University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles
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Number Theory

Palindromic Numbers are numbers that are read the same way when their digits are reversed.

- For example, 12321, 804408 and 7 are palindromic numbers.
- 48, 201, and 13421 are not palindromic numbers.

Questions:

- Are all one-digit numbers palindromic? How many one-digit palindromic numbers are there?
- How many two-digit palindromic numbers are there?
- How many three-digit palindromic numbers are there?
- How many four-digit palindromic numbers are there?
- How many five-digit palindromic numbers are there?
- How many six-digit palindromic numbers are there?

The Palindromic Numbers Party

You are invited to a party of palindromic numbers. Each number comes up to you and introduces himself or herself:

1. The first palindromic number you meet introduces himself as an odd number. He is the smallest four-digit odd palindromic number. What number is he?
2. Walking through the door, you are greeted by a lady. She is an odd number and the sum of her three individual digits is 2. What number is she?
3. Another lady is enjoying her ice cream in one corner. She introduces herself as an odd number and the product of her two digits is 9. What number is she?

4. A man is about to sing on the stage! Before singing, he introduces himself as an even five-digit number and the product of all his five individual digits is 4. What number is he?

5. You meet with an odd couple. The man reveals that he is 6789 (not a palindrome) and he says his wife is the next largest palindromic number that comes after him. What number is his wife?

6. You see three kids who are wearing the same outfits. The first kid says he is 3333 and the second kid introduces himself as 3443. They say the third kid is a palindromic number between them. Is it possible?

7. You meet a giant on your way to the washroom. He is a seven-digit number. The sum of all of his seven digits is 4. How many different numbers could he be?

8. A lady comes and greets you. She says she is a five-digit palindromic number. The sum and the product of all of her five indivdual digits are both 8. She ends in 2. What number is she?

9. You meet a simple guy. He is the largest one-digit palindromic number. What number is he?

10. The party is about to end. The number 7667 comes and asks for your identity. You reveal that the sum of all your four digits is 21. Well, 7667 seems confused since the sum of the four individual digits of any four-digit palindromic number is always a(n) (odd/even/square/perfect) number.

Perfect Squares are integers whose square roots are also integers.

The first 10 perfect squares are:

$1^2 = 1$	$2^2 = 4$	$3^2 = 9$	$4^2 = 16$	$5^2 = 25$
$6^2 = 36$	$7^2 = 49$	$8^2 = 64$	$9^2 = 81$	$10^2 = 100$

Square the following palindromic numbers:

a) $101 \times 101 =$

b) $111 \times 111 =$

c) $121 \times 121 =$

What do you notice about these perfect squares?

Does the pattern continue?

a) $131 \times 131 =$

b) $141 \times 141 =$

c) $151 \times 151 =$

The square of a palindromic number is **not** necessarily a palindromic number.

Is the square root of a palindromic perfect square always a palindromic number? There are three 3-digit palindromic perfect squares. Calculate their square roots:

a) $\sqrt{121} =$

b) $\sqrt{484} =$

c) $\sqrt{676} =$

The square root of a palindromic perfect square is **not** necessarily a palindromic number.

Prime numbers are integers greater than 1 that are only divisible by themselves and 1.

- Numbers that are not prime are called **composite** numbers.
- The number 1 is neither prime nor composite by definition.
- The number 2 is the only even prime number. Why aren't there others?

Cross out all of the composite numbers to determine which numbers between 1 and 100 are prime:

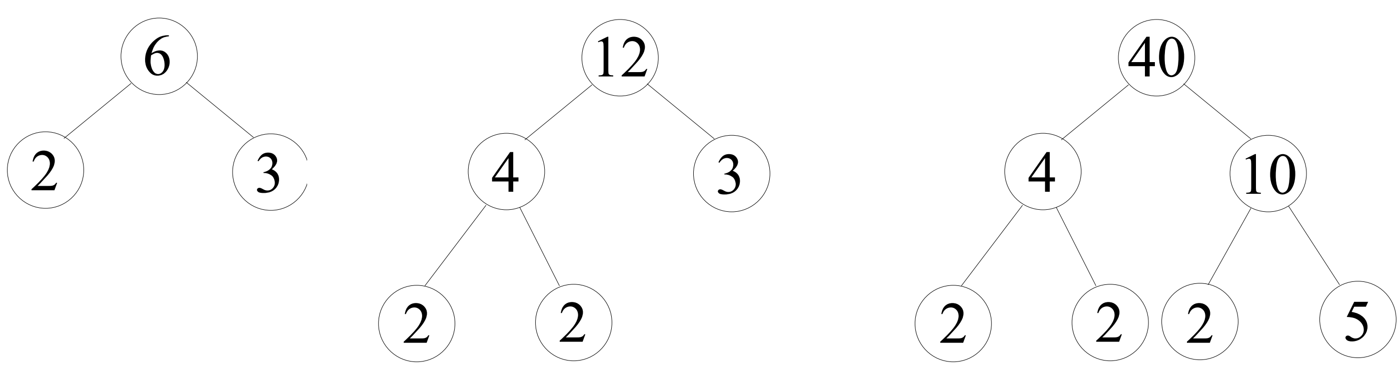
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

How many prime numbers between 1 and 100 are there? How many of those primes are also palindromic numbers? What is the next palimdromic prime number?

Composite numbers can be broken down into their **prime factors**.

- $6 = 2 \times 3$
- $12 = 2 \times 2 \times 3$
- $40 = 2 \times 2 \times 2 \times 5$

An easy way to do this is to construct a **factor tree**.



Exercises

1. How many 7-digit palindromic numbers are there? How many 8-digit palindromic numbers are there?

2. How many 8-digit *non-palindromic* numbers are there?

3. The last palindromic year was 2002. How many years from now (2010) will the next palindromic year occur?

4. Break the following composite numbers down into their prime factors using a factor tree.

   (a) 24

   (b) 80

   (c) 144

5. I am a composite number between 1 and 10. I am not a perfect square, but I have three prime factors. What number am I?

6. a) Paul has forgotten his locker combination which is a 5-digit palindromic number. He knows that the fourth digit is 7. How many combinations does he have to try in order to find the right one?

   b) Paul remembers that his combination is less than 30000. How many combinations does he have to try now?

7. You can create palindromic numbers using the following method:

   - Start with the number 3462 and reverse its digits.
   - Add the two numbers together to get $3462 + 2643 = 6105$
   - Repeat the process with the new number (6105) until you get a palindromic number.
   - You will get $6105 + 5016 = 11121$, then $11121 + 12111 = 23232$, which is a palindromic number.

   Choose a different number and use this method to get another palindromic number. Note: Although this method works with most numbers, it is not certain that it works for all numbers. 196, 887, 1675 and 7436 are some examples of numbers that have yet to produce palindromic numbers using this method.

8. How many palindromic numbers are between 123 and 456?

9. Are there more palindromic numbers or perfect squares under 100? Are there more palindromic numbers or prime numbers under 100?

10. We know that a number can be both a perfect square and palindromic, and both prime and palindromic. Can a number be both a perfect square and prime?

11. Do the following calculations:

    a) $1 \times 1 =$

    b) $11 \times 11 =$

    c) $111 \times 111 =$

    d) $1111 \times 1111 =$

    What do you notice? Do you think the pattern will continue?

    Continue the pattern up to $111\,111\,111 \times 111\,111\,111$. What do you think will happen to the when you square 1 111 111 111?

12. I am a prime number greater than 100 and less than 1000. Each of my digits is different and also prime. I am the smallest prime number such that my hundreds digit is greater than my ones digit. What number am I?

13. I am a two-digit prime number. The sum of my digits is 8. If you add me to 300, I become a palindromic prime number. What number am I?

14. I am a 5-digit palindromic perfect square. Each of my digits is also a perfect square and my last two digits form a palindromic number. My middle digit is the largest of the 5 digits. What number am I?

15. An odd palindromic number has an odd number of digits. Can you tell if the sum of its digits will be odd or even?

16. Jessica is trying to figure out 26^2 without a calculator. She knows that it is a 3-digit palindromic number where the sum of the digits is 19 and the middle digit is one more than the other two. Can you help her out?

17. Beth would like to visit her friend Nathan who is living on the main street of a small town. The main street has 50 houses divided into two blocks numbered from 1 to 20 and 21 to 50. Since Beth has forgotten the number, she asks a passer-by, who replies, "Just try to guess it." Beth likes playing games and asks three questions:

    (a) In which block is it?

(b) Is the number even?

(c) Is it a perfect square?

After Beth has received the answers, she says, "I'm still not sure which house it is, but if you tell me whether the digit 4 is in the number, I will know the answer!" Then she runs to the building in which she thinks Nathan is in. A lady opens the door and it turns out she is wrong. The lady starts laughing and tells Beth, "Your advisor is the biggest liar in the whole town. He never speaks the truth!" Beth thinks for a moment and says, "Thanks, now I know Nathan's real address." What is Nathan's address?

Solutions

1. There are 9 choices for the first digit, and 10 choices for the second, third and forth. Then the fifth digit is the same as the third, the sixth is the same as the second, and the last is the same as the first, so there are 9000 7-digit palindromic numbers; There are also 9000 palidromic 8-digit numbers, since, the only difference is that the fifth digit is the same as the forth, the sixth is the same as the third, the seventh is the same as the second, and the last is the same as the first.

2. There are 9×10^7 8-digit numbers in total, and 9000 8-dgit palidromic numbers, hence there are $9 \times 10^7 - 9000 = 89991000$ non-palidromic numbers.

3. The next palidromic year is 2112, which is in $2112 - 2002 =$102 years

4. (a) $24 = 2$
   (b) $80 = 2 \times 2 \times 2 \times 2 \times 5$
   (c) $144 = 2 \times 2 \times 2 \times 2 \times 3 \times 3$

5. All composite numbers between 1 and 10 are: 4, 6, 8, and 9. 4, 6, and 9 only have 2 prime factors, and $8 = 2 \times 2 \times 2$. Therefore I am 8.

6. (a) The forth digit is 7 which implies the second digit is also 7, there are 9 choices for the first digit (and this sets the last digit), and there are 10 choices for the middle digit. Hence there are $9 \times 10 = 90$ combinations that Paul could try.
   (b) There are 2 choices for the first digit (1 or 2), which sets the last digit, and there are still 10 choices for the middles digit. Hence there are $2 \times 10 = 20$ combinations that Paul could try.

7. Various answers

8. Case 1: first digit is 1:
   There are 7 choices for the middle number $\Rightarrow$ 7 palindromic numbers
   Case 2: first digit is 2 or 3:
   There are 10 choices for the middle number $\Rightarrow$ $2 \times 10 = 20$ palindromic numbers
   Case 3 : first digit is 4:
   There are 6 choices for the middle number $\Rightarrow$ 6 palindromic numbers
   Therefore there is a total of $7 + 6 + 20 = 33$ palidromic numbers between 123 and 456

9. More palindromic numbers than perfect squares since there are 19 palindromic numbers and 9 perfect squares; more prime numbers than palindromic numbers.

10. No since a perfect square has to be a square of a natural number, which gives a factor other than 1 or the number itself, the definition of a prime number.

11. The pattern continues until $1111111111 \times 1111111111$, but stops once another 1 is added to the number.

12. Each of the digits are prime means they can only be 2, 3, 5, or 7. The last digit is small than the first digit. But the last digit cannot be 2, since it would not be prime then. So the last digit must be 3 or 5, and the first digit is 5 or 7. Since we are looking for the smallest prime number, try letting the last digit be 3, and let the first digit be 5, and the middle digit be 2 to get 523, which is prime.

13. The second digit must be 3 since if you add it to 300 you will get a palindromic number. Since the sum of the two digits is eight, this means the first digit must be 5, giving us 53, which is prime.

14. The only choice for the digits is 1, 4, and 9, since the last two digits form a palindromic number, they must be the same, hence the first two and the last two digits are the same. Since the middle digit si the largest of the five digits, possible palindromic numbers are 11411, 11911, and 44944. The only perfect square of these options is 44944. Hence I am 44944

15. You can't tell; it depends whether the middle digit is even or odd.

16. Let the palidromic number be aba, where $b = a + 1$. Since the sum of the digits is 19 we get $a + (a + 1) + a = 19 \Rightarrow a = 6$. Therefore the number is 676.

17. Nathan lives in house 14 or 25.
    We shall break this question down into a table of cases of what the passer-by says. (Table is on next page)
    Referring to the table we can see that the possible addresses for what the passer-by said are 4, 14 16, 36, 49, and 25. But we know he lied about everything he said.
    If Beth thought the address was 4 from what the passer-by said, then the passer-by said it was an even number in block 1-20, and it was a perfect square and 4 is in the number. So this mean it must be an odd number in block 21-50, not a perfect square and must not have a 4 in the number. But this gives more than one answer, so that is not right.

If Beth thought the address was 14, then the passer-by said it was an even number, in block 1-20, not a perfect square and has 4 in the number. So this mean it must be an odd number in block 21-50, a perfect square and not have 4 in the number. This gives 25 as the address. However if we reverse this, we'd get that the address could also be 14.

Going through the rest of the cases similarly we do not get anymore possibilities for the address.

Block	Even or Odd	Perfect Square (Y/N)	4 is in the Number (Y/N)	Possible Address
1-20	Even	Y	Y	4
1-20	Even	Y	N	16
1-20	Even	N	Y	14
1-20	Even	N	N	2,6,8,10,12,18,20
1-20	Odd	Y	Y	N/A
1-20	Odd	Y	N	1, 9
1-20	Odd	N	Y	N/A
1-20	Odd	N	N	3,5,7,11,13,15,17,19
21-50	Even	Y	Y	N/A
21-50	Even	Y	N	36
21-50	Even	N	Y	24,34,40,42,44,46,48
21-50	Even	N	N	22,26,28,30,32,36,38,50
21-50	Odd	Y	Y	49
21-50	Odd	Y	N	25
21-50	Odd	N	Y	41,43,45,47
21-50	Odd	N	N	21,23,27,29,31,33,35,37,39