess
Confess	1,1	0,7
Don't confess	7,0	3,3

If Calvin and Hobbes were allowed to talk about their decision together, then what would be the best thing to do?

Exercise A. **Bake Sale**
Jenny and Max are making treats for a bake sale to help raise money for new books for the school library. They are each having a hard time deciding whether to make cookies or brownies, since the person who sells the most treats will receive a prize. Cookies are more popular than brownies, but if they both bake cookies, a customer will likely only buy one cookie from either Jenny or Max. If one person brings cookies and the other person brings brownies, the customer might buy one of each type of treat:

- If Max and Jenny both bring chocolate chip cookies, then Max will sell 35 cookies and Jenny will sell 45 cookies.
- If Max and Jenny both bring brownies, then Max will sell 20 brownies and Jenny will sell 30 brownies.
- If Max brings cookies and Jenny brings brownies, then Max will sell 60 cookies and Jenny will sell 40 brownies.
- If Max brings brownies and Jenny brings cookies, then Max will sell 30 brownies and Jenny will sell 70 cookies.

Use the information given to fill in the following charts:

Max's Point of View

Max \ Jenny	Cookies	Brownies
Cookies	35	60
Brownies	30	20

Jenny's Point ofView

Max \ Jenny	Cookies	Brownies
Cookies	45	40
Brownies	70	30

Which treat will Jenny choose to make in order to win the prize? Which treat will Max choose to make in order to win the prize?

In the following chart, record the total number of treats sold given Max and Jenny's choices:

TotalTreats Sold

Max \ Jenny	Cookies	Brownies
Cookies	80	100
Brownies	100	50

If Jenny and Max both try to win the prize, how many treats will be sold in total? How can Max and Jenny work together in order to sell the maximum number of treats for the school library?

Game Theory is the study of how people make decisions and the outcomes of those decisions.

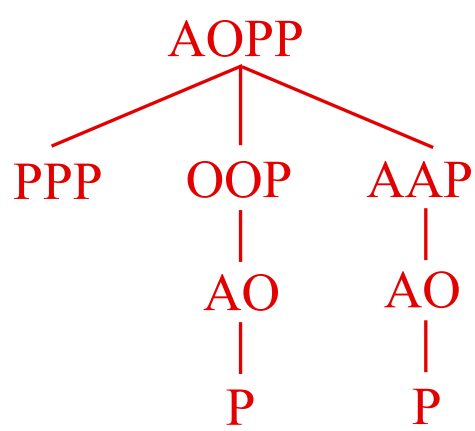
A **game** is a structured activity where one or more **players** take turns completing a set of actions that follow the rules of the game. A single action is called a **move** and the complete set of moves that a player does throughout the game is called a **strategy**.

A **winning strategy** is a strategy that guarantees a win.

Fruits in a Basket

There is 1 apple, 1 orange, and 2 peaches in a basket. Two fruits of a *different* type are selected at random to be taken out of the basket and are replaced with one of the third type of fruit. This continues until there is a single fruit in the basket or there is only one type of fruit in the basket. Which type of fruit is left in the basket?

We can list all of the possible outcomes by constructing a **game tree**.

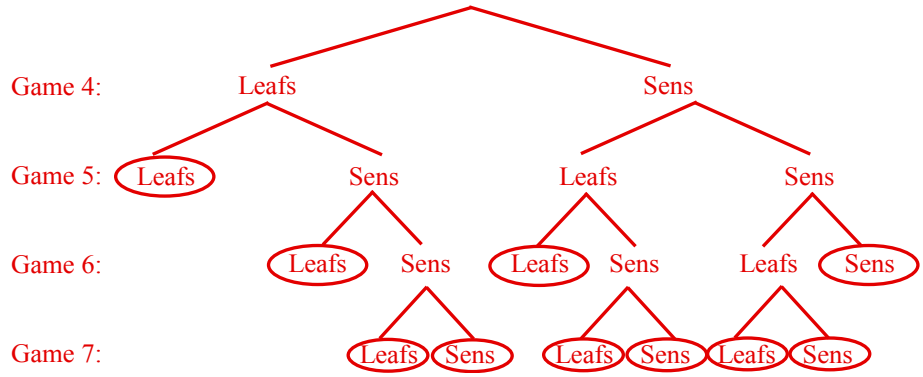


Exercise B.

1. If there are 2 apples, 1 orange, and 3 peaches in the basket and the same rules from the example apply, which type fruit will be left in the basket? apples

2. In the Stanley Cup playoffs, two teams play a maximum of 7 games against each other and the team that wins 4 games moves on to the next round. After a team has won its 4th game, no more games are played; for example, if the Leafs win 2 games, the Senators win 1 game, and the Leafs win 2 more games for a total of 5 games, then the Leafs will move on to the next round and the 6th and 7th games will not be played.

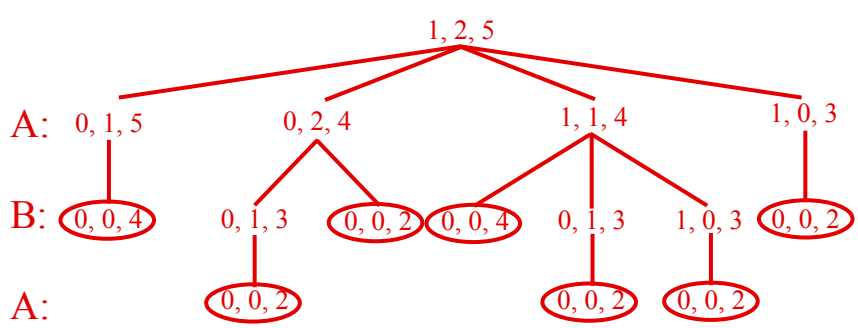
(a) If the Leafs won the first two games against the Senators and the Sens won the third game, construct a game tree to find all of the possible outcomes of the rest of the round.



(b) If each of the outcomes listed in part (a) are equally likely to occur, what is the probability that the Senators will win the round? (Hint: the probability that an event will occur is the number of ways that the event can occur divided by the total number of outcomes.) $\frac{4}{10}$ or $\frac{2}{5}$ or 40%

(c) What is the probability that the Leafs will win the round in six games or less? $\frac{3}{10}$ or 30%

3. Andrew and Bethany are playing a game where they take turns taking marbles out of three jars. Andrew goes first. For each turn, the player must take the same number of marbles out of any two jars (for example, if Bethany takes two marbles out of one jar, she must also take two marbles out of another jar). The person who cannot make any more moves loses (ie. the person who is left with two or more empty jars loses). The first jar has 1 marble in it, the second jar has 2 marbles and the third jar has 5 marbles. Draw a game tree to find all of the possible outcomes. If each of the outcomes you found has the same chance of occurring, then who is more likely to win? Bethany



Twenty-One

In the game twenty-one, you and a partner take turns subtracting numbers from 21 until one of you reaches zero. Each player can only subtract a number from 1 to 4 during each turn, and the person who reaches zero is the winner.

Play this game twice with a partner; once when you go first, and once when your partner goes first. Record your numbers in the charts below:

Game 1:

You:	
Partner:	
You:	
Partner:	
You:	
Partner:	
You:	
Partner:	
You:	
Partner:	
You:	
Partner:	
You:	
Partner:	
You:	
Partner:	
You:	
Partner:	
You:	

Game 2:

Partner:	
You:	
Partner:	
You:	
Partner:	
You:	
Partner:	
You:	
Partner:	
You:	
Partner:	
You:	
Partner:	
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Partner:	
You:	
Partner:	
You:	
Partner:	

If you are the first player to go, there is a way that you can win every time the game is played. What is the winning strategy?

1. Subtract 1 first.
2. Subtract a number such you and your partner (combined) subtract a total of 5.

Exercise C.

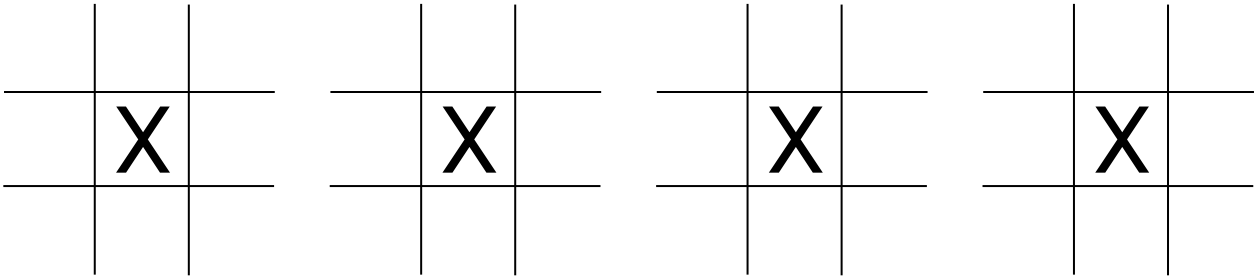
1. (a) What is the winning strategy if the game starts at 24 instead of 21? Subtract 4 (instead of 1) first.

   (b) If the game is played starting at 20, who has a winning strategy and what is it? The person who goes 2nd has the winning strategy: subtract a number such you and your partner (combined) subtract a total of 5. .

   (c) If the you were allowed to only subtract 1 or 2 each time, starting from 21, who has a winning strategy and what is it? The person who goes 2nd has the winning strategy: subtract a total of 3 instead of 5.

2. In the game tic-tac-toe, each player takes a turn to place an X or an O in one of nine spots. The player that forms a straight line of three X's or a straight line of three O's is the winner. The first player usually starts by placing an X in the middle spot on the grid.

   What are all of the places that player two can go next in order for player one to have a winning strategy? Any of the remaining spaces that is not a corner space.



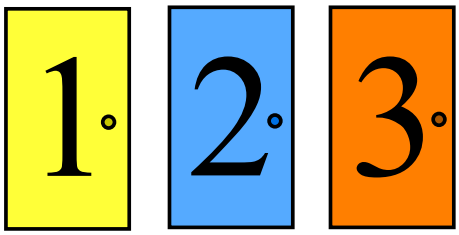
3. Linlin and Harrison are playing a game starting with some coins arranged in piles. Linlin always goes first, and the two players take turns removing one or more coins from any one pile. The player who takes the last coin wins.

   (a) If there are two piles of coins with 3 coins in each pile, show that Harrison can guarantee that he always wins the game. What is his winning strategy? Take the same amount of coins that Linlin takes, but from the other pile.

   (b) If the game starts with three piles of 1, 2 and 3 coins, explain how Harrison can guarantee that he always wins the game. Take coins such that there are only two piles left with the same number of coins, then use the same strategy from (a).

Let's Make a Deal!

In the old game show *Let's Make a Deal!*, contestants are asked to choose between 3 doors. Behind one of the doors is a brand new car, but behind each of the other two doors is a donkey. Once the contestant has chosen a door, one of the other two doors is opened, revealing a donkey. The contestant is then given a choice to keep the door that he or she has chosen, or switch to the other remaining door. What should the contestant do?



Play this game with a partner. Repeat the game 3 times with you as the contestant (record your results on this sheet), and 3 times with your partner as the contestant (your partner will record his or her results on his or her sheet). Make sure you and your partner choose a different strategy that you will stick with for all 3 trials (ie. you choose to switch doors everytime and your partner chooses to keep the same door everytime).

My Strategy (circle one): Switch doors Keep the same door

Trial #	Door picked	Door with a donkey	Switch/don't switch	Win/Lose
1				
2				
3				

I won [ ] out of 3 times.

Sample Game:

- Let's say that you chose door 1. Before any of the doors are opened, what is the probability that the car is behind the door you chose? What is the probability that the car is either behind door 2 or door 3?
- Next, door 2 is opened, revealing a donkey. There are two doors left, door 1 and door 3. Does the probability that the car is behind door 1 change?
- What would be the better strategy to use in order to win the car? Should you keep the door that you started with, or switch to the other door?

Note: The strategy that you chose is not a winning strategy since you wouldn't be guaranteed to win every time you played the game; however, it is the best strategy that gives you the highest chances of winning.

Exercise D.

1. The game is played instead with four doors, three donkeys and a car. You choose a door 1 and door 2 is opened, revealing a donkey.

   You are then given a chance to switch to either door 3 or 4. Do you do it?
   Yes; the probably of getting a car changes from $\frac{1}{4}$ to $\frac{3}{8}$.

2. (a) Bob and Larry have two piles of toothpicks. One pile has 21 toothpicks and the other has 23 toothpicks. They take turns choosing a pile and removing 1 to 4 toothpicks from it. The person who takes the last toothpick is the winner. It is Bob's turn. What is his winning strategy?
   Bob takes two toothpicks from the pile with 23 toothpicks to even out the piles. Then he continues to even out the piles by taking the same number of toothpicks that Larry takes from the opposite pile.

   (b) If the piles have 21 and 26 toothpicks and it is Bob's turn, then Larry has a winning strategy. What is Larry's winning strategy?
   If Bob takes toothpicks from the larger pile at any point during the game, then Larry should even the piles out and use the same strategy from (a). If Bob takes toothpicks from the smaller pile, Larry should take the same amount of toothpicks as Bob does, but from the larger pile.