Mathematical Induction

Brian Forrest

October 1, 2010

Principle of Mathematical Induction

Axiom: (Principle of Mathematical Induction)

Principle of Mathematical Induction

Axiom: (Principle of Mathematical Induction)
Suppose that $S \subseteq \mathbb{N}$ is such that

Principle of Mathematical Induction

Axiom: (Principle of Mathematical Induction)
Suppose that $S \subseteq \mathbb{N}$ is such that

- I1) $1 \in S$

Principle of Mathematical Induction

Axiom: (Principle of Mathematical Induction)
Suppose that $S \subseteq \mathbb{N}$ is such that

- I1) $1 \in S$
- I2) if $n \in S$, then $n + 1 \in S$.

Principle of Mathematical Induction

Axiom: (Principle of Mathematical Induction)
Suppose that $S \subseteq \mathbb{N}$ is such that

- I1) $1 \in S$
- I2) if $n \in S$, then $n + 1 \in S$.

then $S = \mathbb{N}$.

Principle of Mathematical Induction

Question: Why is the Principle of Mathematical Induction reasonable?

Principle of Mathematical Induction

Question: Why is the Principle of Mathematical Induction reasonable?

Test:

Principle of Mathematical Induction

Question: Why is the Principle of Mathematical Induction reasonable?

Test: Suppose S satisfies I1) and I2).

Principle of Mathematical Induction

Question: Why is the Principle of Mathematical Induction reasonable?

Test: Suppose S satisfies I1) and I2). Show that $7 \in S$.

Principle of Mathematical Induction

Question: Why is the Principle of Mathematical Induction reasonable?

Test: Suppose S satisfies I1) and I2). Show that $7 \in S$.

- $1 \in S$ I1)

Principle of Mathematical Induction

Question: Why is the Principle of Mathematical Induction reasonable?

Test: Suppose S satisfies I1) and I2). Show that $7 \in S$.

- $1 \in S$ I1)
- $\Rightarrow 1 + 1 = 2 \in S$ I2)

Principle of Mathematical Induction

Question: Why is the Principle of Mathematical Induction reasonable?

Test: Suppose S satisfies I1) and I2). Show that $7 \in S$.

- $1 \in S$ I1)
- $\Rightarrow 1+1=2 \in S$ I2)
- $\Rightarrow 2+1=3 \in S$ I2)

Principle of Mathematical Induction

Question: Why is the Principle of Mathematical Induction reasonable?

Test: Suppose S satisfies I1) and I2). Show that $7 \in S$.

- $1 \in S$ I1)
- $\Rightarrow 1+1=2 \in S$ I2)
- $\Rightarrow 2+1=3 \in S$ I2)
- $\Rightarrow 3+1=4 \in S$ I2)

Principle of Mathematical Induction

Question: Why is the Principle of Mathematical Induction reasonable?

Test: Suppose S satisfies I1) and I2). Show that $7 \in S$.

- $1 \in S$ I1)
- $\Rightarrow 1+1=2 \in S$ I2)
- $\Rightarrow 2+1=3 \in S$ I2)
- $\Rightarrow 3+1=4 \in S$ I2)
- $\Rightarrow 4+1=5 \in S$ I2)

Principle of Mathematical Induction

Question: Why is the Principle of Mathematical Induction reasonable?

Test: Suppose S satisfies I1) and I2). Show that $7 \in S$.

- $1 \in S$ I1)
- $\Rightarrow 1+1=2 \in S$ I2)
- $\Rightarrow 2+1=3 \in S$ I2)
- $\Rightarrow 3+1=4 \in S$ I2)
- $\Rightarrow 4+1=5 \in S$ I2)
- $\Rightarrow 5+1=6 \in S$ I2)

Principle of Mathematical Induction

Question: Why is the Principle of Mathematical Induction reasonable?

Test: Suppose S satisfies I1) and I2). Show that $7 \in S$.

- $1 \in S$ I1)
- $\Rightarrow 1+1=2 \in S$ I2)
- $\Rightarrow 2+1=3 \in S$ I2)
- $\Rightarrow 3+1=4 \in S$ I2)
- $\Rightarrow 4+1=5 \in S$ I2)
- $\Rightarrow 5+1=6 \in S$ I2)
- $\Rightarrow 6+1=7 \in S$ I2)

Principle of Mathematical Induction

Question:

Principle of Mathematical Induction

Question: How is the Principle of Mathematical Induction useful?

Principle of Mathematical Induction

Question: How is the Principle of Mathematical Induction useful?

- Leads to *Proof by Induction*

Principle of Mathematical Induction

Question: How is the Principle of Mathematical Induction useful?

- Leads to *Proof by Induction* or *Induction* for short.

Proofs by Induction

Question:

Proofs by Induction

Question: How does a proof by induction work?

Proofs by Induction

Question: How does a proof by induction work?

1) The setup:

Proofs by Induction

Question: How does a proof by induction work?

1) The setup:

- Assign to each natural number $n \in \mathbb{N}$

Proofs by Induction

Question: How does a proof by induction work?

1) The setup:

• Assign to each natural number $n \in \mathbb{N}$ the statement $P(n)$ that we want to prove.

Proofs by Induction

Question: How does a proof by induction work?

1) The setup:

• Assign to each natural number $n \in \mathbb{N}$ the statement $P(n)$ that we want to prove.

Example:

Proofs by Induction

Question: How does a proof by induction work?

1) The setup:

• Assign to each natural number $n \in \mathbb{N}$ the statement $P(n)$ that we want to prove.

Example: $P(n)$ is the statement that:

$$\sum_{j=1}^{n} j = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Proofs by Induction

Question: How does a proof by induction work?

1) The setup:

• Assign to each natural number $n \in \mathbb{N}$ the statement $P(n)$ that we want to prove.

Example: $P(n)$ is the statement that:

$$\sum_{j=1}^{n} j = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

• Let S be the set of all natural numbers n such that the statement $P(n)$ is true.

Proofs by Induction

Question: How does a proof by induction work?

1) The setup:

• Assign to each natural number $n \in \mathbb{N}$ the statement $P(n)$ that we want to prove.

Example: $P(n)$ is the statement that:

$$\sum_{j=1}^{n} j = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

• Let S be the set of all natural numbers n such that the statement $P(n)$ is true.

2) The Goal:

Proofs by Induction

Question: How does a proof by induction work?

1) The setup:

• Assign to each natural number $n \in \mathbb{N}$ the statement $P(n)$ that we want to prove.

Example: $P(n)$ is the statement that:

$$\sum_{j=1}^{n} j = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

• Let S be the set of all natural numbers n such that the statement $P(n)$ is true.

2) The Goal: Show that $S = \mathbb{N}$.

Proofs by Induction

3) The Method:

Proofs by Induction

3) The Method:

- Step 0)

3) The Method:

- Step 0)

Clearly state what $P(n)$ represents.

3) The Method:

- Step 0)

Clearly state what $P(n)$ represents.

- Step 1) Initial Step:

Proofs by Induction

3) The Method:

- Step 0)

Clearly state what $P(n)$ represents.

- Step 1) Initial Step:

Show that $P(1)$ is true.

Proofs by Induction

3) The Method:

- Step 0)

Clearly state what $P(n)$ represents.

- Step 1) Initial Step:

Show that $P(1)$ is true.

- Step 2) Inductive Step:

Proofs by Induction

3) The Method:

- Step 0)

Clearly state what $P(n)$ represents.

- Step 1) Initial Step:

Show that $P(1)$ is true.

- Step 2) Inductive Step:

Assume that $P(k)$ holds true for some k

Proofs by Induction

3) The Method:

- Step 0)

Clearly state what $P(n)$ represents.

- Step 1) Initial Step:

Show that $P(1)$ is true.

- Step 2) Inductive Step:

Assume that $P(k)$ holds true for some k (this is called *the induction hypothesis)*

Proofs by Induction

3) The Method:

- Step 0)

Clearly state what $P(n)$ represents.

- Step 1) Initial Step:

Show that $P(1)$ is true.

- Step 2) Inductive Step:

Assume that $P(k)$ holds true for some k (this is called *the induction hypothesis)* and then use this assumption to show that $P(k+1)$ also holds.

Example

Example:

Example

Example: Prove that

$$\sum_{j=1}^{n} j = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

for all $n \in \mathbb{N}$.

Example

Example: Prove that

$$\sum_{j=1}^{n} j = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

for all $n \in \mathbb{N}$.

Step 0): Let $P(n)$ be the statement that

$$\sum_{j=1}^{n} j = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Initial Step

Step 1): Show that $P(1)$ holds.

Initial Step

Step 1): Show that $P(1)$ holds.

$$\sum_{j=1}^{1} j$$

Initial Step

Step 1): Show that $P(1)$ holds.

$$\sum_{j=1}^{1} j = 1$$

Initial Step

Step 1): Show that $P(1)$ holds.

$$\begin{aligned} \sum_{j=1}^{1} j &= 1 \\ &= \frac{1(1+1)}{2} \end{aligned}$$

Initial Step

Step 1): Show that $P(1)$ holds.

$$\begin{aligned} \sum_{j=1}^{1} j &= 1 \\ &= \frac{1(1+1)}{2} \end{aligned}$$

This establishes $P(1)$.

Inductive Step

Step 2):

Inductive Step

Step 2): Assume that $P(k)$ holds.

Inductive Step

Step 2): Assume that $P(k)$ holds. Consider

$$\sum_{j=1}^{k+1} j$$

Inductive Step

Step 2): Assume that $P(k)$ holds. Consider

$$\sum_{j=1}^{k+1} j = \left(\sum_{j=1}^{k} j \right) + (k+1)$$

Inductive Step

Step 2): Assume that $P(k)$ holds. Consider

$$\begin{aligned}\sum_{j=1}^{k+1} j &= \left(\sum_{j=1}^{k} j\right) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \qquad (by\ P(k))\end{aligned}$$

Inductive Step

Step 2): Assume that $P(k)$ holds. Consider

$$\begin{aligned}
\sum_{j=1}^{k+1} j &= \left(\sum_{j=1}^{k} j\right) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \qquad (by\ P(k)) \\
&= \left(\frac{k}{2} + \frac{2}{2}\right)(k+1)
\end{aligned}$$

Inductive Step

Step 2): Assume that $P(k)$ holds. Consider

$$\begin{aligned}
\sum_{j=1}^{k+1} j &= \left(\sum_{j=1}^{k} j\right) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \qquad (\textit{by } P(k)) \\
&= \left(\frac{k}{2} + \frac{2}{2}\right)(k+1) \\
&= \frac{(k+2)(k+1)}{2}
\end{aligned}$$

Inductive Step

Step 2): Assume that $P(k)$ holds. Consider

$$\begin{aligned}
\sum_{j=1}^{k+1} j &= \left(\sum_{j=1}^{k} j\right) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \qquad (\textit{by } P(k)) \\
&= \left(\frac{k}{2} + \frac{2}{2}\right)(k+1) \\
&= \frac{(k+2)(k+1)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2}.
\end{aligned}$$

Inductive Step

Step 2): Assume that $P(k)$ holds. Consider

$$\begin{aligned}
\sum_{j=1}^{k+1} j &= \left(\sum_{j=1}^{k} j\right) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \qquad (\textit{by } P(k)) \\
&= \left(\frac{k}{2} + \frac{2}{2}\right)(k+1) \\
&= \frac{(k+2)(k+1)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2}.
\end{aligned}$$

This shows that $P(k+1)$ also holds.

Conclusion

We have shown that:

Conclusion

We have shown that:

- $P(1)$ holds.

Conclusion

We have shown that:

- $P(1)$ holds.
- If $P(k)$ holds,

Conclusion

We have shown that:

- $P(1)$ holds.
- If $P(k)$ holds, so does $P(k+1)$.

Conclusion

We have shown that:

- $P(1)$ holds.
- If $P(k)$ holds, so does $P(k+1)$.

We conclude by induction that

$$\sum_{j=1}^{n} j = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for all $n \in \mathbb{N}$.

Shaggy Dogs

Important Note:

Shaggy Dogs

Important Note: Care must be taken with induction proofs!

Shaggy Dogs

Important Note: Care must be taken with induction proofs!

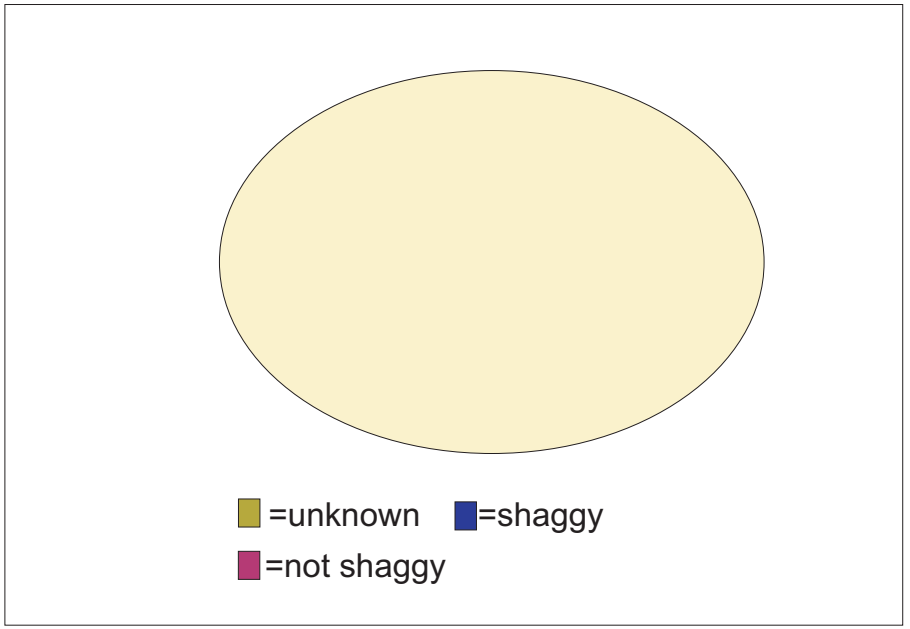
Problem:

Shaggy Dogs

Important Note: Care must be taken with induction proofs!

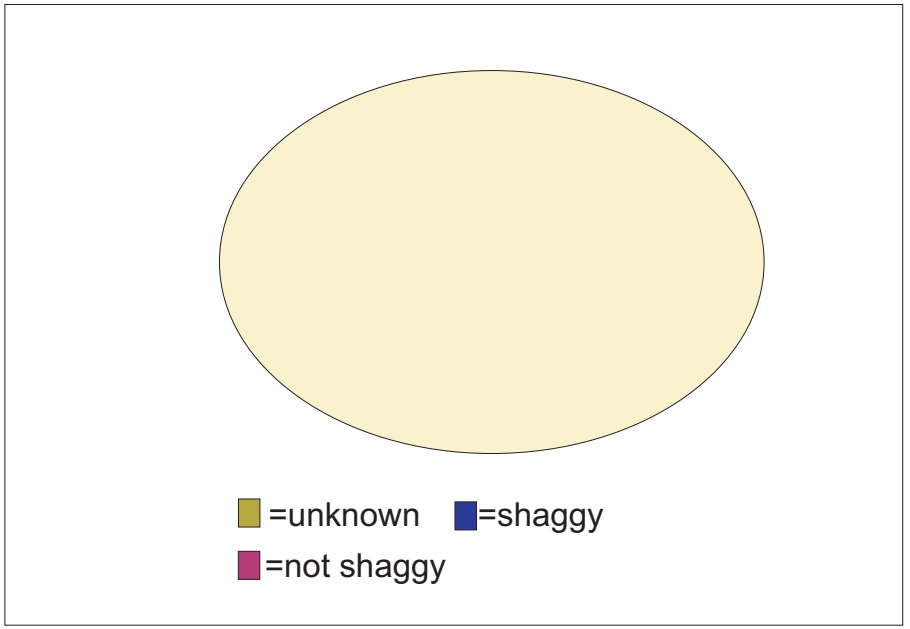
Problem: Prove that all dogs are shaggy!

Shaggy Dogs



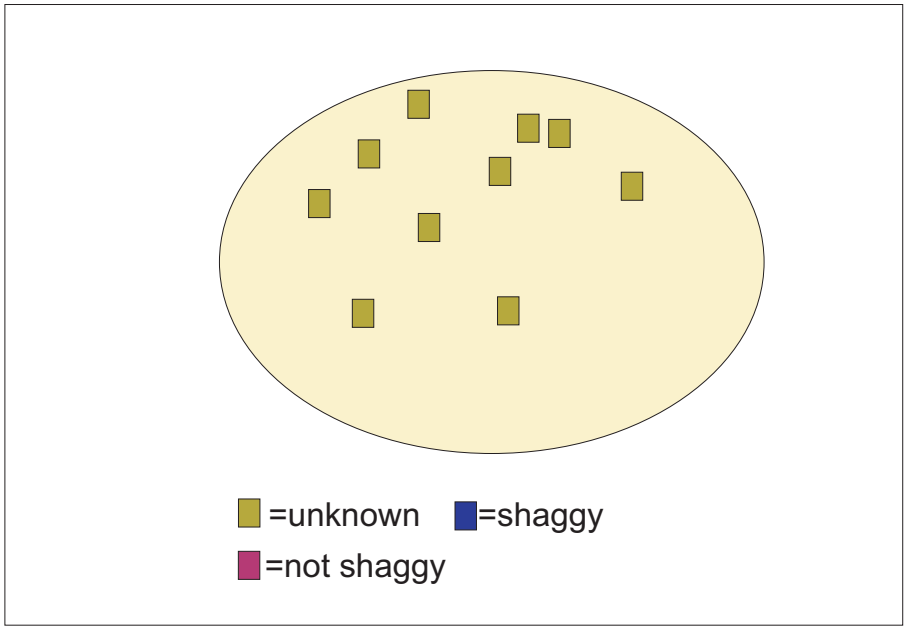
Proof:

Shaggy Dogs



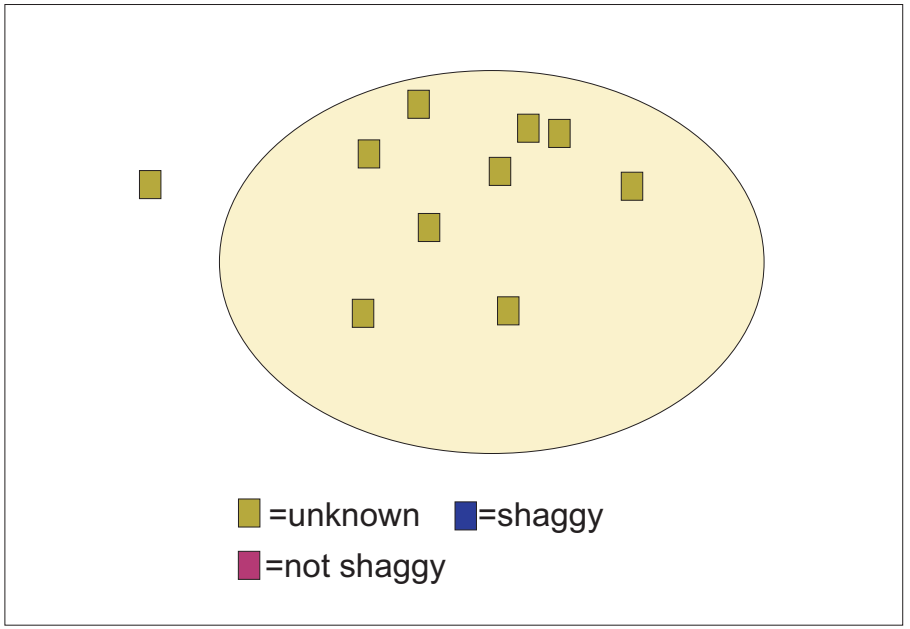
Proof: Assume that $P(k)$=*every group of k dogs is shaggy*.

Shaggy Dogs



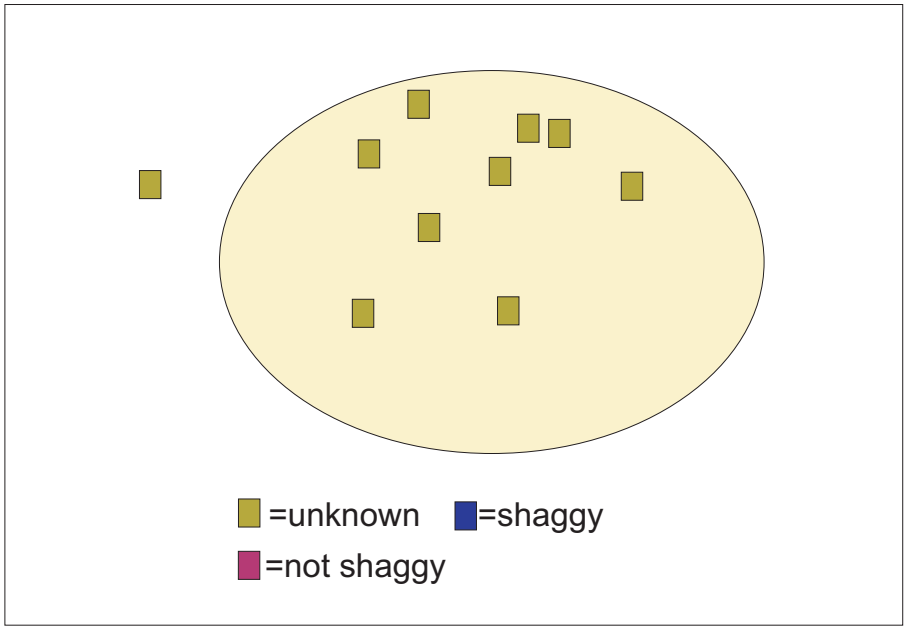
Proof: Assume that $P(k)$=*every group of k dogs is shaggy*.
Consider a group of $k+1$ dogs.

Shaggy Dogs



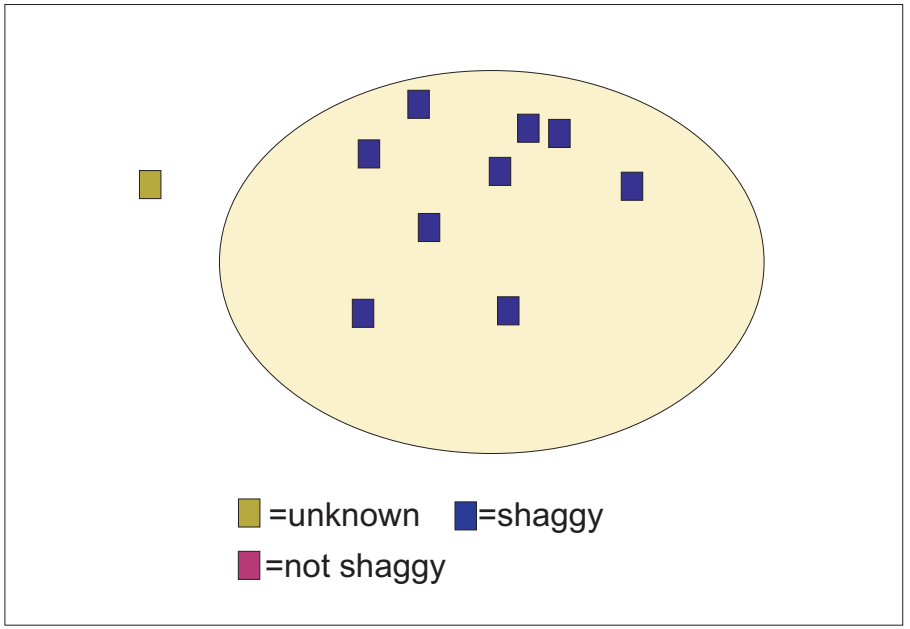
Proof: Assume that $P(k)$=*every group of k dogs is shaggy*.
Consider a group of $k+1$ dogs.
Remove one dog from the group.

Shaggy Dogs



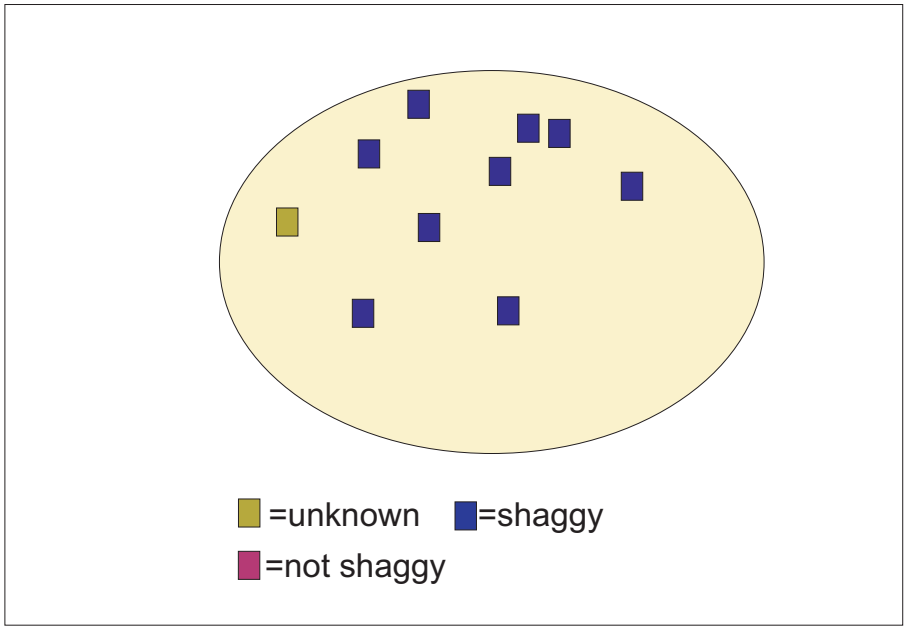
Proof: Assume that $P(k)$=*every group of k dogs is shaggy*.
Consider a group of $k+1$ dogs.
Remove one dog from the group.
The new group contains k dogs

Shaggy Dogs



Proof: Assume that $P(k)$=*every group of k dogs is shaggy*.
Consider a group of $k+1$ dogs.
Remove one dog from the group.
The new group contains k dogs so all are shaggy.

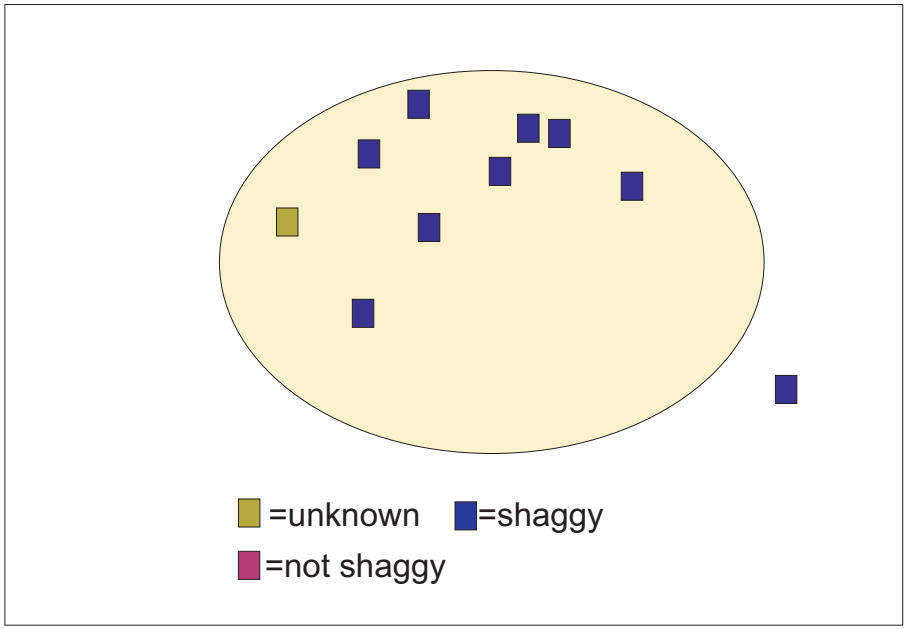
Shaggy Dogs



Proof: Assume that $P(k)$=*every group of k dogs is shaggy*.
Consider a group of $k+1$ dogs.
Remove one dog from the group.
The new group contains k dogs so all are shaggy.

Next return the original dog to the group

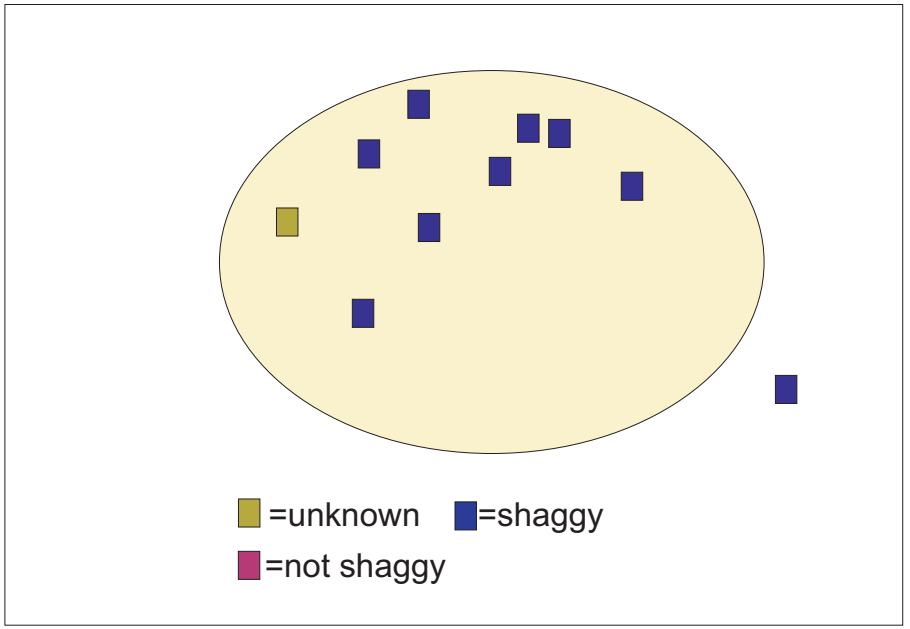
Shaggy Dogs



Proof: Assume that $P(k)$=*every group of k dogs is shaggy*.
Consider a group of $k+1$ dogs.
Remove one dog from the group.
The new group contains k dogs so all are shaggy.

Next return the original dog to the group and remove one shaggy dog.

Shaggy Dogs

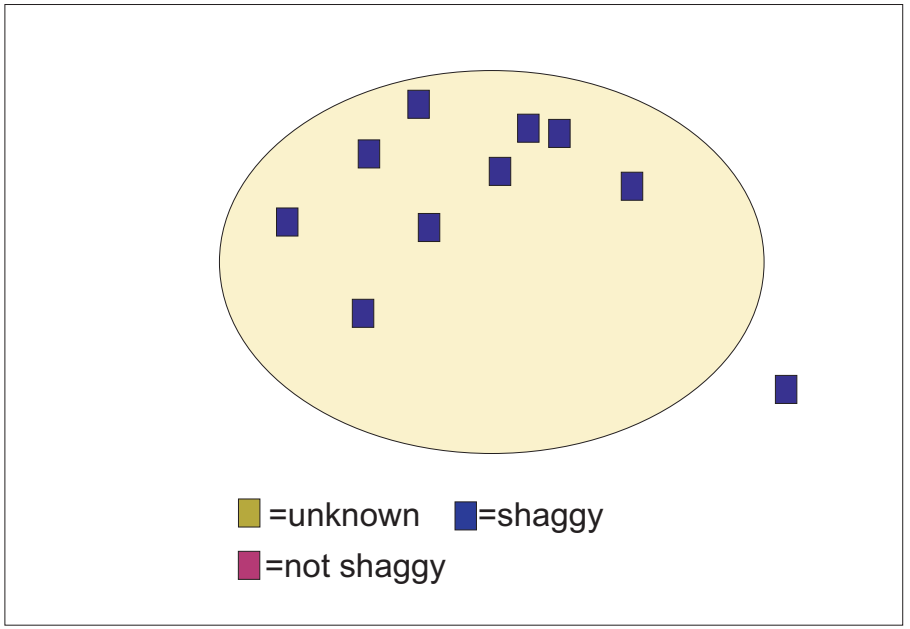


Again we have a group of k dogs,

Proof: Assume that $P(k)$=*every group of k dogs is shaggy*.
Consider a group of $k+1$ dogs.
Remove one dog from the group.
The new group contains k dogs so all are shaggy.

Next return the original dog to the group and remove one shaggy dog.

Shaggy Dogs

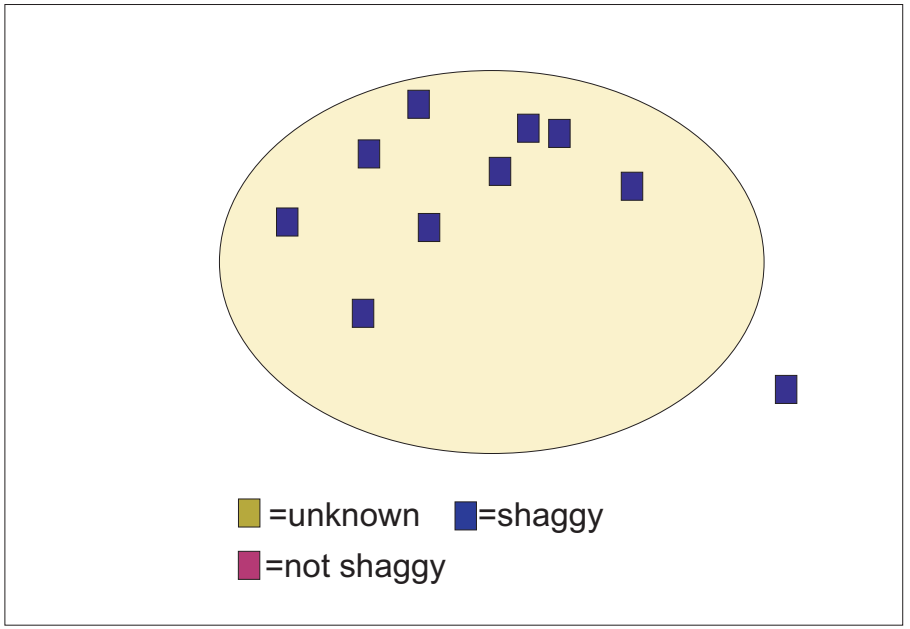


Proof: Assume that $P(k)$=*every group of k dogs is shaggy*.
Consider a group of $k+1$ dogs.
Remove one dog from the group.
The new group contains k dogs so all are shaggy.

Next return the original dog to the group and remove one shaggy dog.

Again we have a group of k dogs, so all of these dogs are shaggy.

Shaggy Dogs

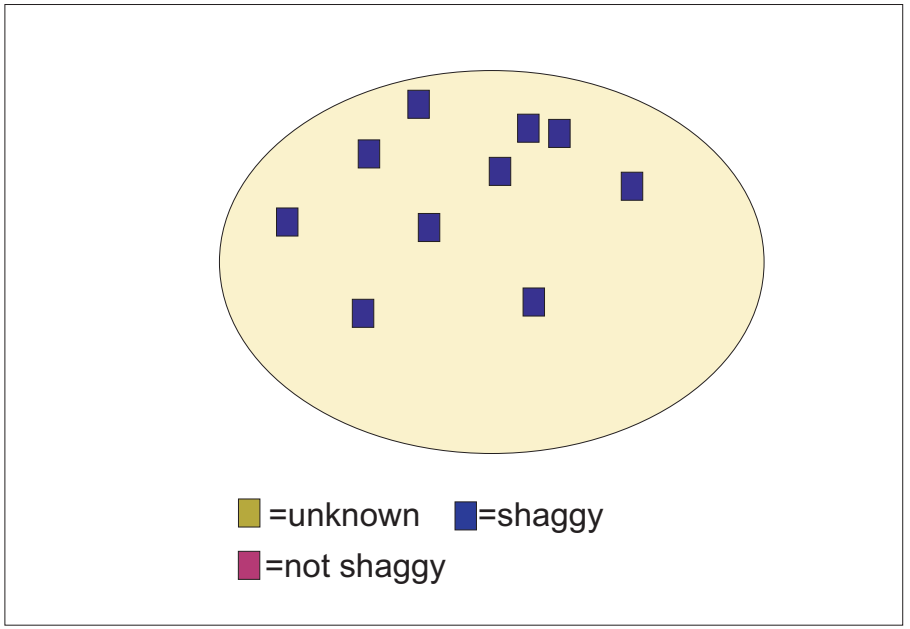


Proof: Assume that $P(k)$=*every group of k dogs is shaggy*.
Consider a group of $k+1$ dogs.
Remove one dog from the group.
The new group contains k dogs so all are shaggy.

Next return the original dog to the group and remove one shaggy dog.

Again we have a group of k dogs, so all of these dogs are shaggy. .
Therefore, the original dog is shaggy.

Shaggy Dogs

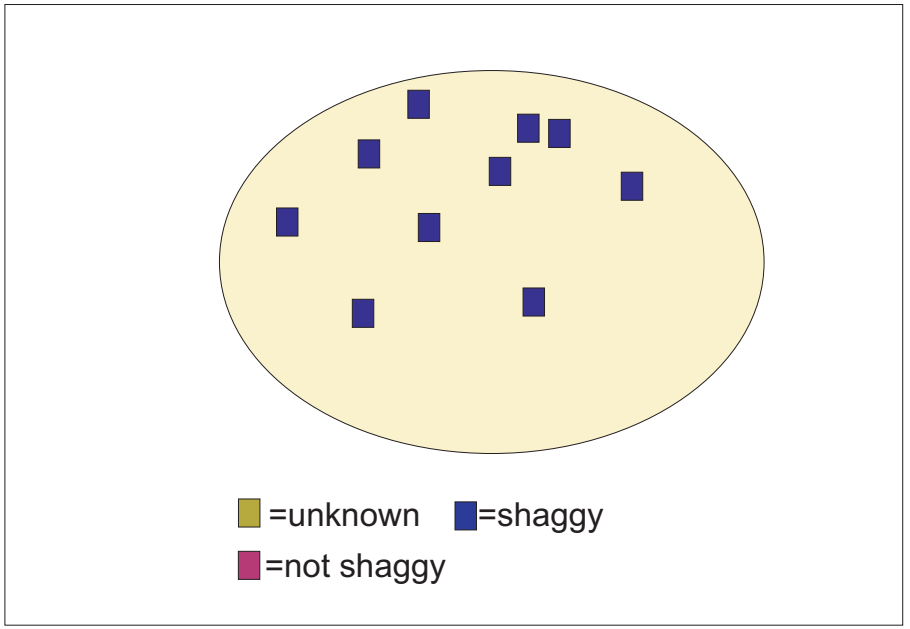


Proof: Assume that $P(k)$=*every group of k dogs is shaggy*.
Consider a group of $k+1$ dogs.
Remove one dog from the group.
The new group contains k dogs so all are shaggy.

Next return the original dog to the group and remove one shaggy dog.

Again we have a group of k dogs, so all of these dogs are shaggy. .
Therefore, the original dog is shaggy.
Hence the group of $k+1$ dogs are all shaggy.

Shaggy Dogs

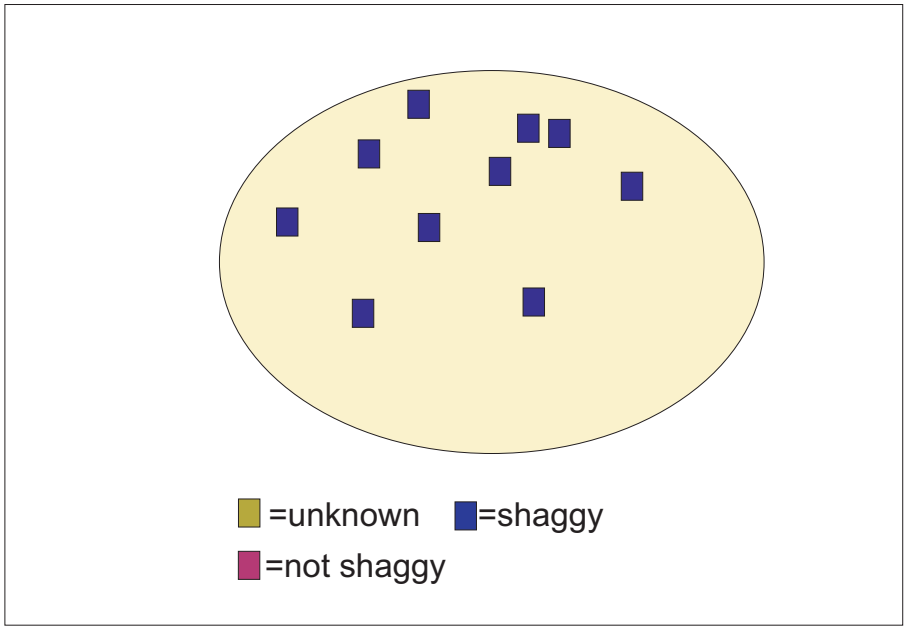


Proof: Assume that $P(k)$=*every group of k dogs is shaggy*.
Consider a group of $k+1$ dogs.
Remove one dog from the group.
The new group contains k dogs so all are shaggy.

Next return the original dog to the group and remove one shaggy dog.

Again we have a group of k dogs, so all of these dogs are shaggy. .
Therefore, the original dog is shaggy.
Hence the group of $k+1$ dogs are all shaggy. By induction we have proved that all dogs are shaggy!!!

Shaggy Dogs



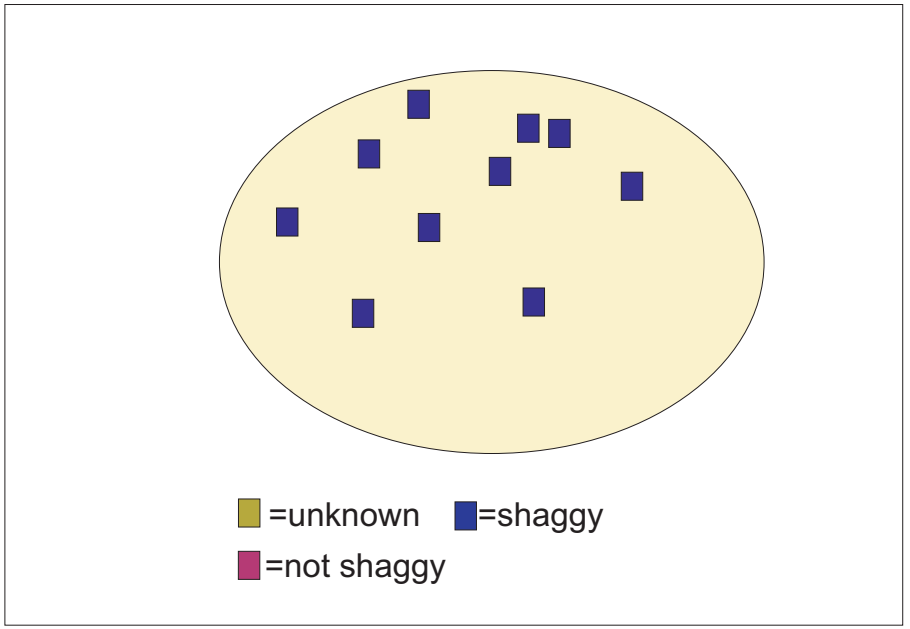
Proof: Assume that $P(k)$=*every group of k dogs is shaggy*.
Consider a group of $k+1$ dogs.
Remove one dog from the group.
The new group contains k dogs so all are shaggy.

Next return the original dog to the group and remove one shaggy dog.

Again we have a group of k dogs, so all of these dogs are shaggy. .
Therefore, the original dog is shaggy.
Hence the group of $k+1$ dogs are all shaggy. By induction we have proved that all dogs are shaggy!!!

Question:

Shaggy Dogs



Proof: Assume that $P(k)$=*every group of k dogs is shaggy*.
Consider a group of $k+1$ dogs.
Remove one dog from the group.
The new group contains k dogs so all are shaggy.

Next return the original dog to the group and remove one shaggy dog.

Again we have a group of k dogs, so all of these dogs are shaggy. .
Therefore, the original dog is shaggy.
Hence the group of $k+1$ dogs are all shaggy. By induction we have proved that all dogs are shaggy!!!

Question: What's wrong with this proof?

Tower of Hanoi

Problem: (Tower of Hanoi)

Tower of Hanoi

Problem: (Tower of Hanoi) You are given three pegs. On one of the pegs is a tower made up of n rings

Tower of Hanoi

Problem: (Tower of Hanoi) You are given three pegs. On one of the pegs is a tower made up of n rings placed on top of one another so that as you move down the tower each successive ring has a larger diameter than the previous ring.

Tower of Hanoi

Problem: (Tower of Hanoi) You are given three pegs. On one of the pegs is a tower made up of n rings placed on top of one another so that as you move down the tower each successive ring has a larger diameter than the previous ring. The object of this puzzle is to reconstruct the tower on one of the other pegs by moving one ring at a time, from one peg to another, in such a manner that you **never have a ring above any smaller ring** on any of the three pegs.

Tower of Hanoi

Problem: (Tower of Hanoi) You are given three pegs. On one of the pegs is a tower made up of n rings placed on top of one another so that as you move down the tower each successive ring has a larger diameter than the previous ring. The object of this puzzle is to reconstruct the tower on one of the other pegs by moving one ring at a time, from one peg to another, in such a manner that you **never have a ring above any smaller ring** on any of the three pegs.

Task:

Tower of Hanoi

Problem: (Tower of Hanoi) You are given three pegs. On one of the pegs is a tower made up of n rings placed on top of one another so that as you move down the tower each successive ring has a larger diameter than the previous ring. The object of this puzzle is to reconstruct the tower on one of the other pegs by moving one ring at a time, from one peg to another, in such a manner that you **never have a ring above any smaller ring** on any of the three pegs.

Task: Prove that for any $n \in \mathbb{N}$, if you begin with n rings, **then the puzzle can be completed in** $2^n - 1$ **moves.**

Tower of Hanoi

Problem: (Tower of Hanoi) You are given three pegs. On one of the pegs is a tower made up of n rings placed on top of one another so that as you move down the tower each successive ring has a larger diameter than the previous ring. The object of this puzzle is to reconstruct the tower on one of the other pegs by moving one ring at a time, from one peg to another, in such a manner that you **never have a ring above any smaller ring** on any of the three pegs.

Task: Prove that for any $n \in \mathbb{N}$, if you begin with n rings, **then the puzzle can be completed in** $2^n - 1$ **moves.**

Moreover, prove that for each n this is the **minimum number of moves** necessary to complete the task.

Tower of Hanoi

Problem: (Tower of Hanoi) You are given three pegs. On one of the pegs is a tower made up of n rings placed on top of one another so that as you move down the tower each successive ring has a larger diameter than the previous ring. The object of this puzzle is to reconstruct the tower on one of the other pegs by moving one ring at a time, from one peg to another, in such a manner that you **never have a ring above any smaller ring** on any of the three pegs.

Task: Prove that for any $n \in \mathbb{N}$, if you begin with n rings, **then the puzzle can be completed in** $2^n - 1$ **moves.**

Moreover, prove that for each n this is the **minimum number of moves** necessary to complete the task.

Question:

Tower of Hanoi

Problem: (Tower of Hanoi) You are given three pegs. On one of the pegs is a tower made up of n rings placed on top of one another so that as you move down the tower each successive ring has a larger diameter than the previous ring. The object of this puzzle is to reconstruct the tower on one of the other pegs by moving one ring at a time, from one peg to another, in such a manner that you **never have a ring above any smaller ring** on any of the three pegs.

Task: Prove that for any $n \in \mathbb{N}$, if you begin with n rings, **then the puzzle can be completed in** $2^n - 1$ **moves.**

Moreover, prove that for each n this is the **minimum number of moves** necessary to complete the task.

Question: What is $P(n)$?

Introduction to Sequences

Brian Forrest

October 1, 2010

What is a sequence?

What is a sequence?

What is a sequence?

What is a sequence?

Informally, a sequence is an ordered list of real numbers.

What is a sequence?

What is a sequence?

Informally, a sequence is an ordered list of real numbers.

For example a phone number 519-555-1234 is a *finite* sequence of 1-digit numbers

What is a sequence?

What is a sequence?

Informally, a sequence is an ordered list of real numbers.

For example a phone number 519-555-1234 is a *finite* sequence of 1-digit numbers

$$5 -$$

What is a sequence?

What is a sequence?

Informally, a sequence is an ordered list of real numbers.

For example a phone number 519-555-1234 is a *finite* sequence of 1-digit numbers

$$5 - 1 -$$

What is a sequence?

What is a sequence?

Informally, a sequence is an ordered list of real numbers.

For example a phone number 519-555-1234 is a *finite* sequence of 1-digit numbers

$$5 - 1 - 9 -$$

What is a sequence?

What is a sequence?

Informally, a sequence is an ordered list of real numbers.

For example a phone number 519-555-1234 is a *finite* sequence of 1-digit numbers

$$5 - 1 - 9 - 5 -$$

What is a sequence?

What is a sequence?

Informally, a sequence is an ordered list of real numbers.

For example a phone number 519-555-1234 is a *finite* sequence of 1-digit numbers

$$5 - 1 - 9 - 5 - 5 -$$

What is a sequence?

What is a sequence?

Informally, a sequence is an ordered list of real numbers.

For example a phone number 519-555-1234 is a *finite* sequence of 1-digit numbers

$$5 - 1 - 9 - 5 - 5 - 5 -$$

What is a sequence?

What is a sequence?

Informally, a sequence is an ordered list of real numbers.

For example a phone number 519-555-1234 is a *finite* sequence of 1-digit numbers

$$5 - 1 - 9 - 5 - 5 - 5 - 1 -$$

What is a sequence?

What is a sequence?

Informally, a sequence is an ordered list of real numbers.

For example a phone number 519-555-1234 is a *finite* sequence of 1-digit numbers

$$5 - 1 - 9 - 5 - 5 - 5 - 1 - 2 -$$

What is a sequence?

What is a sequence?

Informally, a sequence is an ordered list of real numbers.

For example a phone number 519-555-1234 is a *finite* sequence of 1-digit numbers

$$5 - 1 - 9 - 5 - 5 - 5 - 1 - 2 - 3 -$$

What is a sequence?

What is a sequence?

Informally, a sequence is an ordered list of real numbers.

For example a phone number 519-555-1234 is a *finite* sequence of 1-digit numbers

$$5 - 1 - 9 - 5 - 5 - 5 - 1 - 2 - 3 - 4$$

What is a sequence?

What is a sequence?

Informally, a sequence is an ordered list of real numbers.

For example a phone number 519-555-1234 is a *finite* sequence of 1-digit numbers

$$5 - 1 - 9 - 5 - 5 - 5 - 1 - 2 - 3 - 4.$$

Note: Order is important.

What is a sequence?

What is a sequence?

Informally, a sequence is an ordered list of real numbers.

For example a phone number 519-555-1234 is a *finite* sequence of 1-digit numbers

$$5 - 1 - 9 - 5 - 5 - 5 - 1 - 2 - 3 - 4.$$

Note: Order is important.

For us sequences will be *infinite ordered lists of real numbers* of the form $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$.

What is a sequence?

More formally:

What is a sequence?

More formally:

Definition: A *sequence* is a function $f : \mathbb{N} \to \mathbb{R}$.

What is a sequence?

More formally:

Definition: A *sequence* is a function $f : \mathbb{N} \to \mathbb{R}$.

For $n \in \mathbb{N}$, we call $f(n)$ the *nth term* of the sequence and n is called *the index*.

What is a sequence?

More formally:

Definition: A *sequence* is a function $f : \mathbb{N} \to \mathbb{R}$.

For $n \in \mathbb{N}$, we call $f(n)$ the *nth term* of the sequence and n is called *the index*.

Notation:

What is a sequence?

More formally:

Definition: A *sequence* is a function $f : \mathbb{N} \to \mathbb{R}$.

For $n \in \mathbb{N}$, we call $f(n)$ the *nth term* of the sequence and n is called *the index*.

Notation: We often write a_n,

What is a sequence?

More formally:

Definition: A *sequence* is a function $f : \mathbb{N} \to \mathbb{R}$.

For $n \in \mathbb{N}$, we call $f(n)$ the *nth term* of the sequence and n is called *the index*.

Notation: We often write a_n, or b_n,

What is a sequence?

More formally:

Definition: A *sequence* is a function $f : \mathbb{N} \to \mathbb{R}$.

For $n \in \mathbb{N}$, we call $f(n)$ the *nth term* of the sequence and n is called *the index*.

Notation: We often write a_n, or b_n, or x_n

What is a sequence?

More formally:

Definition: A *sequence* is a function $f : \mathbb{N} \to \mathbb{R}$.

For $n \in \mathbb{N}$, we call $f(n)$ the *nth term* of the sequence and n is called *the index*.

Notation: We often write a_n, or b_n, or x_n instead of $f(n)$ to denote the nth term.

Ways to specify a sequence

Ways to specify a sequence:

Ways to specify a sequence

Ways to specify a sequence:

1) Give the explicit function:

Ways to specify a sequence

Ways to specify a sequence:

1) Give the explicit function:

- $f(n) = \frac{1}{n}$

Ways to specify a sequence

Ways to specify a sequence:

1) Give the explicit function:

- $f(n) = \frac{1}{n}$
- $a_n = \frac{1}{n}$

Ways to specify a sequence

Ways to specify a sequence:

1) Give the explicit function:

- $f(n) = \frac{1}{n}$
- $a_n = \frac{1}{n}$
- $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$

Ways to specify a sequence

Ways to specify a sequence:

1) Give the explicit function:

- $f(n) = \frac{1}{n}$
- $a_n = \frac{1}{n}$
- $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$
- $\{\frac{1}{n}\}$

Ways to specify a sequence

2) Recursion:

Ways to specify a sequence

2) Recursion: Future terms in a sequence are defined from existing terms.

Ways to specify a sequence

2) Recursion: Future terms in a sequence are defined from existing terms.

Example: Let $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.

Ways to specify a sequence

2) Recursion: Future terms in a sequence are defined from existing terms.

Example: Let $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.

What is a_5?

Ways to specify a sequence

2) Recursion: Future terms in a sequence are defined from existing terms.

Example: Let $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.

What is a_5?

- $a_1 = 1$

Ways to specify a sequence

2) Recursion: Future terms in a sequence are defined from existing terms.

Example: Let $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.

What is a_5?

- $a_1 = 1$
- $a_2 = \frac{1}{1+a_1}$

Ways to specify a sequence

2) Recursion: Future terms in a sequence are defined from existing terms.

Example: Let $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.

What is a_5?

- $a_1 = 1$
- $a_2 = \frac{1}{1+a_1} = \frac{1}{1+1} = \frac{1}{2}$

Ways to specify a sequence

2) Recursion: Future terms in a sequence are defined from existing terms.

Example: Let $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.

What is a_5?

- $a_1 = 1$
- $a_2 = \frac{1}{1+a_1} = \frac{1}{1+1} = \frac{1}{2}$
- $a_3 = \frac{1}{1+a_2}$

Ways to specify a sequence

2) Recursion: Future terms in a sequence are defined from existing terms.

Example: Let $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.

What is a_5?

- $a_1 = 1$
- $a_2 = \frac{1}{1+a_1} = \frac{1}{1+1} = \frac{1}{2}$
- $a_3 = \frac{1}{1+a_2} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$

Ways to specify a sequence

2) Recursion: Future terms in a sequence are defined from existing terms.

Example: Let $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.

What is a_5?

- $a_1 = 1$
- $a_2 = \frac{1}{1+a_1} = \frac{1}{1+1} = \frac{1}{2}$
- $a_3 = \frac{1}{1+a_2} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$
- $a_4 = \frac{1}{1+a_3}$

Ways to specify a sequence

2) Recursion: Future terms in a sequence are defined from existing terms.

Example: Let $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.

What is a_5?

- $a_1 = 1$
- $a_2 = \frac{1}{1+a_1} = \frac{1}{1+1} = \frac{1}{2}$
- $a_3 = \frac{1}{1+a_2} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$
- $a_4 = \frac{1}{1+a_3} = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}$

Ways to specify a sequence

2) Recursion: Future terms in a sequence are defined from existing terms.

Example: Let $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.

What is a_5?

- $a_1 = 1$
- $a_2 = \frac{1}{1+a_1} = \frac{1}{1+1} = \frac{1}{2}$
- $a_3 = \frac{1}{1+a_2} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$
- $a_4 = \frac{1}{1+a_3} = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}$
- $a_5 = \frac{1}{1+a_4}$

Ways to specify a sequence

2) Recursion: Future terms in a sequence are defined from existing terms.

Example: Let $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.

What is a_5?

- $a_1 = 1$
- $a_2 = \frac{1}{1+a_1} = \frac{1}{1+1} = \frac{1}{2}$
- $a_3 = \frac{1}{1+a_2} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$
- $a_4 = \frac{1}{1+a_3} = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}$
- $a_5 = \frac{1}{1+a_4} = \frac{1}{1+\frac{3}{5}} = \frac{5}{8}$

Ways to specify a sequence

2) Recursion: Future terms in a sequence are defined from existing terms.

Example: Let $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.

What is a_5?

- $a_1 = 1$
- $a_2 = \frac{1}{1+a_1} = \frac{1}{1+1} = \frac{1}{2}$
- $a_3 = \frac{1}{1+a_2} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$
- $a_4 = \frac{1}{1+a_3} = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}$
- $a_5 = \frac{1}{1+a_4} = \frac{1}{1+\frac{3}{5}} = \frac{5}{8}$

Exercise:

Ways to specify a sequence

2) Recursion: Future terms in a sequence are defined from existing terms.

Example: Let $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.

What is a_5?

- $a_1 = 1$
- $a_2 = \frac{1}{1+a_1} = \frac{1}{1+1} = \frac{1}{2}$
- $a_3 = \frac{1}{1+a_2} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$
- $a_4 = \frac{1}{1+a_3} = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}$
- $a_5 = \frac{1}{1+a_4} = \frac{1}{1+\frac{3}{5}} = \frac{5}{8}$

Exercise: Can you find an explicit formula for a_n?

Subsequences: New sequences from old

Definition: Given a sequence $\{a_n\}$ and a sequence $\{n_k\} \in \mathbb{N}$ such that

$$n_1 < n_2 < n_2 \ldots < n_k \ldots$$

Subsequences: New sequences from old

Definition: Given a sequence $\{a_n\}$ and a sequence $\{n_k\} \in \mathbb{N}$ such that

$$n_1 < n_2 < n_2 \ldots < n_k \ldots$$

define

$$b_k = a_{n_k}.$$

Subsequences: New sequences from old

Definition: Given a sequence $\{a_n\}$ and a sequence $\{n_k\} \in \mathbb{N}$ such that

$$n_1 < n_2 < n_2 \ldots < n_k \ldots$$

define

$$b_k = a_{n_k}.$$

The sequence $\{b_k\} = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots, a_{n_1}, \ldots\}$ is called a *subsequence* of $\{a_n\}$.

Subsequences: New sequences from old

Definition: Given a sequence $\{a_n\}$ and a sequence $\{n_k\} \in \mathbb{N}$ such that

$$n_1 < n_2 < n_2 \ldots < n_k \ldots$$

define

$$b_k = a_{n_k}.$$

The sequence $\{b_k\} = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots, a_{n_1}, \ldots\}$ is called a *subsequence* of $\{a_n\}$.

Example: Let $a_n = \frac{1}{n}$ and $n_k = 2k$.

Subsequences: New sequences from old

Definition: Given a sequence $\{a_n\}$ and a sequence $\{n_k\} \in \mathbb{N}$ such that

$$n_1 < n_2 < n_2 \ldots < n_k \ldots$$

define

$$b_k = a_{n_k}.$$

The sequence $\{b_k\} = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots, a_{n_1}, \ldots\}$ is called a *subsequence* of $\{a_n\}$.

Example: Let $a_n = \frac{1}{n}$ and $n_k = 2k$. Then

$$b_k = a_{n_k} = a_{2k} = \frac{1}{2k}.$$

Subsequences: New sequences from old

$$\{a_n\} = \{a_1, a_2, a_3, a_4, \ldots, a_{2k}, \ldots\}$$

$$\{b_k\} = \{$$

$$= \{$$

Subsequences: New sequences from old

$$\{a_n\} = \{a_1, a_2, a_3, a_4, \ldots, a_{2k}, \ldots\}$$

$$\{b_k\} = \{b_1,$$

$$= \{a_2,$$

Subsequences: New sequences from old

$$\{a_n\} = \{a_1, a_2, a_3, a_4, \ldots, a_{2k}, \ldots\}$$

$$\{b_k\} = \{b_1, b_2,$$

$$= \{a_2, a_4,$$

Subsequences: New sequences from old

$$\{a_n\} = \{a_1, a_2, a_3, a_4, \ldots, a_{2k}, \ldots\}$$

$$\{b_k\} = \{b_1, b_2, \ldots, b_k,$$

$$= \{a_2, a_4, \ldots, a_{2k},$$

Subsequences: New sequences from old

$$\{a_n\} = \{a_1, a_2, a_3, a_4, \ldots, a_{2k}, \ldots\}$$

$$\{b_k\} = \{b_1, b_2, \ldots, b_k, \ldots\}$$

$$= \{a_2, a_4, \ldots, a_{2k}, \ldots\}$$

Subsequences: New sequences from old

Example: Given $\{a_n\}$ and $j \in \mathbb{N} \cup \{0\}$,

Subsequences: New sequences from old

Example: Given $\{a_n\}$ and $j \in \mathbb{N} \cup \{0\}$, let

$$b_k = a_{k+j} = \{a_{1+j}, a_{2+j}, a_{3+j}, \ldots\}.$$

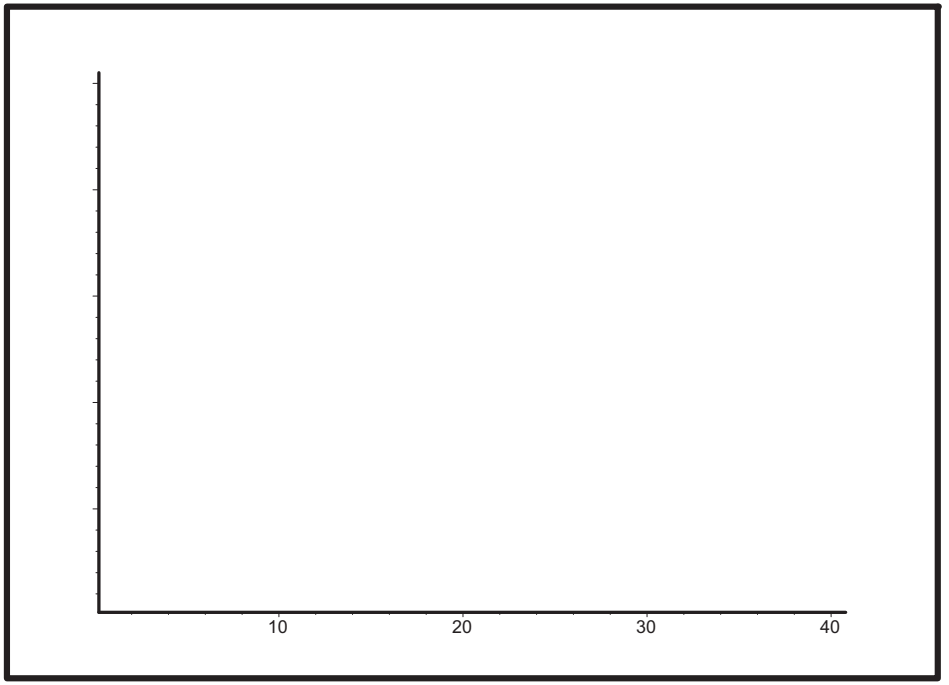
Subsequences: New sequences from old

Example: Given $\{a_n\}$ and $j \in \mathbb{N} \cup \{0\}$, let

$$b_k = a_{k+j} = \{a_{1+j}, a_{2+j}, a_{3+j}, \ldots\}.$$

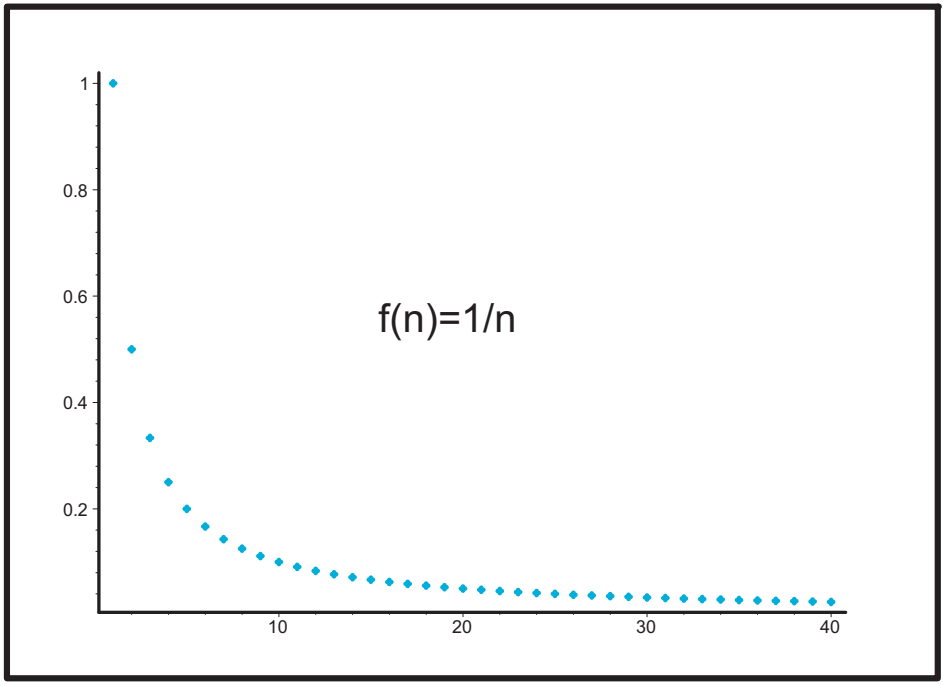
Note: Such a subsequence is called a *tail* of $\{a_n\}$.

Graphical Representation of a Sequence



1) Standard 2-dimensional graph.

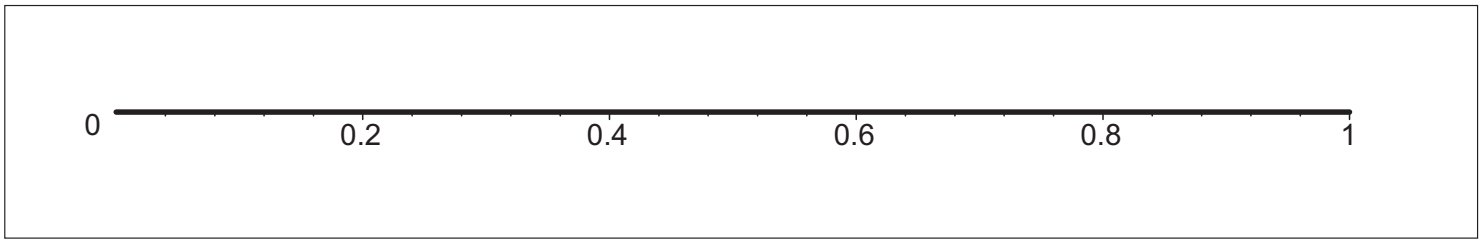
Graphical Representation of a Sequence



1) Standard 2-dimensional graph.

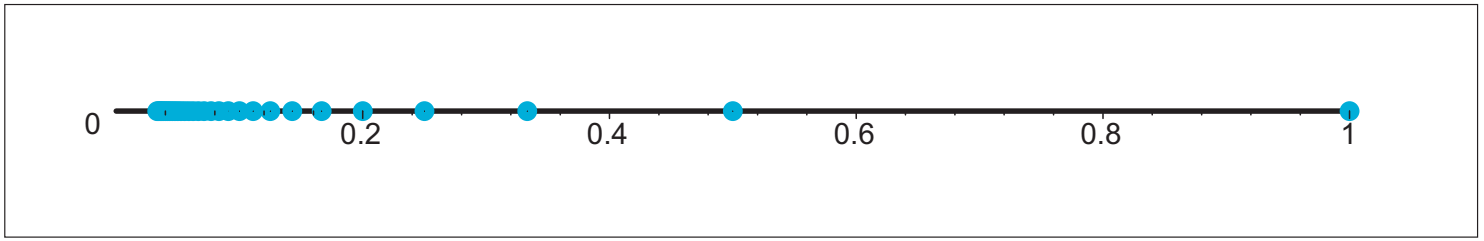
Example: $\{\frac{1}{n}\}$.

Graphical Representation of a Sequence



2) 1-dimensional plot.

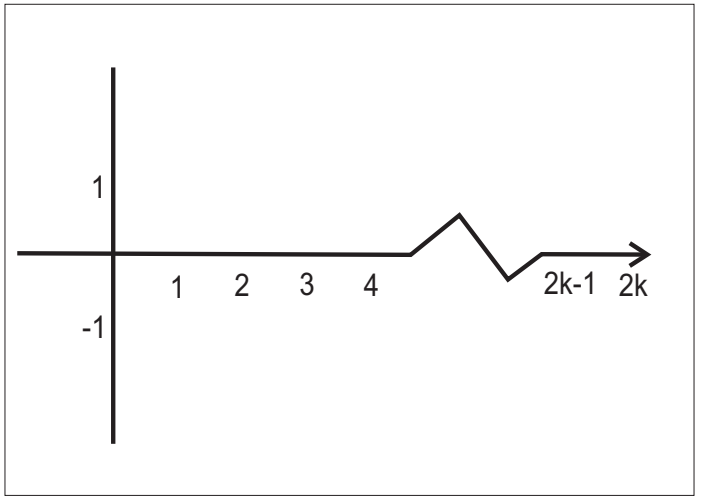
Graphical Representation of a Sequence

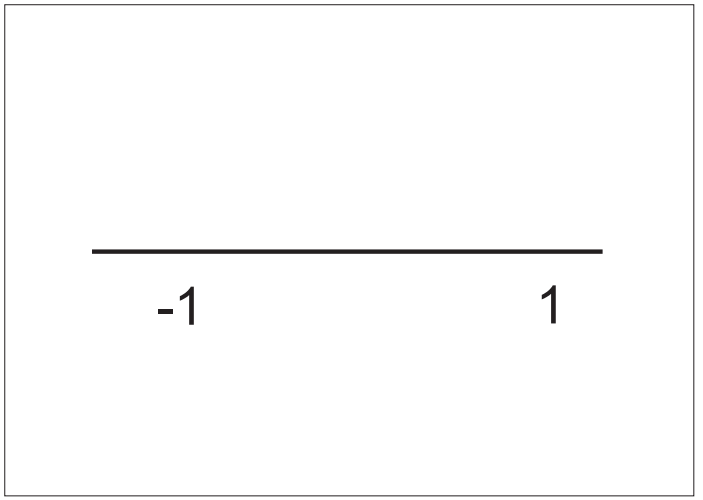


2) 1-dimensional plot.

Example: $\{\frac{1}{n}\}$.

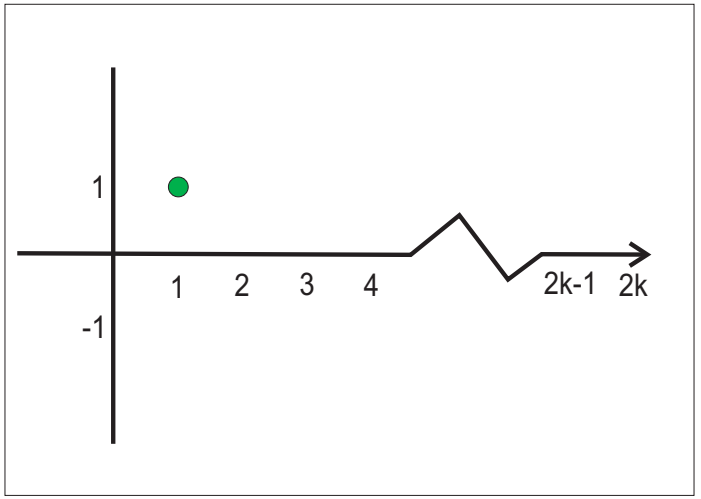
Graphical Representation of a Sequence

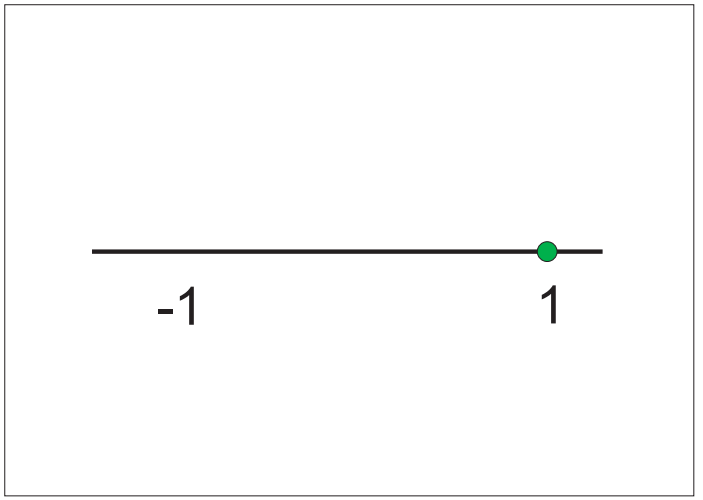




Example: $a_n = (-1)^{n+1}$

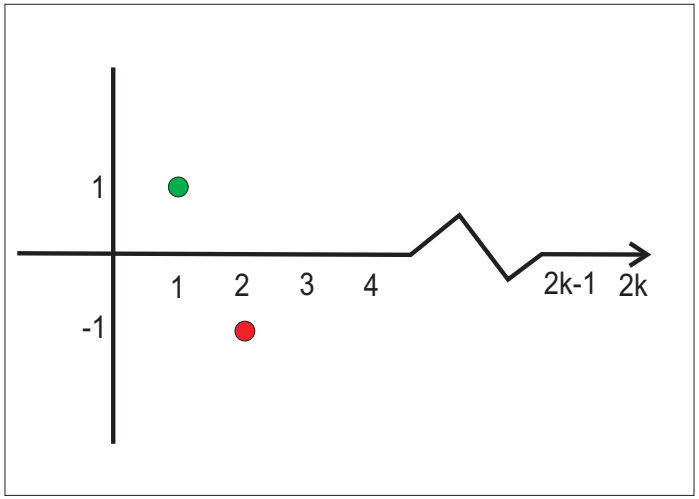
Graphical Representation of a Sequence

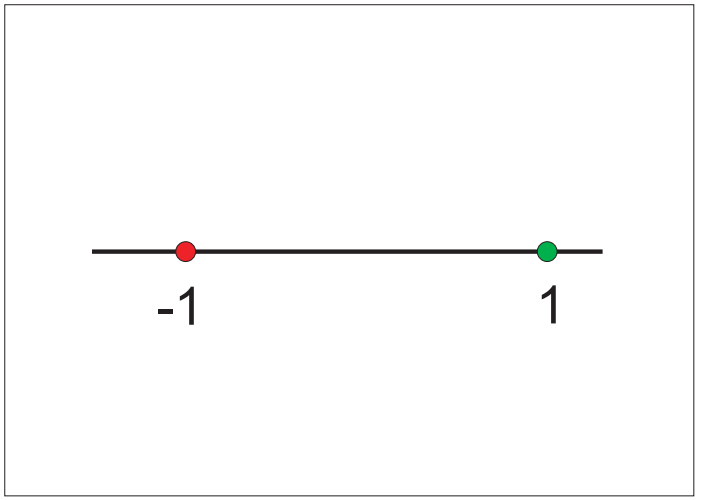




Example: $a_n = (-1)^{n+1} \Rightarrow \{1,$

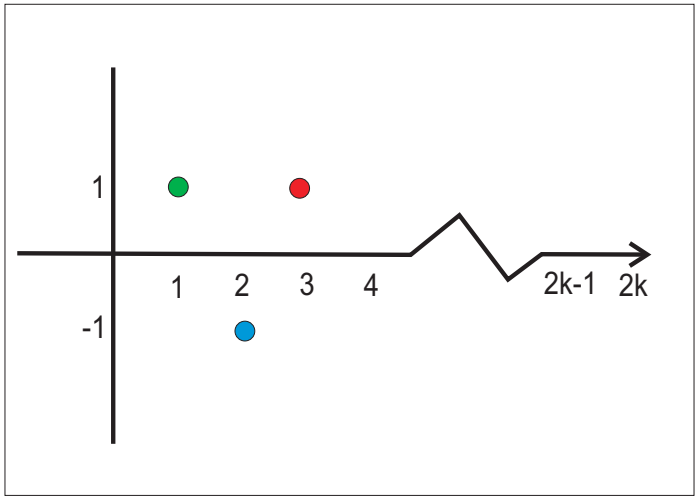
Graphical Representation of a Sequence

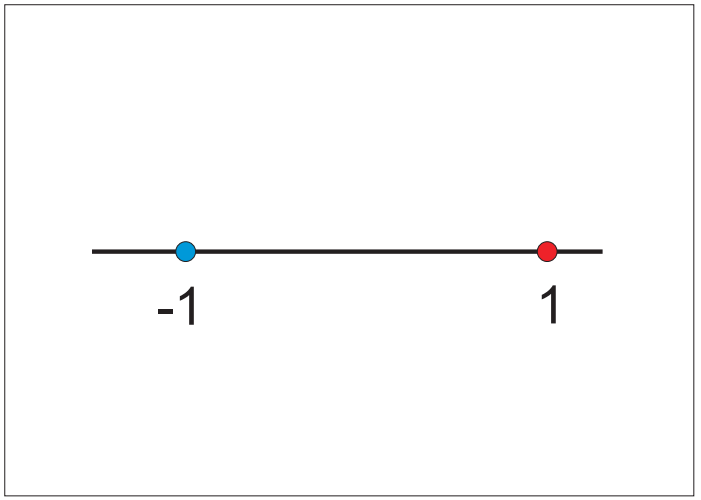




Example: $a_n = (-1)^{n+1} \Rightarrow \{1, -1,$

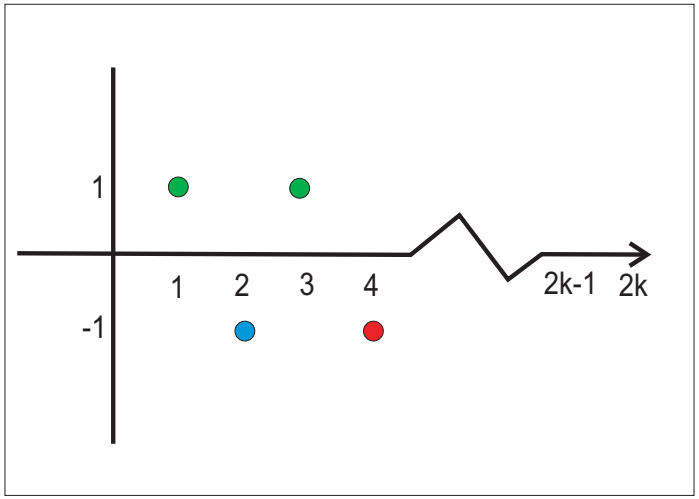
Graphical Representation of a Sequence

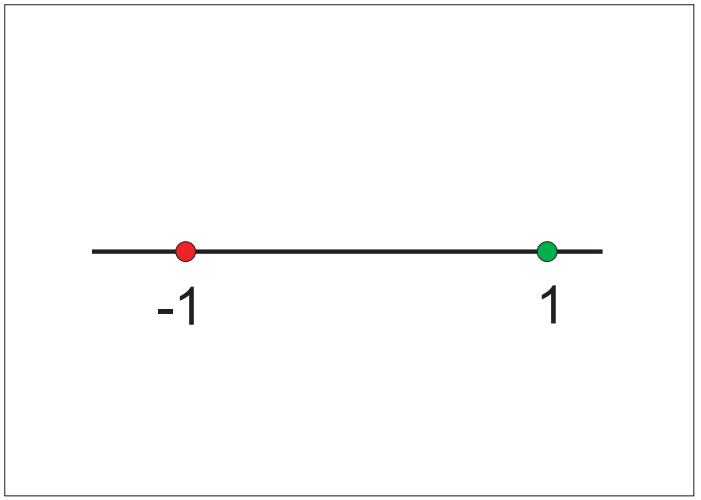




Example: $a_n = (-1)^{n+1} \Rightarrow \{1, -1, 1,$

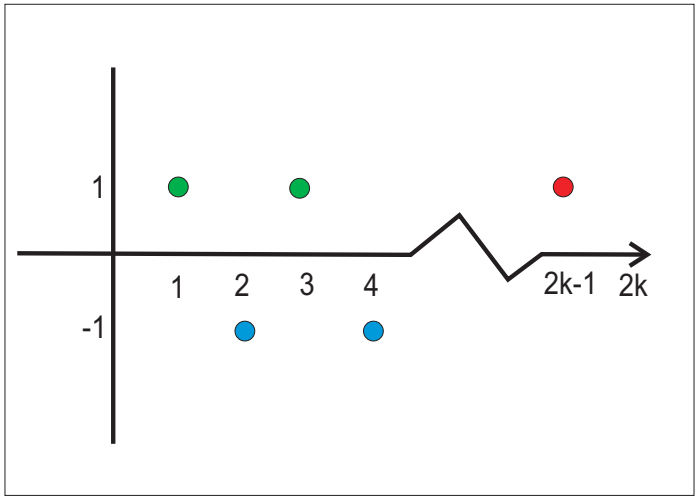
Graphical Representation of a Sequence

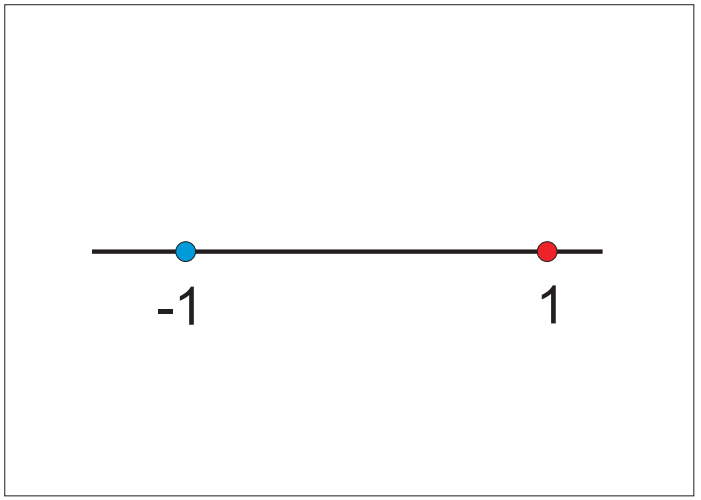




Example: $a_n = (-1)^{n+1} \Rightarrow \{1, -1, 1, -1,$

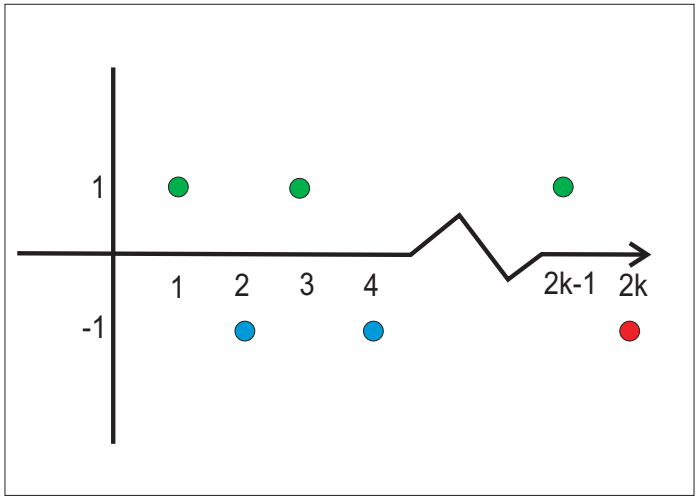
Graphical Representation of a Sequence

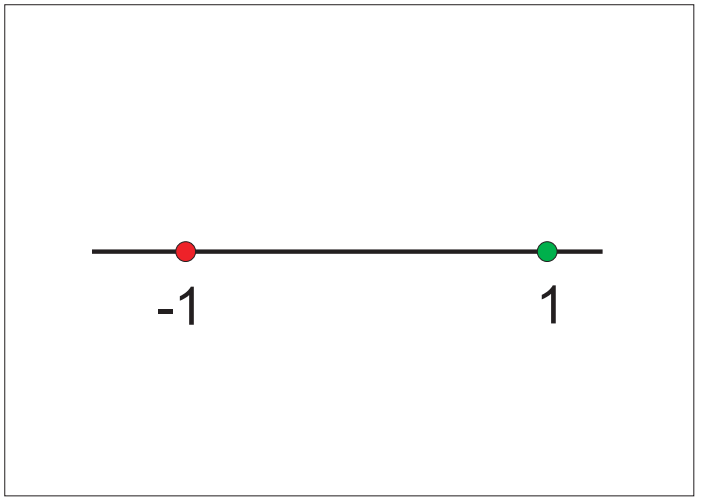




Example: $a_n = (-1)^{n+1} \Rightarrow \{1, -1, 1, -1, \ldots 1,$

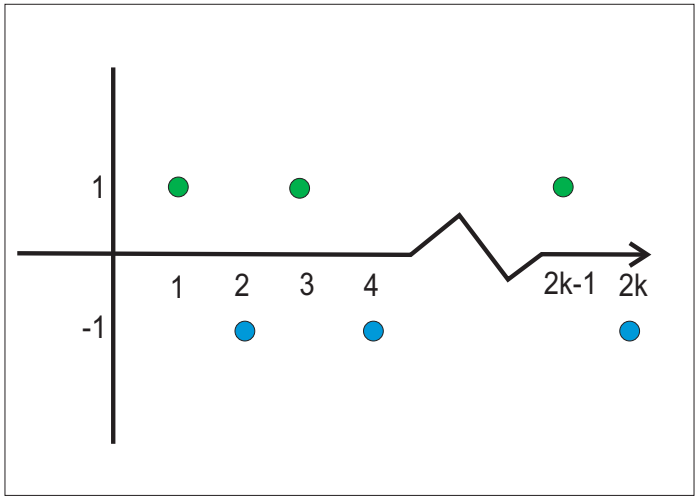
Graphical Representation of a Sequence

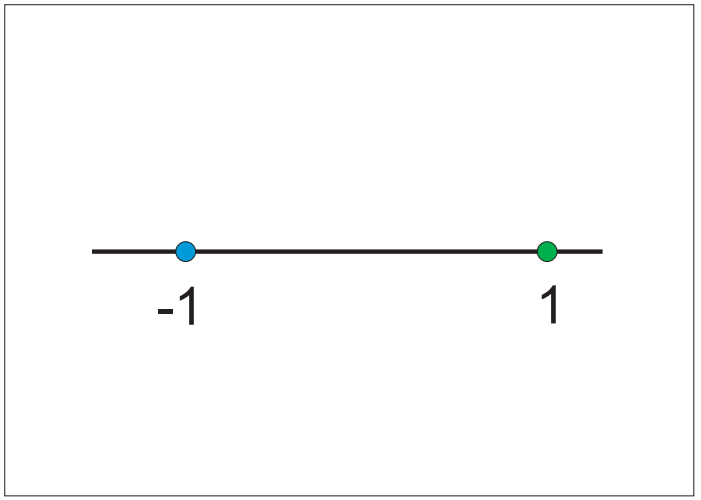




Example: $a_n = (-1)^{n+1} \Rightarrow \{1, -1, 1, -1, \ldots 1, -1,$

Graphical Representation of a Sequence





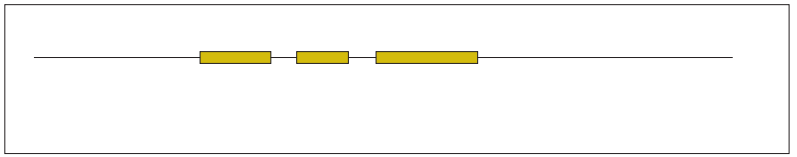
Example: $a_n = (-1)^{n+1} \Rightarrow \{1, -1, 1, -1, \ldots 1, -1, \ldots\}$

Least Upper Bound Property

Brian Forrest

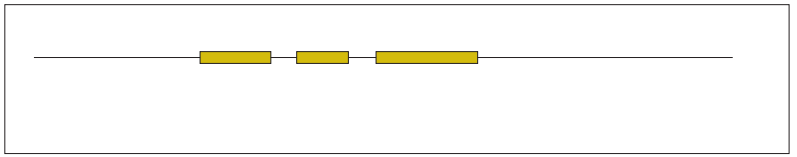
September 5, 2010

Bounded Sets



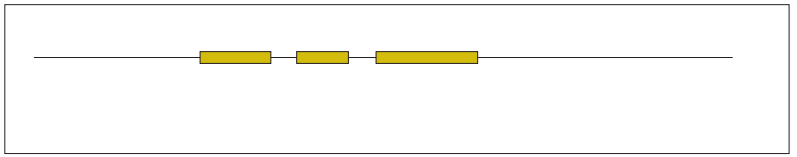
Definition:

Bounded Sets



Definition: Let $S \subset \mathbb{R}$

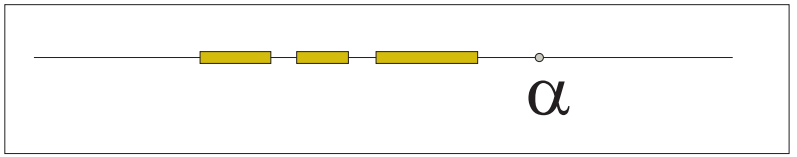
Bounded Sets



Definition: Let $S \subset \mathbb{R}$

1. α is an *upper bound* for S

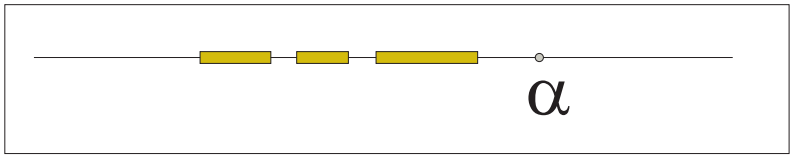
Bounded Sets



Definition: Let $S \subset \mathbb{R}$

1. α is an *upper bound* for S if $x \leq \alpha$ for all $x \in S$.

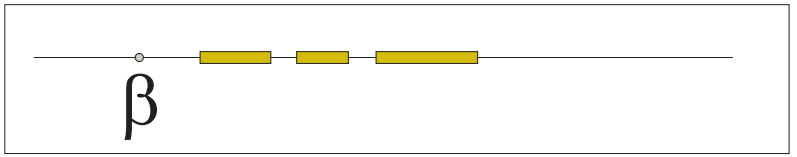
Bounded Sets



Definition: Let $S \subset \mathbb{R}$

1. α is an *upper bound* for S if $x \leq \alpha$ for all $x \in S$.
2. β is an *lower bound* for S

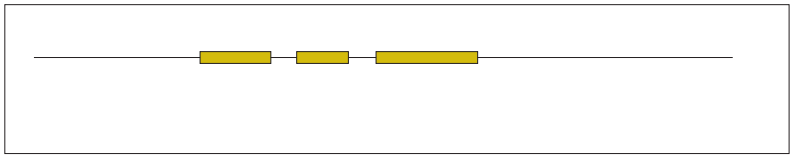
Bounded Sets



Definition: Let $S \subset \mathbb{R}$

1. α is an *upper bound* for S if $x \leq \alpha$ for all $x \in S$.
2. β is an *lower bound* for S if $\beta \leq x$ for all $x \in S$.

Bounded Sets

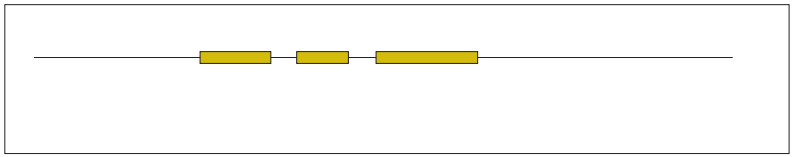


Definition: Let $S \subset \mathbb{R}$

1. α is an *upper bound* for S if $x \leq \alpha$ for all $x \in S$.
2. β is an *lower bound* for S if $\beta \leq x$ for all $x \in S$.

Definition::

Bounded Sets

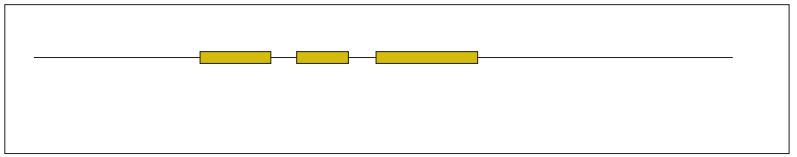


Definition: Let $S \subset \mathbb{R}$

1. α is an *upper bound* for S if $x \leq \alpha$ for all $x \in S$.
2. β is an *lower bound* for S if $\beta \leq x$ for all $x \in S$.

Definition::Let $S \subset \mathbb{R}$

Bounded Sets



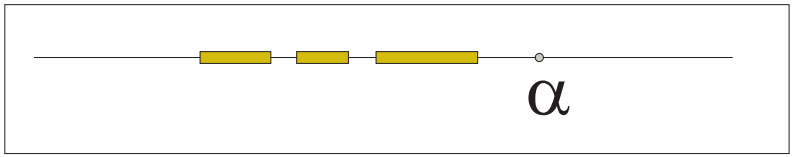
Definition: Let $S \subset \mathbb{R}$

1. α is an *upper bound* for S if $x \leq \alpha$ for all $x \in S$.
2. β is an *lower bound* for S if $\beta \leq x$ for all $x \in S$.

Definition::Let $S \subset \mathbb{R}$

1. S is *bounded above*

Bounded Sets



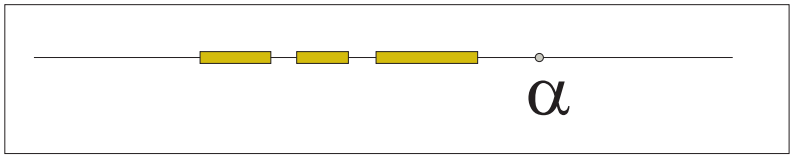
Definition: Let $S \subset \mathbb{R}$

1. α is an *upper bound* for S if $x \leq \alpha$ for all $x \in S$.
2. β is an *lower bound* for S if $\beta \leq x$ for all $x \in S$.

Definition::Let $S \subset \mathbb{R}$

1. S is *bounded above* if S has an upper bound α.

Bounded Sets



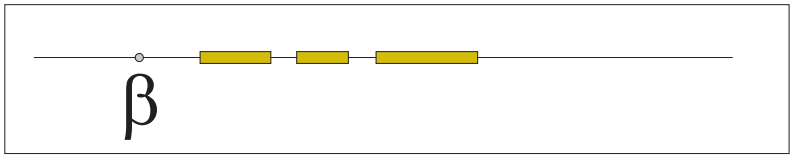
Definition: Let $S \subset \mathbb{R}$

1. α is an *upper bound* for S if $x \leq \alpha$ for all $x \in S$.
2. β is an *lower bound* for S if $\beta \leq x$ for all $x \in S$.

Definition::Let $S \subset \mathbb{R}$

1. S is *bounded above* if S has an upper bound α.
2. S is *bounded below*

Bounded Sets



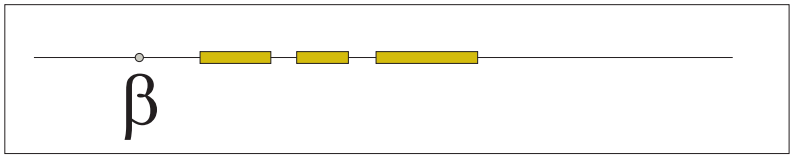
Definition: Let $S \subset \mathbb{R}$

1. α is an *upper bound* for S if $x \leq \alpha$ for all $x \in S$.
2. β is an *lower bound* for S if $\beta \leq x$ for all $x \in S$.

Definition::Let $S \subset \mathbb{R}$

1. S is *bounded above* if S has an upper bound α.
2. S is *bounded below* if S has an lower bound β.

Bounded Sets



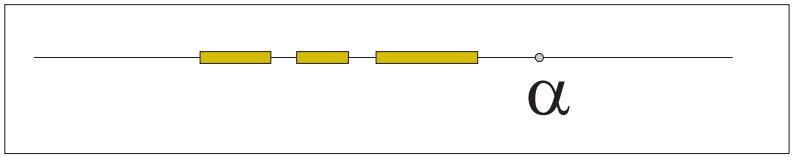
Definition: Let $S \subset \mathbb{R}$

1. α is an *upper bound* for S if $x \leq \alpha$ for all $x \in S$.
2. β is an *lower bound* for S if $\beta \leq x$ for all $x \in S$.

Definition::Let $S \subset \mathbb{R}$

1. S is *bounded above* if S has an upper bound α.
2. S is *bounded below* if S has an lower bound β.
3. S is *bounded*

Bounded Sets



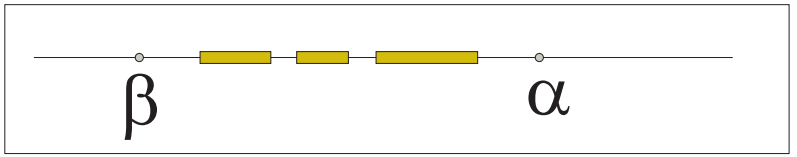
Definition: Let $S \subset \mathbb{R}$

1. α is an *upper bound* for S if $x \leq \alpha$ for all $x \in S$.
2. β is an *lower bound* for S if $\beta \leq x$ for all $x \in S$.

Definition::Let $S \subset \mathbb{R}$

1. S is *bounded above* if S has an upper bound α.
2. S is *bounded below* if S has an lower bound β.
3. S is *bounded* if S is bound above

Bounded Sets



Definition: Let $S \subset \mathbb{R}$

1. α is an *upper bound* for S if $x \leq \alpha$ for all $x \in S$.
2. β is an *lower bound* for S if $\beta \leq x$ for all $x \in S$.

Definition::Let $S \subset \mathbb{R}$

1. S is *bounded above* if S has an upper bound α.
2. S is *bounded below* if S has an lower bound β.
3. S is *bounded* if S is bound above and bounded below.

Is $\mathbb{N}$ bounded?

Problem:

Is $\mathbb{N}$ bounded?

Problem: Is $\mathbb{N}$ bounded?

Is $\mathbb{N}$ bounded?

Problem: Is $\mathbb{N}$ bounded?

$\mathbb{N}$ is bounded below by 1.

Is $\mathbb{N}$ bounded?

Problem: Is $\mathbb{N}$ bounded?

$\mathbb{N}$ is bounded below by 1.

Problem:

Is $\mathbb{N}$ bounded?

Problem: Is $\mathbb{N}$ bounded?

$\mathbb{N}$ is bounded below by 1.

Problem: Is $\mathbb{N}$ bounded above?

Is $\mathbb{N}$ bounded?

Argument 1:

Is $\mathbb{N}$ bounded?

Argument 1: $\mathbb{N}$ is bounded above because it has no largest element!

Is $\mathbb{N}$ bounded?

Argument 1: $\mathbb{N}$ is bounded above because it has no largest element!

False!

Is $\mathbb{N}$ bounded?

Argument 1: $\mathbb{N}$ is bounded above because it has no largest element!

False! $[0, 1)$ is bounded above,

Is $\mathbb{N}$ bounded?

Argument 1: $\mathbb{N}$ is bounded above because it has no largest element!

False! $[0, 1)$ is bounded above, but it has no largest element.

Is $\mathbb{N}$ bounded?

Argument 2:

Is $\mathbb{N}$ bounded?

Argument 2: Assume $\alpha \in \mathbb{R}$ is an upper bound for $\mathbb{N}$.

Is $\mathbb{N}$ bounded?

Argument 2: Assume $\alpha \in \mathbb{R}$ is an upper bound for $\mathbb{N}$. Write

$$\alpha = m.a_1 a_2 a_3 \cdots$$

where $m \in \mathbb{N}$.

Is $\mathbb{N}$ bounded?

Argument 2: Assume $\alpha \in \mathbb{R}$ is an upper bound for $\mathbb{N}$. Write

$$\alpha = m.a_1a_2a_3\cdots$$

where $m \in \mathbb{N}$.

Then

- $m+1 \in \mathbb{N}$

Is $\mathbb{N}$ bounded?

Argument 2: Assume $\alpha \in \mathbb{R}$ is an upper bound for $\mathbb{N}$. Write

$$\alpha = m.a_1 a_2 a_3 \cdots$$

where $m \in \mathbb{N}$.

Then

- $m + 1 \in \mathbb{N}$
- $m + 1 > \alpha$.

Is $\mathbb{N}$ bounded?

Argument 2: Assume $\alpha \in \mathbb{R}$ is an upper bound for $\mathbb{N}$. Write

$$\alpha = m.a_1a_2a_3\cdots$$

where $m \in \mathbb{N}$.

Then

- $m+1 \in \mathbb{N}$
- $m+1 > \alpha$.

This shows that no such α could be an upper bound.

Is $\mathbb{N}$ bounded?

Problem:

Is $\mathbb{N}$ bounded?

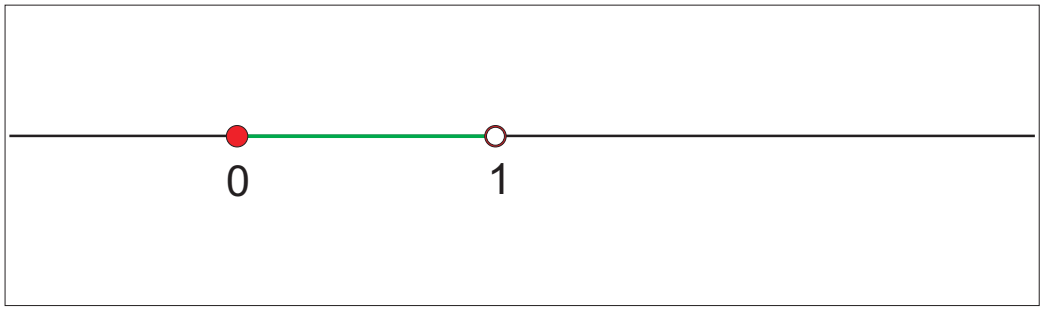
Problem: How do we know that every real number has a decimal expansion?

Is $\mathbb{N}$ bounded?

Problem: How do we know that every real number has a decimal expansion?

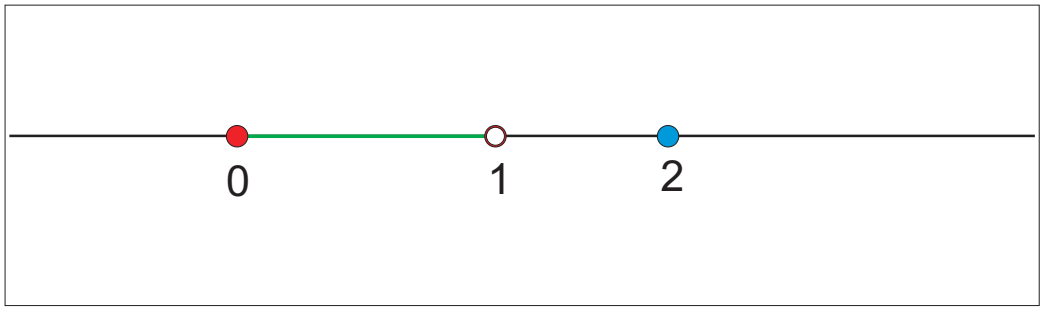
Fact: The existence of a decimal expansion is essentially equivalent to $\mathbb{N}$ being bounded.

$[0, 1)$



Let $S = [0, 1)$.

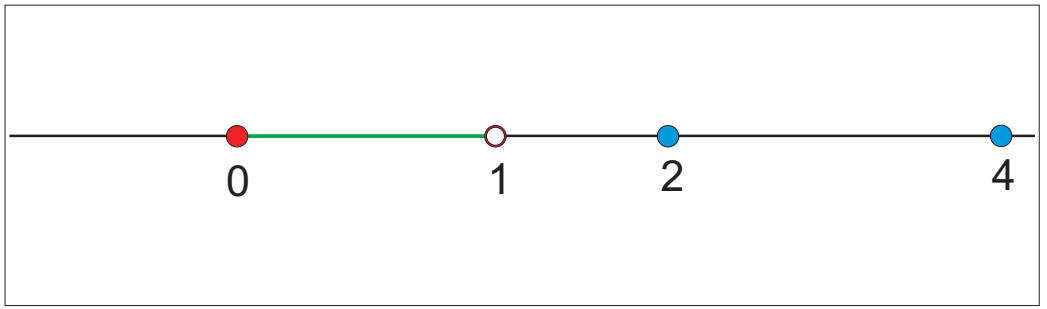
$[0, 1)$



Let $S = [0, 1)$.

- S is bounded above by 2.

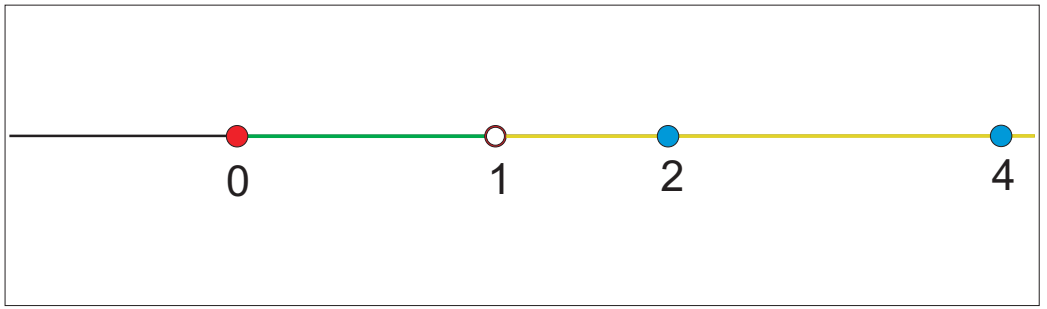
$[0, 1)$



Let $S = [0, 1)$.

- S is bounded above by 2.
- S is also bounded above by 4.

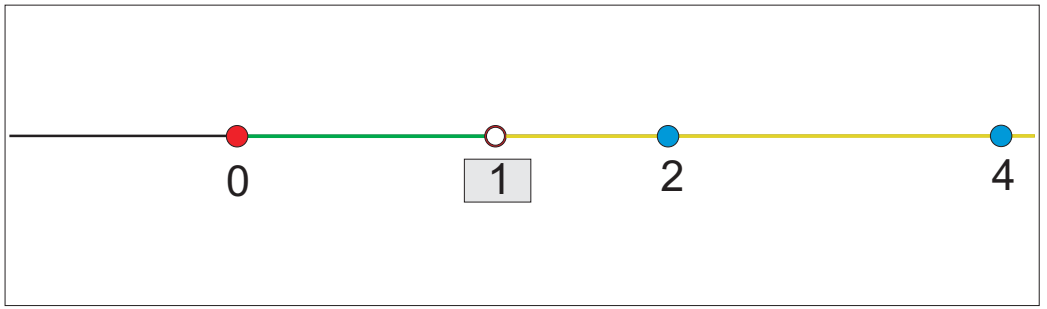
$[0, 1)$



Let $S = [0, 1)$.

- S is bounded above by 2.
- S is also bounded above by 4.
- S has infinitely many upper bounds.

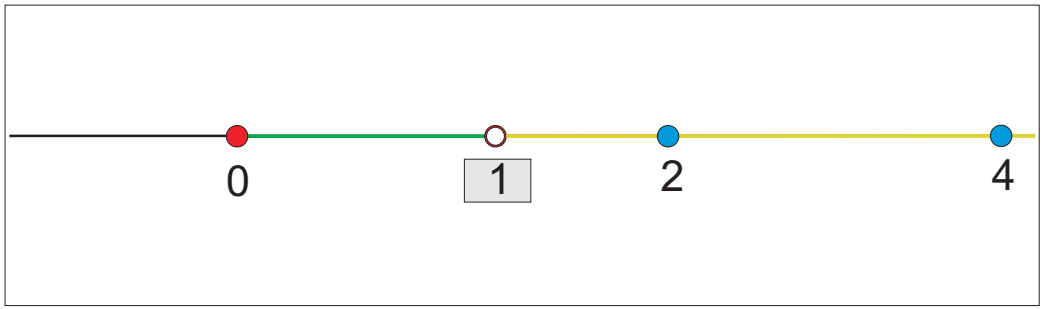
$[0, 1)$



Let $S = [0, 1)$.

- S is bounded above by 2.
- S is also bounded above by 4.
- S has infinitely many upper bounds.
- 1 is a *special* upper bound for S.

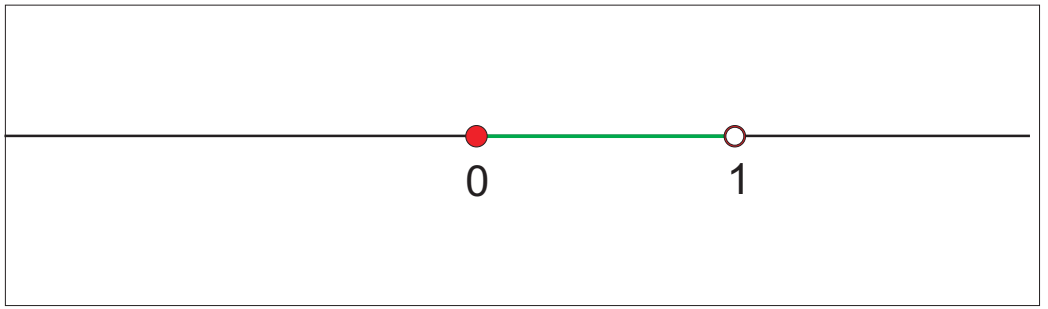
$[0, 1)$



Let $S = [0, 1)$.

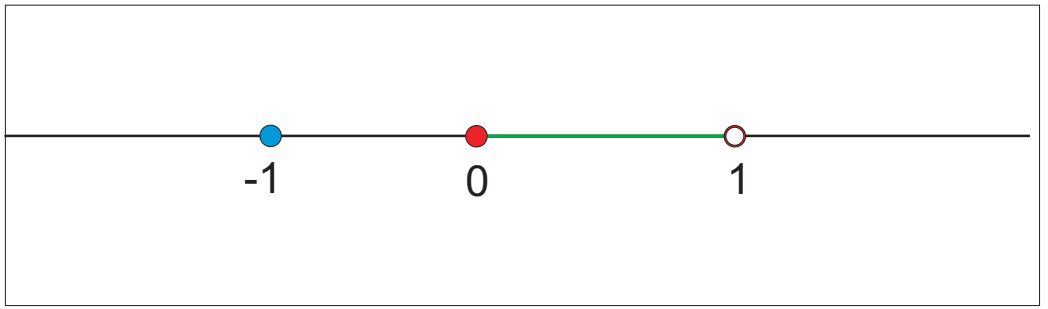
- S is bounded above by 2.
- S is also bounded above by 4.
- S has infinitely many upper bounds.
- 1 is a *special* upper bound for S.
- 1 is the smallest or *least upper bound* for S.

$[0, 1)$



Let $S = [0, 1)$.

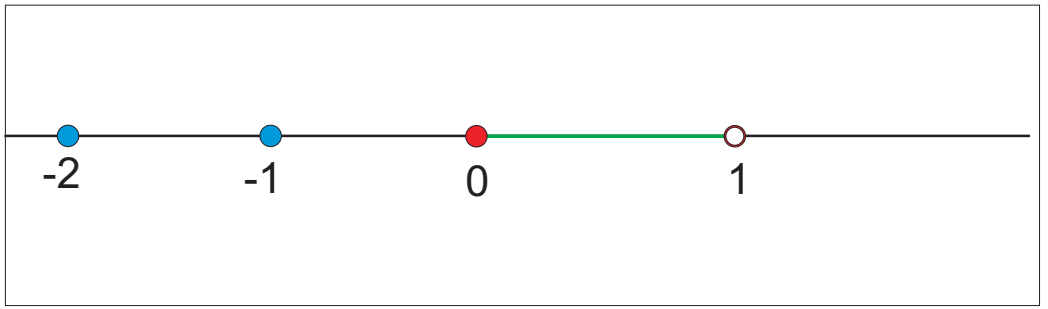
$[0, 1)$



Let $S = [0, 1)$.

- S is bounded below by -1.

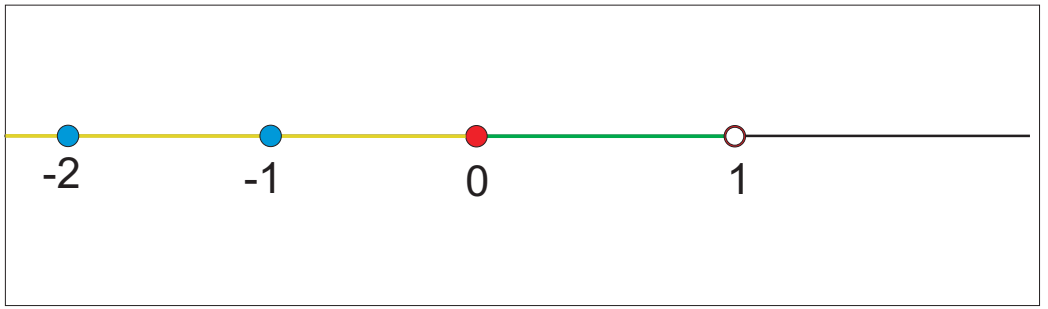
$[0, 1)$



Let $S = [0, 1)$.

- S is bounded below by -1.
- S is also bounded below by -2.

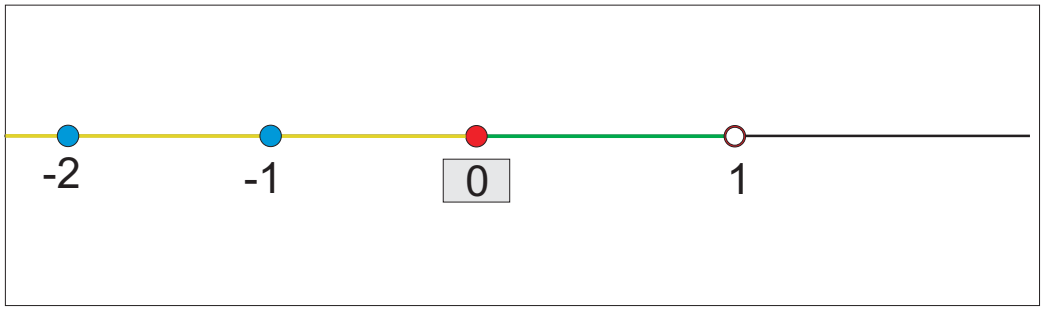
$[0, 1)$



Let $S = [0, 1)$.

- S is bounded below by -1.
- S is also bounded below by -2.
- S has infinitely many lower bounds.

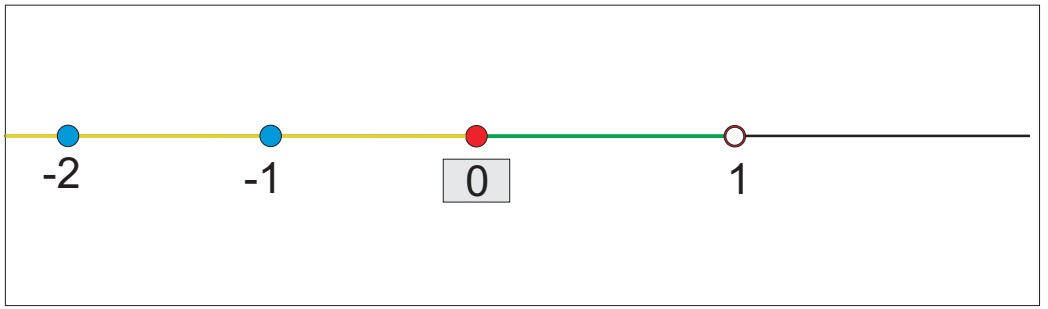
$[0, 1)$



Let $S = [0, 1)$.

- S is bounded below by -1.
- S is also bounded below by -2.
- S has infinitely many lower bounds.
- 0 is a *special* lower bound for S.

$[0, 1)$



Let $S = [0, 1)$.

- S is bounded below by -1.
- S is also bounded below by -2.
- S has infinitely many lower bounds.
- 0 is a *special* lower bound for S.
- 0 is the largest or *greatest lower bound* for S.

Least Upper Bound

Definition:

Least Upper Bound

Definition: We say that $\alpha \in \mathbb{R}$ is the *least upper bound* for a set $S \subset \mathbb{R}$ if

Least Upper Bound

Definition: We say that $\alpha \in \mathbb{R}$ is the *least upper bound* for a set $S \subset \mathbb{R}$ if

1. α is an upper bound for S, and

Least Upper Bound

Definition: We say that $\alpha \in \mathbb{R}$ is the *least upper bound* for a set $S \subset \mathbb{R}$ if

1. α is an upper bound for S, and
2. if γ is an upper bound for S,

Least Upper Bound

Definition: We say that $\alpha \in \mathbb{R}$ is the *least upper bound* for a set $S \subset \mathbb{R}$ if

1. α is an upper bound for S, and
2. if γ is an upper bound for S, then $\alpha \leq \gamma$.

Least Upper Bound

Definition: We say that $\alpha \in \mathbb{R}$ is the *least upper bound* for a set $S \subset \mathbb{R}$ if

1. α is an upper bound for S, and
2. if γ is an upper bound for S, then $\alpha \leq \gamma$.

If a set S has a least upper bound

Least Upper Bound

Definition: We say that $\alpha \in \mathbb{R}$ is the *least upper bound* for a set $S \subset \mathbb{R}$ if

1. α is an upper bound for S, and
2. if γ is an upper bound for S, then $\alpha \leq \gamma$.

If a set S has a least upper bound , then we denote it by $lub(S)$.

Least Upper Bound

Definition: We say that $\alpha \in \mathbb{R}$ is the *least upper bound* for a set $S \subset \mathbb{R}$ if

1. α is an upper bound for S, and
2. if γ is an upper bound for S, then $\alpha \leq \gamma$.

If a set S has a least upper bound , then we denote it by $lub(S)$.

The least upper bound of S is often called the *supremum* of S,

Least Upper Bound

Definition: We say that $\alpha \in \mathbb{R}$ is the *least upper bound* for a set $S \subset \mathbb{R}$ if

1. α is an upper bound for S, and
2. if γ is an upper bound for S, then $\alpha \leq \gamma$.

If a set S has a least upper bound , then we denote it by $lub(S)$.

The least upper bound of S is often called the *supremum* of S, denoted by $sup(S)$.

Greatest Lower Bound

Definition:

Greatest Lower Bound

Definition: We say that $\beta \in \mathbb{R}$ is the *greatest lower bound* for a set $S \subset \mathbb{R}$ if

Greatest Lower Bound

Definition: We say that $\beta \in \mathbb{R}$ is the *greatest lower bound* for a set $S \subset \mathbb{R}$ if

1. β is a lower bound for S, and

Greatest Lower Bound

Definition: We say that $\beta \in \mathbb{R}$ is the *greatest lower bound* for a set $S \subset \mathbb{R}$ if

1. β is a lower bound for S, and
2. if γ is a lower bound for S,

Greatest Lower Bound

Definition: We say that $\beta \in \mathbb{R}$ is the *greatest lower bound* for a set $S \subset \mathbb{R}$ if

1. β is a lower bound for S, and
2. if γ is a lower bound for S, then $\beta \geq \gamma$.

Greatest Lower Bound

Definition: We say that $\beta \in \mathbb{R}$ is the *greatest lower bound* for a set $S \subset \mathbb{R}$ if

1. β is a lower bound for S, and
2. if γ is a lower bound for S, then $\beta \geq \gamma$.

If a set S has a greatest bound

Greatest Lower Bound

Definition: We say that $\beta \in \mathbb{R}$ is the *greatest lower bound* for a set $S \subset \mathbb{R}$ if

1. β is a lower bound for S, and
2. if γ is a lower bound for S, then $\beta \geq \gamma$.

If a set S has a greatest bound , then we denote it by $glb(S)$.

Greatest Lower Bound

Definition: We say that $\beta \in \mathbb{R}$ is the *greatest lower bound* for a set $S \subset \mathbb{R}$ if

1. β is a lower bound for S, and
2. if γ is a lower bound for S, then $\beta \geq \gamma$.

If a set S has a greatest bound , then we denote it by $glb(S)$.

The greatest lower bound of S is often called the *infimum* of S,

Greatest Lower Bound

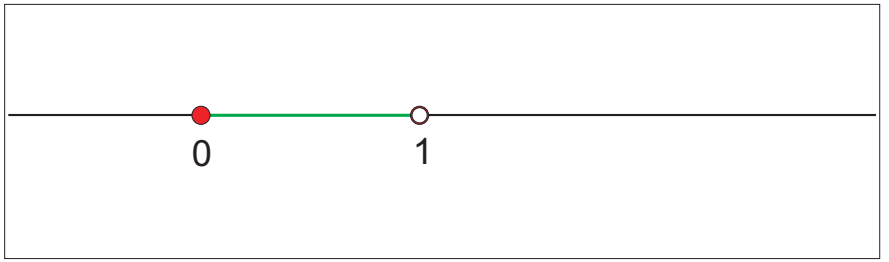
Definition: We say that $\beta \in \mathbb{R}$ is the *greatest lower bound* for a set $S \subset \mathbb{R}$ if

1. β is a lower bound for S, and
2. if γ is a lower bound for S, then $\beta \geq \gamma$.

If a set S has a greatest bound , then we denote it by $glb(S)$.

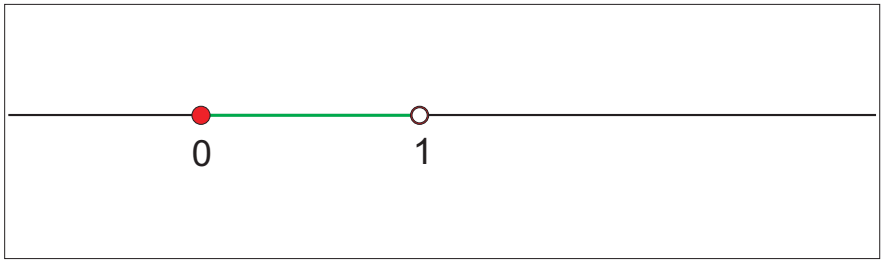
The greatest lower bound of S is often called the *infimum* of S, denoted by $inf(S)$.

$[0, 1)$



Let $S = [0, 1)$.

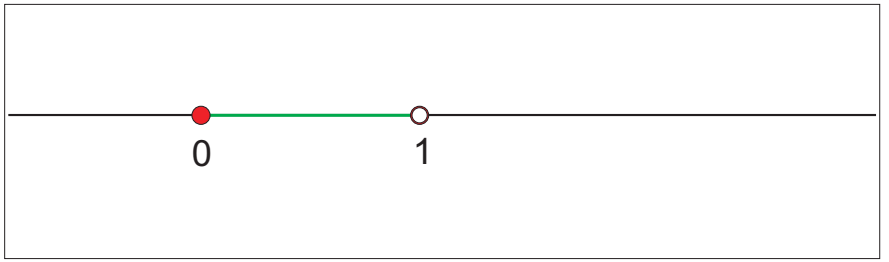
$[0, 1)$



Let $S = [0, 1)$.

- $lub(S) = 1$.

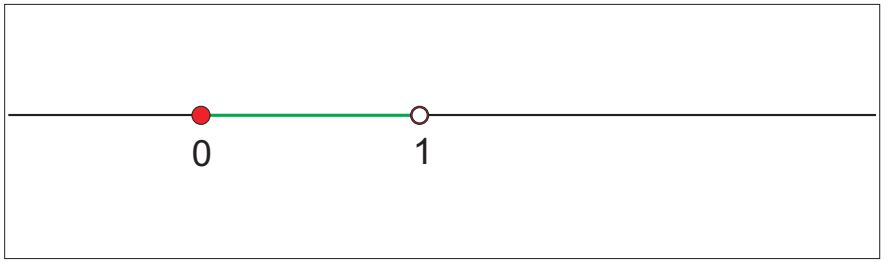
$[0, 1)$



Let $S = [0, 1)$.

- $lub(S) = 1$.
- $glb(S) = 0$.

$[0, 1)$

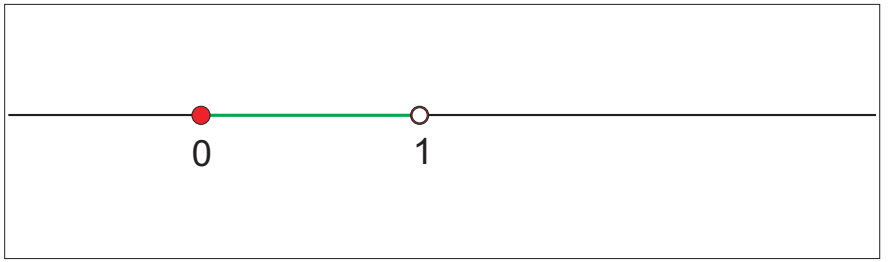


Let $S = [0, 1)$.

- $lub(S) = 1$.
- $glb(S) = 0$.

Note:

$[0, 1)$

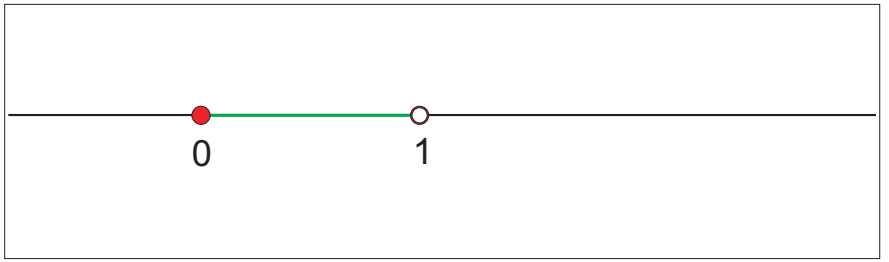


Let $S = [0, 1)$.

- $lub(S) = 1$.
- $glb(S) = 0$.

Note: $glb(S) = 0 \in S$

$[0, 1)$



Let $S = [0, 1)$.

- $lub(S) = 1$.
- $glb(S) = 0$.

Note: $glb(S) = 0 \in S$ but $lub(S) = 1 \notin S$.

Max and Min

Definition:

Max and Min

Definition:

1. If S contains $\alpha = lub(S)$, then

Max and Min

Definition:

1. If S contains $\alpha = lub(S)$, then α is called the *maximum* of S

Max and Min

Definition:

1. If S contains $\alpha = lub(S)$, then α is called the *maximum* of S ($\Rightarrow \alpha = max(S)$).

Max and Min

Definition:

1. If S contains $\alpha = lub(S)$, then α is called the *maximum* of S ($\Rightarrow \alpha = max(S)$).
2. If S contains $\beta = glb(S)$,

Max and Min

Definition:

1. If S contains $\alpha = lub(S)$, then α is called the *maximum* of S ($\Rightarrow \alpha = max(S)$).
2. If S contains $\beta = glb(S)$, then β is called the *minimum* of S

Max and Min

Definition:

1. If S contains $\alpha = lub(S)$, then α is called the *maximum* of S ($\Rightarrow \alpha = max(S)$).
2. If S contains $\beta = glb(S)$, then β is called the *minimum* of S ($\Rightarrow \beta = min(S)$).

Max and Min

Definition:

1. If S contains $\alpha = lub(S)$, then α is called the *maximum* of S ($\Rightarrow \alpha = max(S)$).
2. If S contains $\beta = glb(S)$, then β is called the *minimum* of S ($\Rightarrow \beta = min(S)$).

If S is a finite set with n elements

$$S = \{a_1 < a_2 < \cdots < a_n\},$$

Max and Min

Definition:

1. If S contains $\alpha = lub(S)$, then α is called the *maximum* of S ($\Rightarrow \alpha = max(S)$).
2. If S contains $\beta = glb(S)$, then β is called the *minimum* of S ($\Rightarrow \beta = min(S)$).

If S is a finite set with n elements

$$S = \{a_1 < a_2 < \cdots < a_n\},$$

then

- $a_n = lub(S) = max(S)$,

Max and Min

Definition:

1. If S contains $\alpha = lub(S)$, then α is called the *maximum* of S ($\Rightarrow \alpha = max(S)$).
2. If S contains $\beta = glb(S)$, then β is called the *minimum* of S ($\Rightarrow \beta = min(S)$).

If S is a finite set with n elements

$$S = \{a_1 < a_2 < \cdots < a_n\},$$

then

- $a_n = lub(S) = max(S)$,
- $a_1 = glb(S) = min(S)$.

Empty Set

Problem:

Empty Set

Problem: Does every set S that is bounded above have a LUB?

Empty Set

Problem: Does every set S that is bounded above have a LUB?

Problem: Is $\emptyset$ bounded?

Empty Set

Problem: Does every set S that is bounded above have a LUB?

Problem: Is $\emptyset$ bounded?

Claim:

Empty Set

Problem: Does every set S that is bounded above have a LUB?

Problem: Is $\emptyset$ bounded?

Claim: -13 is an upper bound for $\emptyset$.

Empty Set

Problem: Does every set S that is bounded above have a LUB?

Problem: Is $\emptyset$ bounded?

Claim: -13 is an upper bound for $\emptyset$.

Suppose not.

Empty Set

Problem: Does every set S that is bounded above have a LUB?

Problem: Is $\emptyset$ bounded?

Claim: -13 is an upper bound for $\emptyset$.

Suppose not.

- Then there exists $x \in \emptyset$ such that $x > -13$.

Empty Set

Problem: Does every set S that is bounded above have a LUB?

Problem: Is $\emptyset$ bounded?

Claim: -13 is an upper bound for $\emptyset$.

Suppose not.

- Then there exists $x \in \emptyset$ such that $x > -13$.
- No such x could exist!!!

Empty Set

Problem: Does every set S that is bounded above have a LUB?

Problem: Is $\emptyset$ bounded?

Claim: -13 is an upper bound for $\emptyset$.

Suppose not.

- Then there exists $x \in \emptyset$ such that $x > -13$.
- No such x could exist!!!
- So -13 **is** an upper bound of $\emptyset$.

Empty Set

In fact:

Empty Set

In fact:

- Every real number α is an upper bound of $\emptyset$.

Empty Set

In fact:

- Every real number α is an upper bound of $\emptyset$.
- $\emptyset$ is bounded above, but it has no least upper bound.

Empty Set

We saw that -13 is an upper bound for $\emptyset$.

Empty Set

We saw that -13 is an upper bound for $\emptyset$.

Claim:

Empty Set

We saw that -13 is an upper bound for $\emptyset$.

Claim: 7 is a lower bound for $\emptyset$.

Empty Set

We saw that -13 is an upper bound for $\emptyset$.

Claim: 7 is a lower bound for $\emptyset$.

Again, suppose not.

Empty Set

We saw that -13 is an upper bound for $\emptyset$.

Claim: 7 is a lower bound for $\emptyset$.

Again, suppose not.

- Then there exists $x \in \emptyset$ such that $x < 7$.

Empty Set

We saw that -13 is an upper bound for $\emptyset$.

Claim: 7 is a lower bound for $\emptyset$.

Again, suppose not.

- Then there exists $x \in \emptyset$ such that $x < 7$.
- No such x could exist!!!

Empty Set

We saw that -13 is an upper bound for $\emptyset$.

Claim: 7 is a lower bound for $\emptyset$.

Again, suppose not.

- Then there exists $x \in \emptyset$ such that $x < 7$.
- No such x could exist!!!
- So 7 **is** a lower bound of $\emptyset$.

Empty Set

We saw that -13 is an upper bound for $\emptyset$.

Claim: 7 is a lower bound for $\emptyset$.

Again, suppose not.

- Then there exists $x \in \emptyset$ such that $x < 7$.
- No such x could exist!!!
- So 7 **is** a lower bound of $\emptyset$.
- Every real number β is a lower of $\emptyset$.

Empty Set

We saw that -13 is an upper bound for $\emptyset$.

Claim: 7 is a lower bound for $\emptyset$.

Again, suppose not.

- Then there exists $x \in \emptyset$ such that $x < 7$.
- No such x could exist!!!
- So 7 **is** a lower bound of $\emptyset$.
- Every real number β is a lower of $\emptyset$.
- $\emptyset$ is bounded.

Empty Set

We saw that -13 is an upper bound for $\emptyset$.

Claim: 7 is a lower bound for $\emptyset$.

Again, suppose not.

- Then there exists $x \in \emptyset$ such that $x < 7$.
- No such x could exist!!!
- So 7 **is** a lower bound of $\emptyset$.
- Every real number β is a lower of $\emptyset$.
- $\emptyset$ is bounded.

Note:

Empty Set

We saw that -13 is an upper bound for $\emptyset$.

Claim: 7 is a lower bound for $\emptyset$.

Again, suppose not.

- Then there exists $x \in \emptyset$ such that $x < 7$.
- No such x could exist!!!
- So 7 **is** a lower bound of $\emptyset$.
- Every real number β is a lower of $\emptyset$.
- $\emptyset$ is bounded.

Note: $\emptyset$ is the only subset of $\mathbb{R}$ with an upper bound α and a lower bound β with

$$\alpha < \beta.$$

Least Upper Bound Property

Axiom: (Least Upper Bound Property or LUBP)

Least Upper Bound Property

Axiom: (Least Upper Bound Property or LUBP)

A **nonempty** subset $S \subset \mathbb{R}$ that is bounded above always has a least upper bound.

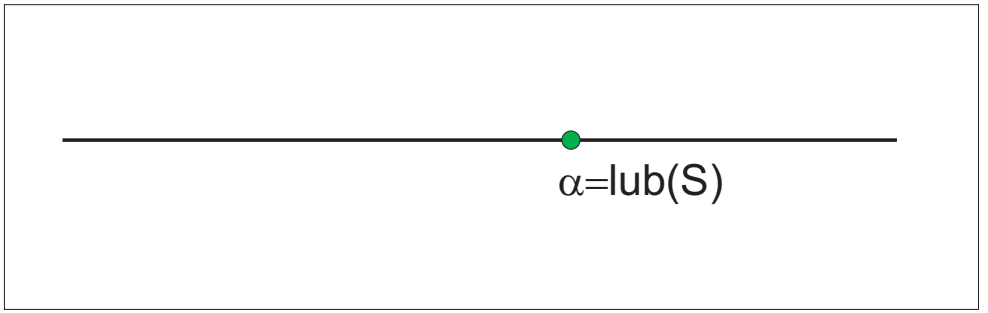
Least Upper Bound Property

Axiom: (Least Upper Bound Property or LUBP)

A **nonempty** subset $S \subset \mathbb{R}$ that is bounded above always has a least upper bound.

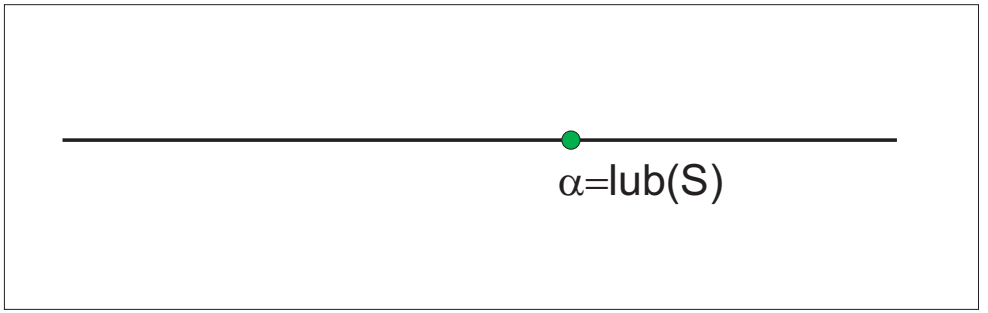
This Axiom plays a central role in the course!

Important Note



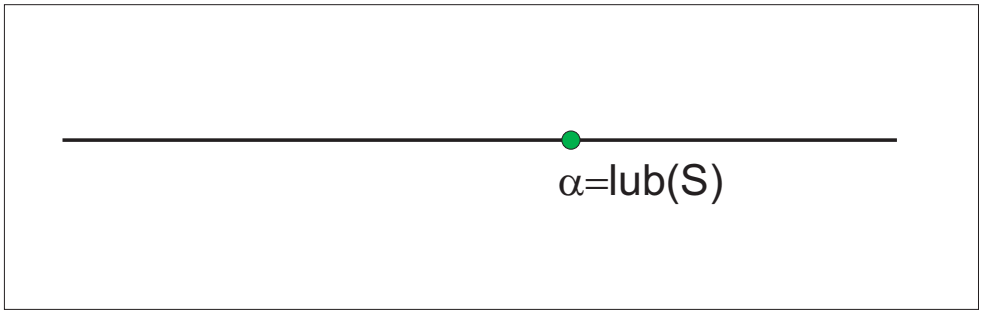
Note:

Important Note



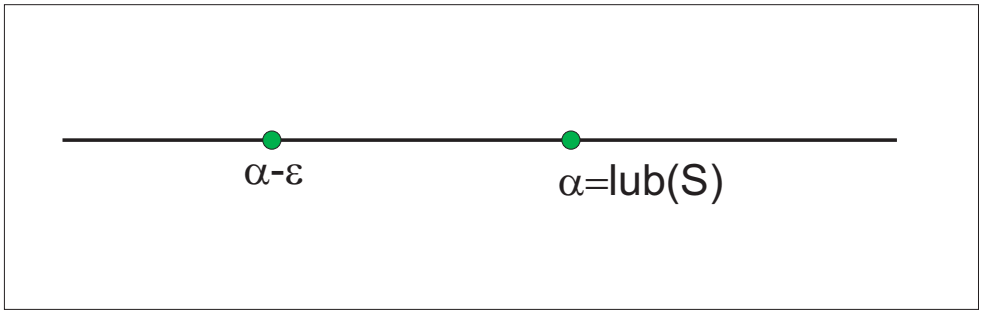
Note: If $\alpha = lub(S)$

Important Note



Note: If $\alpha = lub(S)$ and $\epsilon > 0$

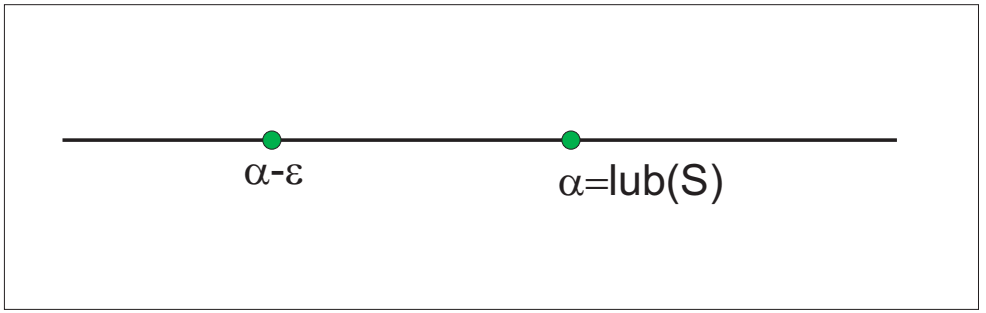
Important Note



Note: If $\alpha = lub(S)$ and $\epsilon > 0$ then:

- $\alpha - \epsilon < \alpha$

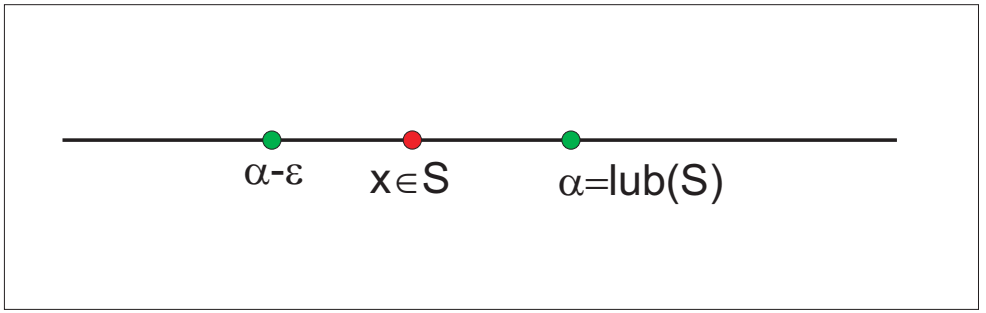
Important Note



Note: If $\alpha = lub(S)$ and $\epsilon > 0$ then:

- $\alpha - \epsilon < \alpha \implies \alpha$ is not an upper bound of S.

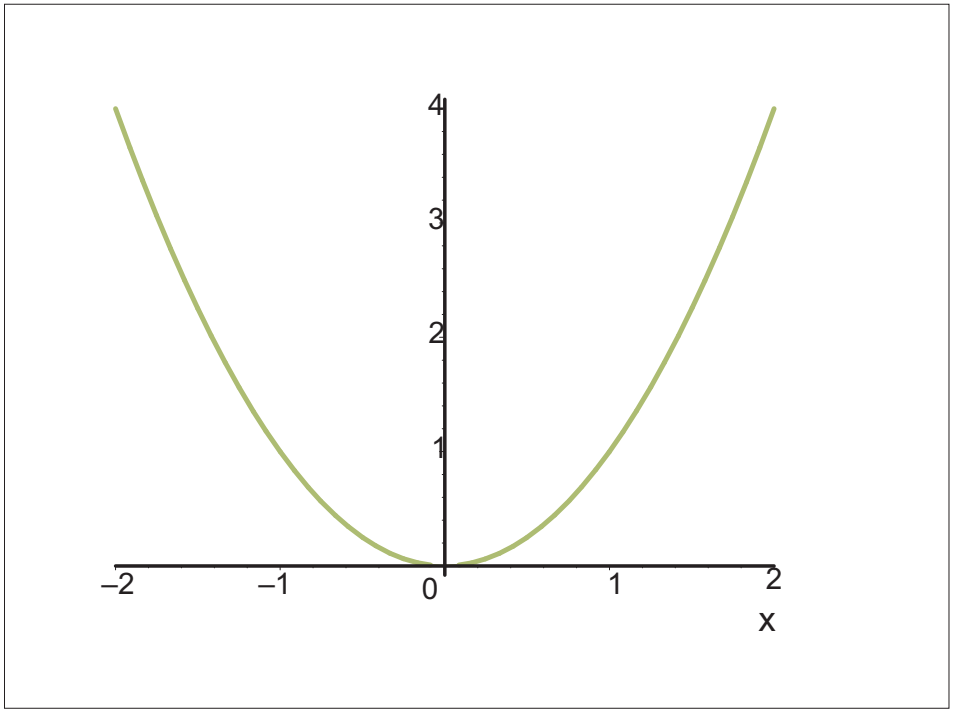
Important Note



Note: If $\alpha = lub(S)$ and $\epsilon > 0$ then:

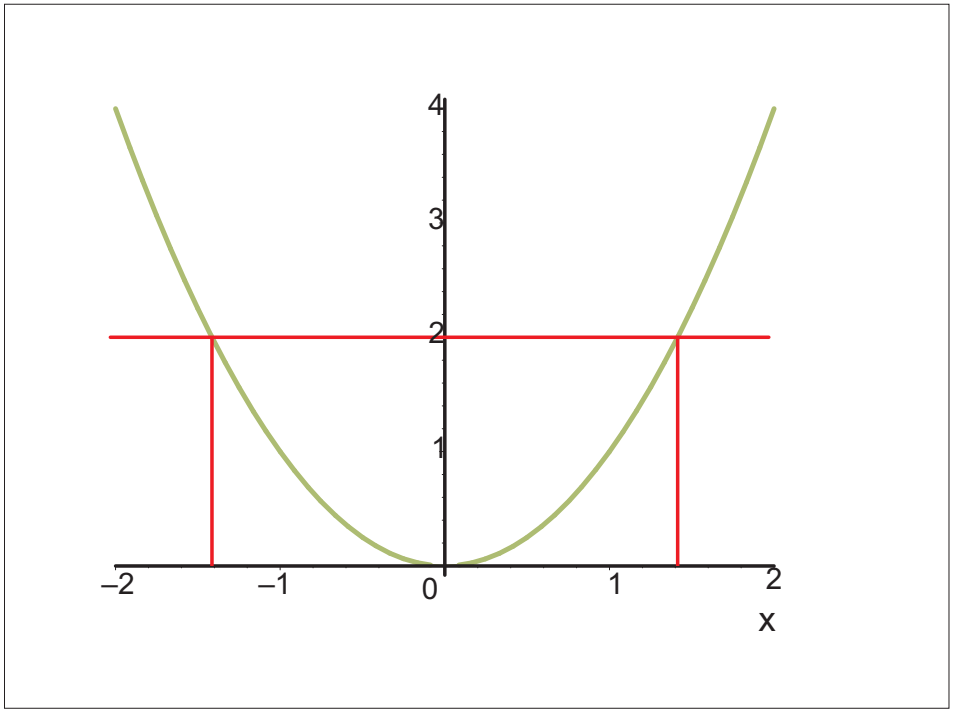
- $\alpha - \epsilon < \alpha \implies \alpha$ is not an upper bound of S.
- There is an $x \in S$ with $\alpha - \epsilon < x \leq \alpha$.

Example



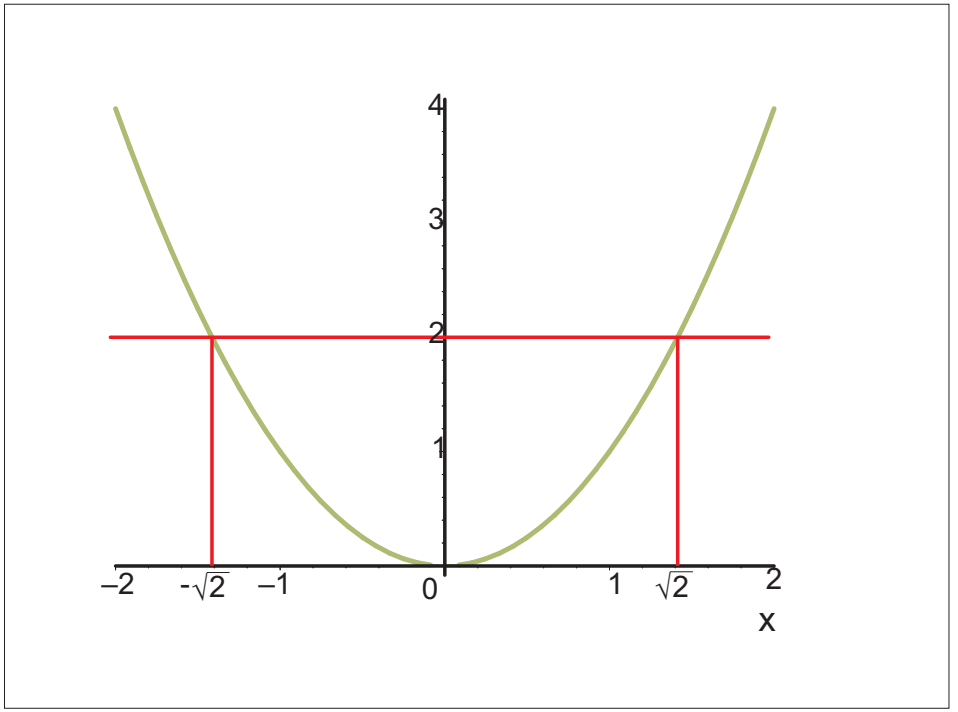
$$S = \{x \in \mathbb{Q} \mid x^2 < 2\}$$

Example



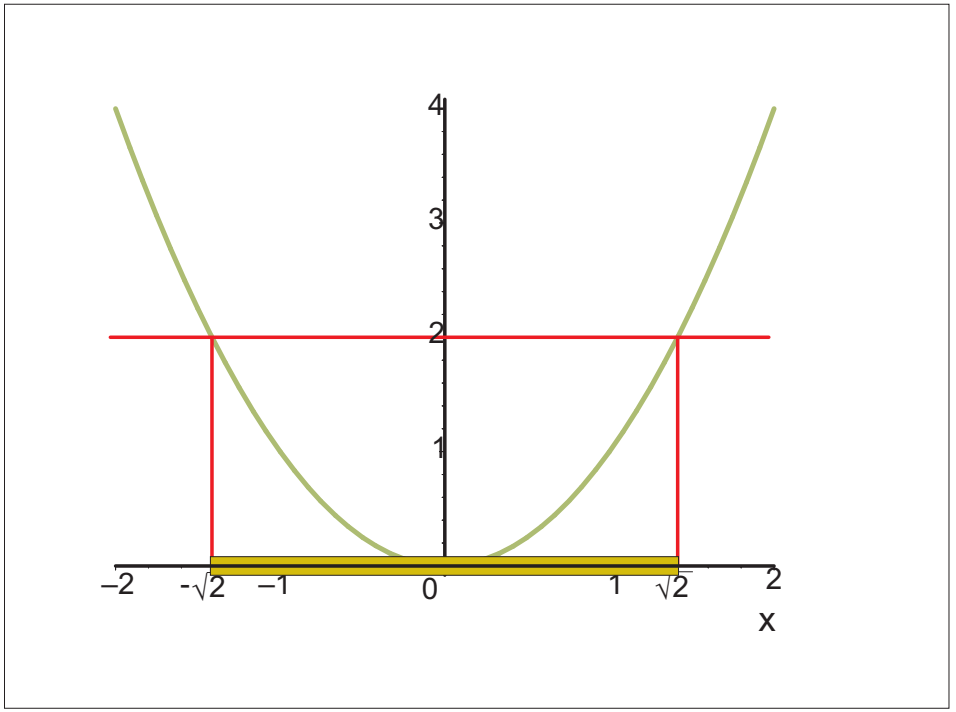
$$S = \{x \in \mathbb{Q} \mid x^2 < 2\}$$

Example



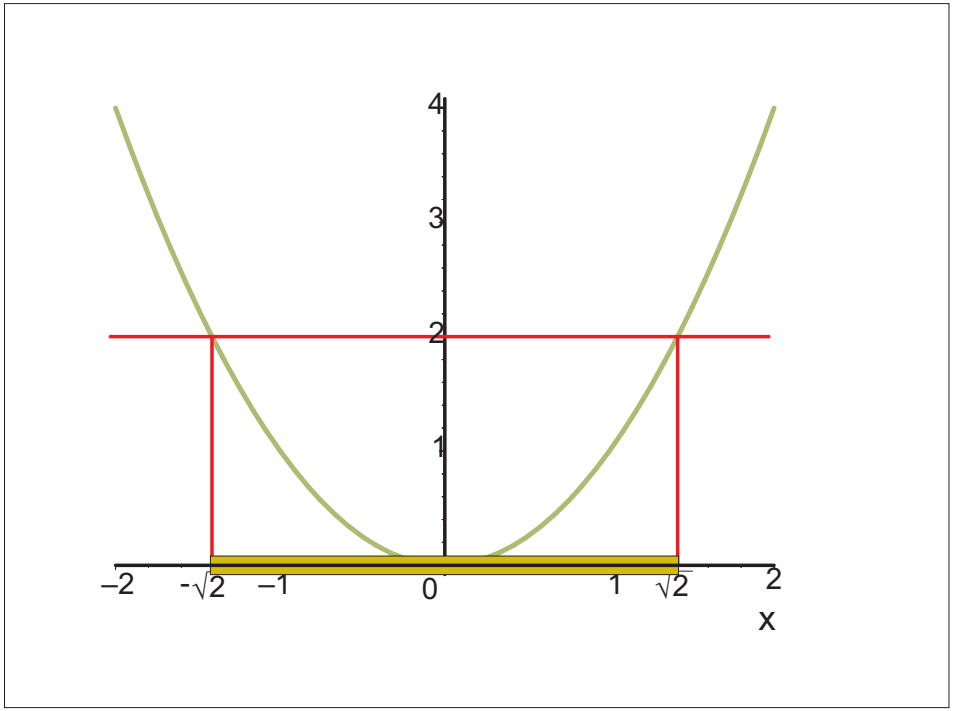
$$S = \{x \in \mathbb{Q} \mid x^2 < 2\}$$

Example



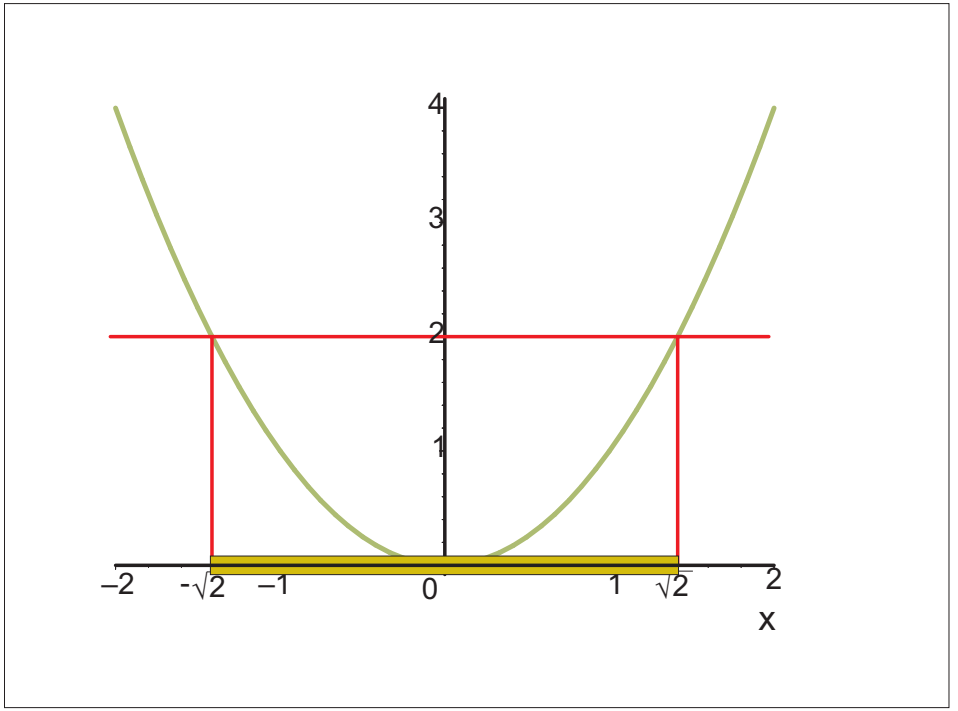
$$\begin{aligned} S &= \{x \in \mathbb{Q} \mid x^2 < 2\} \\ &\Rightarrow \quad lub(S) = \sqrt{2}, glb(S) = -\sqrt{2}. \end{aligned}$$

Example



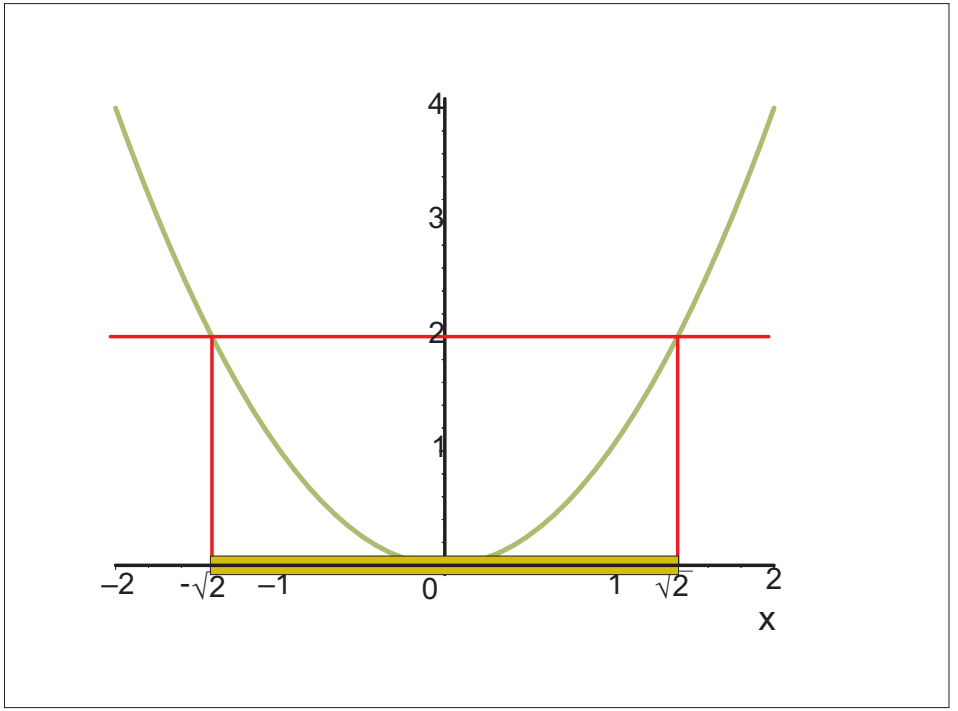
Problem:

Example



Problem: What happens to $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ if we only consider rational numbers rather than real numbers?

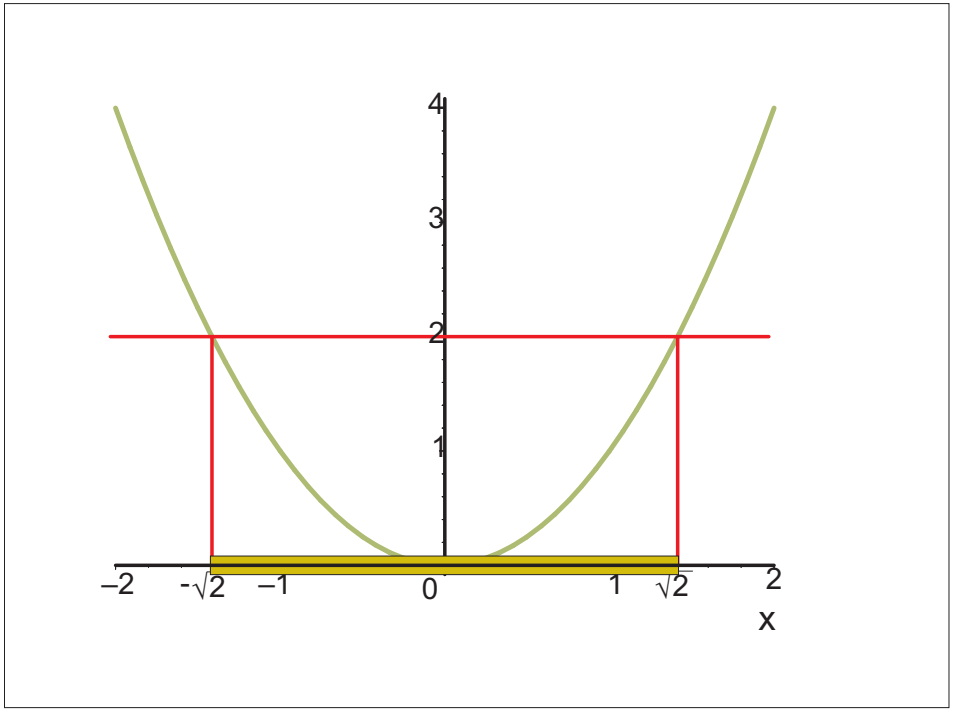
Example



Problem: What happens to $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ if we only consider rational numbers rather than real numbers?

- S has no lub in $\mathbb{Q}$.

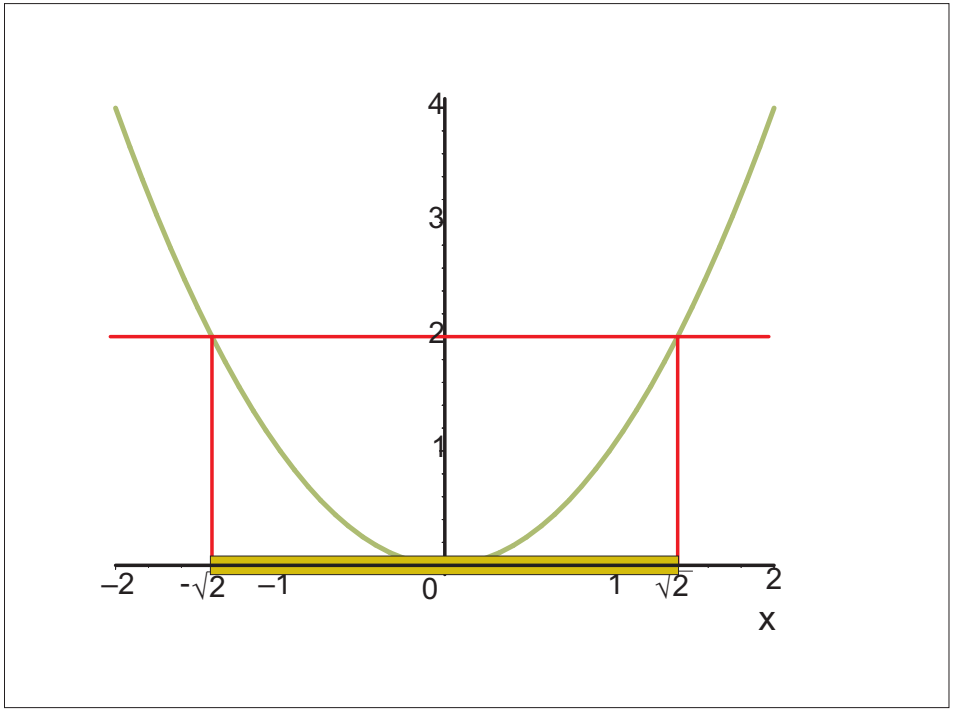
Example



Problem: What happens to $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ if we only consider rational numbers rather than real numbers?

- S has no lub in $\mathbb{Q}$.
- S has no glb in $\mathbb{Q}$.

Example

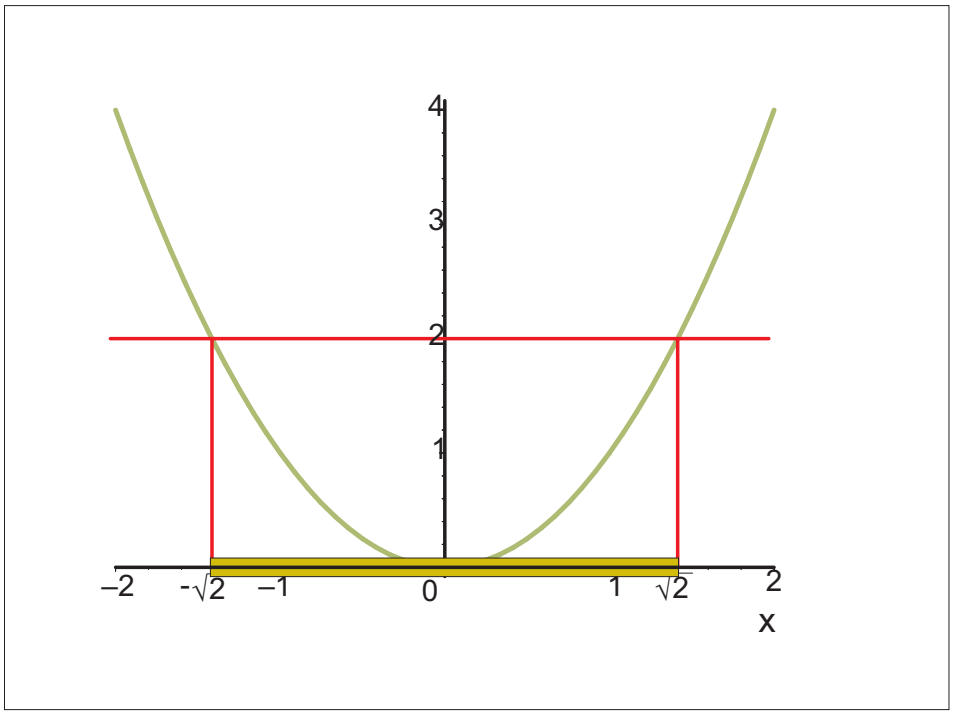


Problem: What happens to $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ if we only consider rational numbers rather than real numbers?

- S has no lub in $\mathbb{Q}$.
- S has no glb in $\mathbb{Q}$.

Conclusion:

Example



Problem: What happens to $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ if we only consider rational numbers rather than real numbers?

- S has no lub in $\mathbb{Q}$.
- S has no glb in $\mathbb{Q}$.

Conclusion: $\mathbb{Q}$ does not have the Least Upper Bound Property.

Is $\mathbb{N}$ bounded?

Problem:

Is $\mathbb{N}$ bounded?

Problem: Is $\mathbb{N}$ bounded?