Limits of Sequences II

Brian Forrest

September 27, 2010

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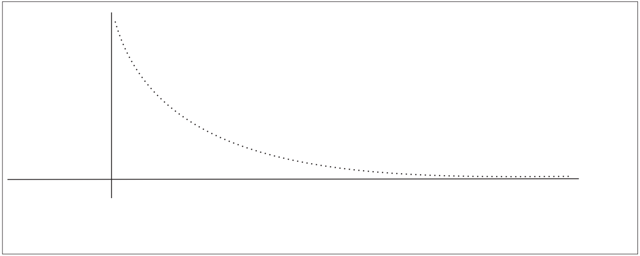
We may also say $\{a_n\}$ converges to L and write $a_n \to L$.

If no such L exits, we say that $\{a_n\}$ *diverges*.

Example



Example: Show that $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$.

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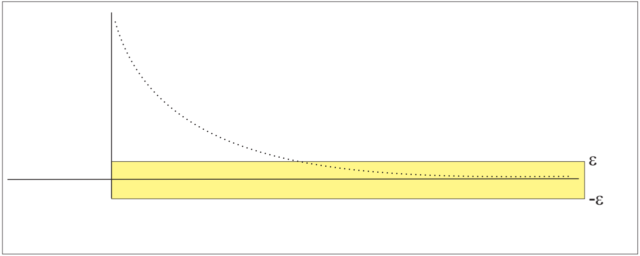

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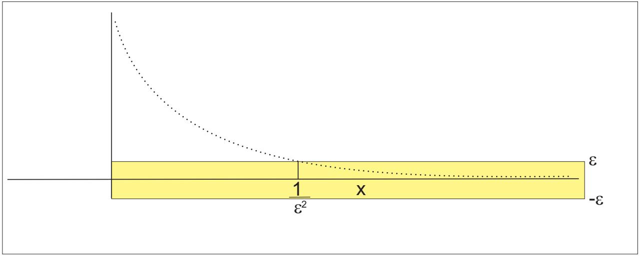

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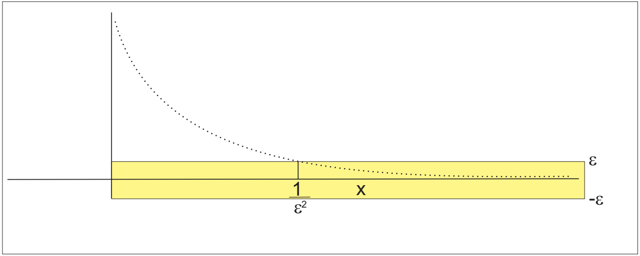

Example: Show that $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$. Let $\epsilon > 0$.

- $x > \frac{1}{\epsilon^2} \Rightarrow \epsilon^2 > \frac{1}{x}$

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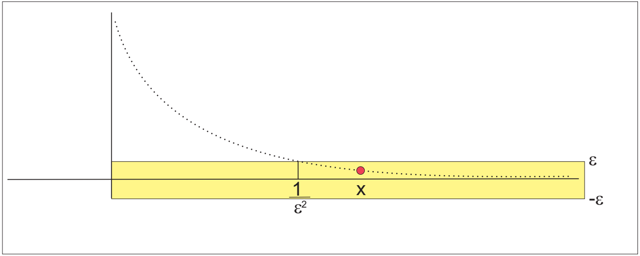

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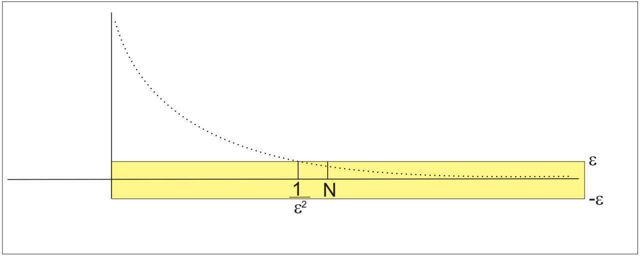

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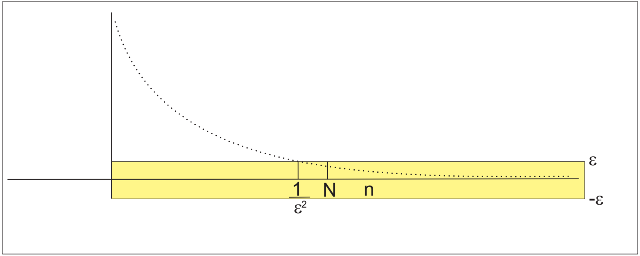

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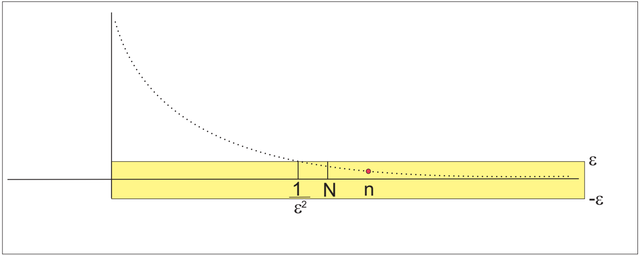

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- If $\frac{1}{\epsilon^2} < N \leq n \Rightarrow \mid \frac{1}{\sqrt{n}} - 0 \mid < \epsilon$.

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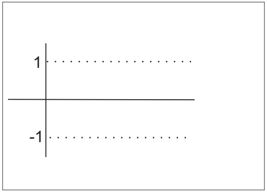

Example:

Consider $\{(-1)^{n+1}\} = \{1, -1, 1, -1, \ldots\}$.

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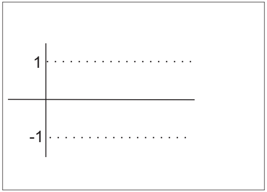

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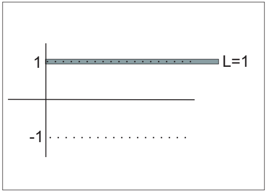

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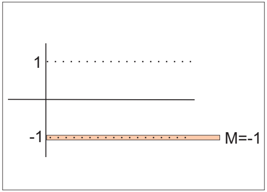

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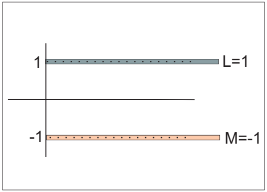

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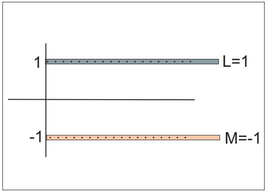

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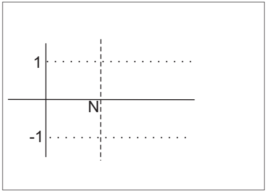

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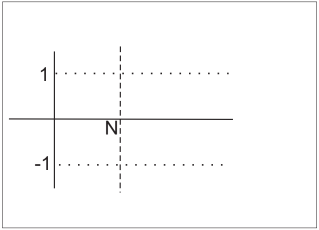

Pick $k_0 \in \mathbb{N}$ such that $2k_0 + 1 \geq N$.

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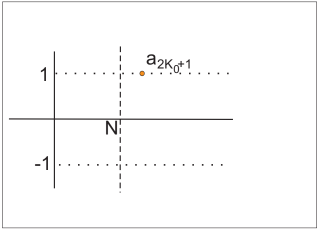

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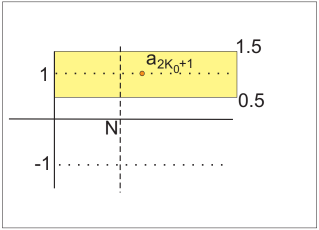

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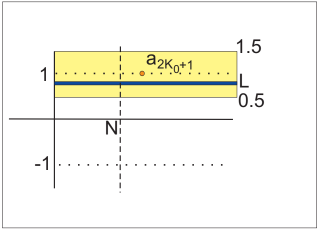

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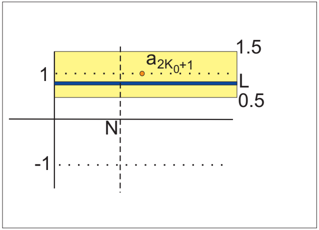

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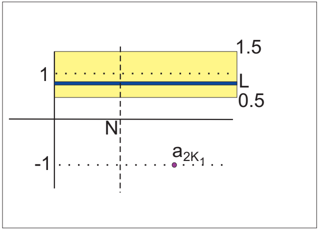

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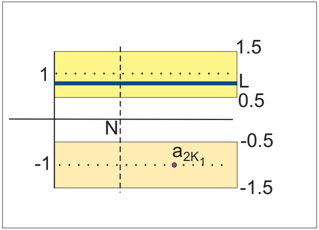

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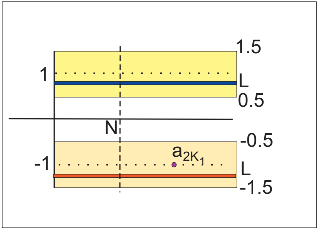

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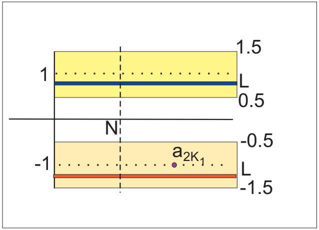

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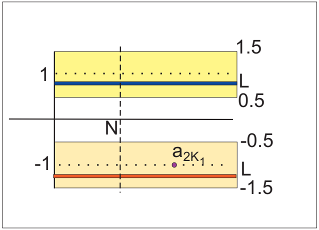

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Therefore, $\{(-1)^{n+1}\}$ has no limit!

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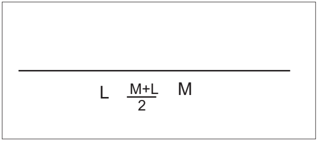

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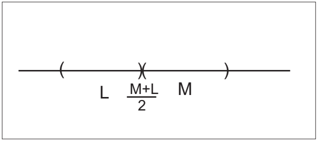

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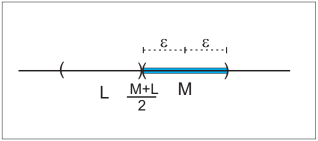

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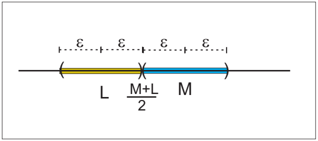

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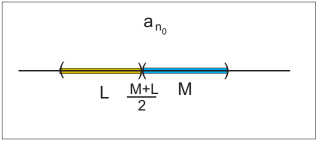

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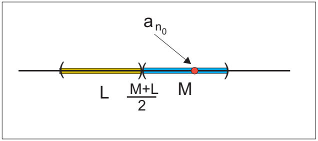

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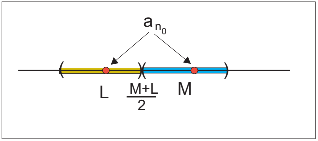

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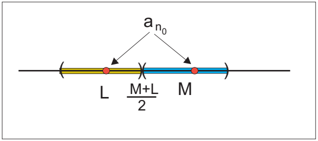

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which is impossible!

Uniqueness of Limits

Theorem: (Uniqueness of Limits)

Assume that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$. Then

$$L = M.$$

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Does $\{a_n\}$ converge? If so, what is $\lim_{n\to\infty} a_n$?

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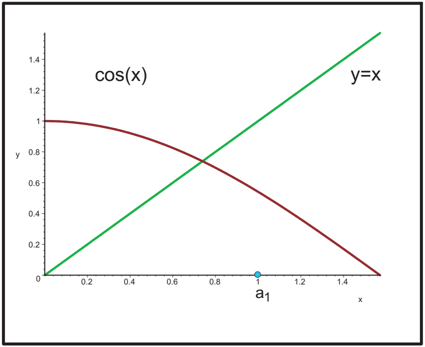

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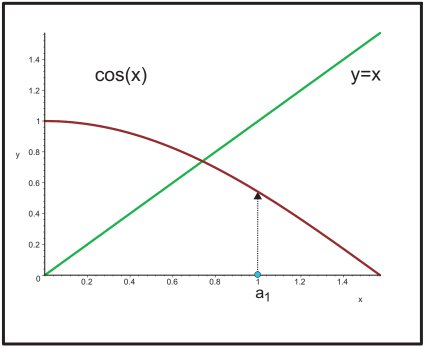

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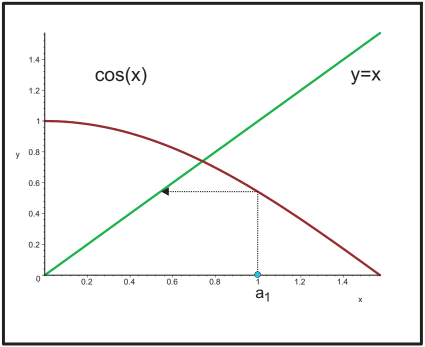

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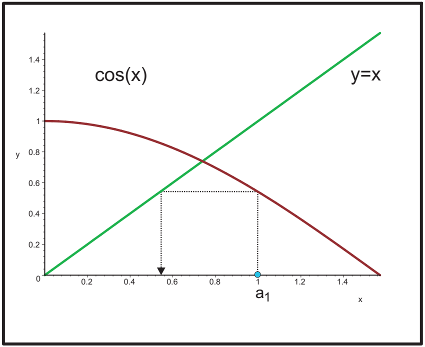

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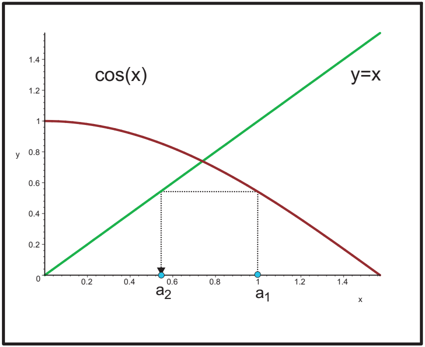

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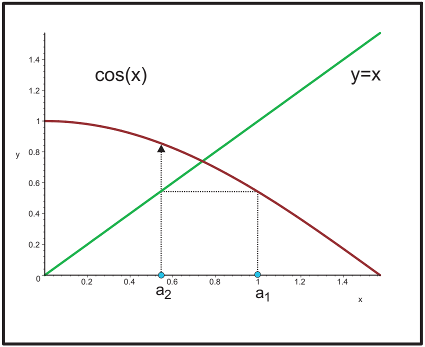

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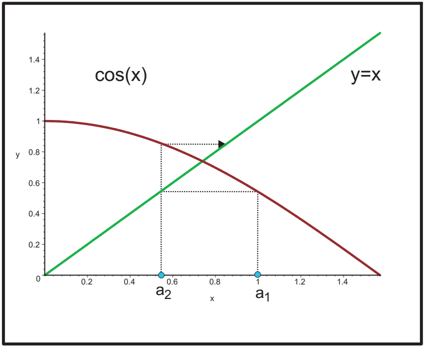
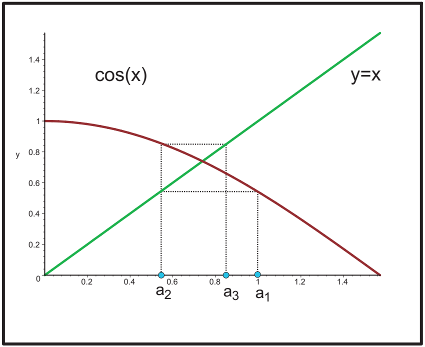
Example



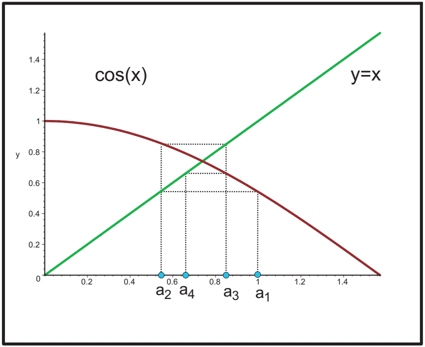
$$a_1 = 1,$$
$$a_2 = 0.5403023059,$$

Example



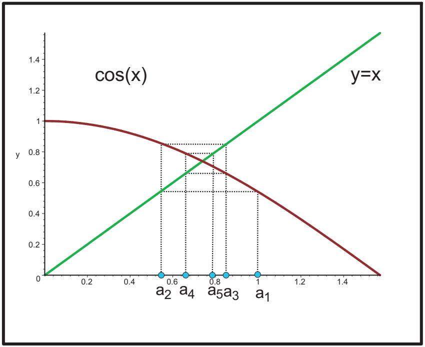
$$a_1 = 1,$$
$$a_2 = 0.5403023059,$$
$$a_3 = 0.8575532158,$$

Example



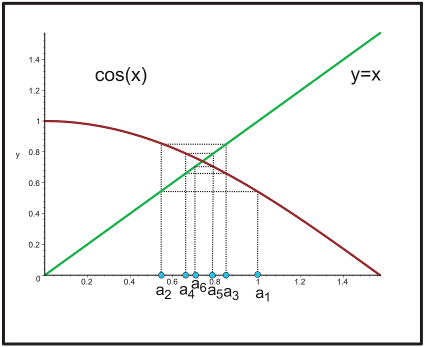
$a_1 = 1,$
$a_2 = 0.5403023059,$
$a_3 = 0.8575532158,$
$a_4 = 0.6542897905,$

Example



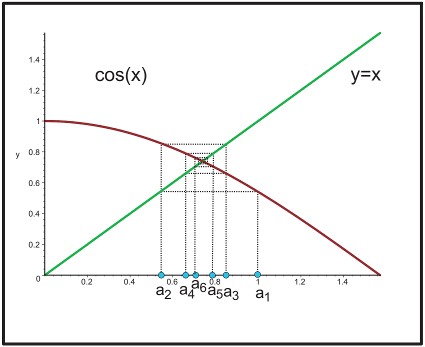
$a_1 = 1,$
$a_2 = 0.5403023059,$
$a_3 = 0.8575532158,$
$a_4 = 0.6542897905,$
$a_5 = 0.7934803587,$

Example



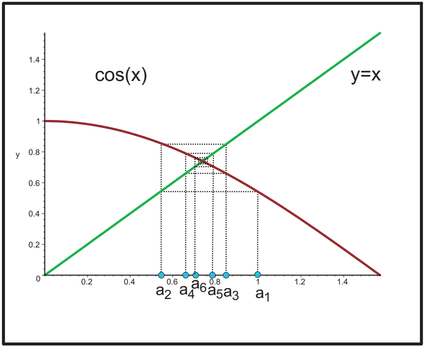
$$a_1 = 1,$$
$$a_2 = 0.5403023059,$$
$$a_3 = 0.8575532158,$$
$$a_4 = 0.6542897905,$$
$$a_5 = 0.7934803587,$$
$$a_6 = 0.7013687737,$$

Example



$$
\begin{aligned}
a_1 &= 1,\\
a_2 &= 0.5403023059,\\
a_3 &= 0.8575532158,\\
a_4 &= 0.6542897905,\\
a_5 &= 0.7934803587,\\
a_6 &= 0.7013687737,\\
a_7 &= 0.7639596829,\\
a_8 &= 0.7221024250,\\
a_9 &= 0.7504177618,
\end{aligned}
$$

Example

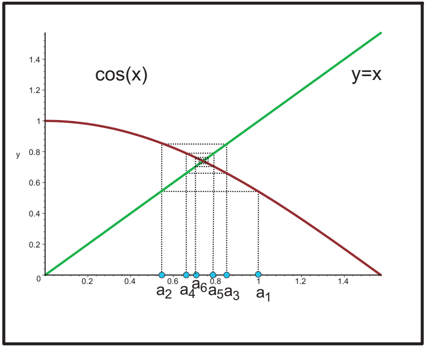


$$a_1 = 1,$$
$$a_2 = 0.5403023059,$$
$$a_3 = 0.8575532158,$$
$$a_4 = 0.6542897905,$$
$$a_5 = 0.7934803587,$$
$$a_6 = 0.7013687737,$$
$$a_7 = 0.7639596829,$$
$$a_8 = 0.7221024250,$$
$$a_9 = 0.7504177618,$$

0.7314040424, 0.7442373549, 0.7356047404, 0.7414250866,
0.7375068905, 0.7401473356, 0.7383692041, 0.7395672022,
0.7387603199, 0.7393038924, 0.7389377567, 0.7391843998,

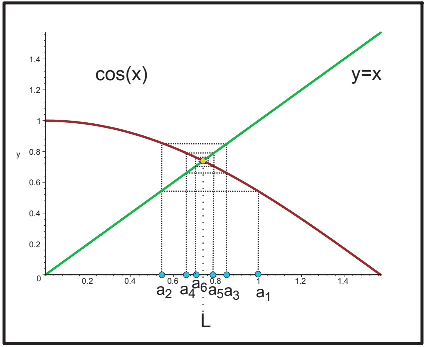
...

Example



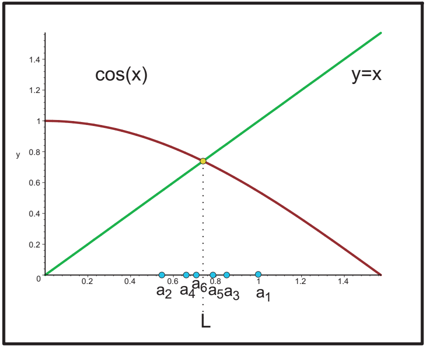
$a_{72} = 0.7390851332,\ a_{73} = 0.7390851332,\ a_{74} = 0.7390851332,$

Example



$a_{72} = 0.7390851332$, $a_{73} = 0.7390851332$, $a_{74} = 0.7390851332$, suggests that $\{a_n\}$ converges to some L.

Example



$a_{72} = 0.7390851332$, $a_{73} = 0.7390851332$, $a_{74} = 0.7390851332$, suggests that $\{a_n\}$ converges to some L.
In fact,

$$cos(L) = L.$$