Problems Involving Length and Area in the Plane

1: Approximate the length of the curve $y = e^x$ from the point $(0,0)$ to the point $(1,e)$ by choosing the 6 points $(x_k, y_k) = \left(\frac{k}{5}, e^{k/5}\right)$ with $k = 0,1,2,3,4,5$, then using a calculator to find the sum of the legths of the 5 line segments between these points. We remark that it can be shown, using the methods of first year university calculus, that the exact length of this curve is equal to $l = \sqrt{1+e^2} + \ln(\sqrt{1+e^2} - 1) + \ln(1+\sqrt{2}) - (1+\sqrt{2})$.

2: Let $A = (1,0)$, $B = (4,1)$ and $C = (6,5)$.
(a) Find the area of triangle ABC.
(b) Find the angles at A, B and C in triangle ABC.

3: Find the area of an equilateral triangle inscribed in a circle which is inscribed in a square which is inscribed in a circle of radius 1.

4: Let $A = (0,0)$ and $B = (7,0)$. Find a point $C = (x,y)$ such that triangle ABC has area and perimeter both equal to 21.

5: A circle of radius 1 and a circle of radius 2 intersect at right angles. Find the area of the region which lies inside both circles.

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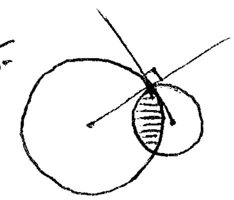

6: Find the area and the perimeter of the region shown at the right, where the larger circle has radius 3 and the smaller circle has radius 1.

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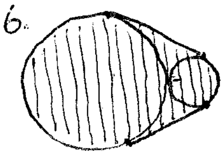

7: (a) Use similar triangles to determine the value of x in the figure shown at the right.
(b) Use part (a) to find the exact values of $\cos\frac{\pi}{5}$ and $\sin\frac{\pi}{5}$.
(c) Use part (b) to find the area and the perimeter of a regular pentagon inscribed in a circle of radius 1.

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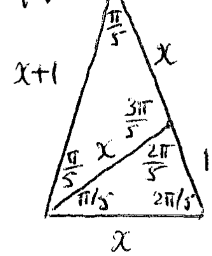

8: (a) A point P is chosen at random inside the square $ABCD$. Find the probability that P lies within a distance of 1 from each of the vertices A, B, C and D.
(b) A point P is chosen at random inside an equilateral triangle ABC. Find the probability that one of the triangles PAB, PAC and PBC is acute-angled.
(c) A point P is chosen at random in the square $ABCD$. Find the probability that, in the triangle PAB, the angle at each vertex is at most $\frac{5\pi}{12}$.

9: A sphere of radius $\sqrt{5}$ has its center at the point $(0,0,5)$.
(a) Find the area of the circular shadow which is cast on the xy-plane by a light at $(0,0,8)$.
(b) Find the area of the elliptic shadow which is cast on the xy-plane by a light at $(0,1,8)$.

10: (a) Let n be a positve integer. Show that $\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{n\pi}{n} = \dfrac{1+\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}$.
(b) Use part (a) to find the exact area under one arch of the curve $y = \sin x$.

Problems Involving Areas and Volumes of Polyhedra

1: (a) Find the volume of the tetrahedron with vertices at $(0,0,0)$, $(2,3,0)$, $(0,5,0)$ and $(1,2,6)$.
(b) Find the volume of the octahedron with vertices at $(1,3,0)$, $(3,1,0)$, $(4,3,0)$, $(2,7,0)$, $(3,2,4)$ and $(2,4,-6)$.

2: (a) Find the volume and surface area of the regular tetrahedron with sides of length 1 unit.
(b) Find the volume and surface area of the regular octahedron with sides of length 1 unit.

3: Let $ABCDEFGH$ be a solid rectangular prism, with vertices as shown, with $AB = 2$, $AD = 4$ and $AE = 6$. Let O be the cetre of the face $ABCD$ and let P, Q, R and S be the midpoints of the sides AB, BC, CD and DA respectively. The box is cut into four pieces by slicing it once along the rectangle $PRHE$ and once along the rectangle $QSEF$. Find the volume of each of the four pieces.

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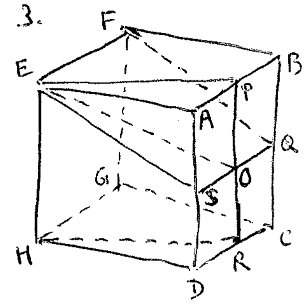

4: Two rectangle-based cones (pyramids) overlap in space to form a solid. They share the common rectangular base with vertices at $(\pm 2, \pm 1, 0)$, and one of the pyramids has its top vertex at $(0,3,4)$ and the other has its top vertex at $(0,-3,4)$. Find the volume and surface area of the resulting solid.

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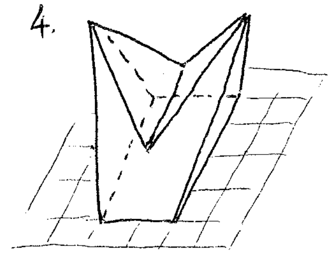

5: For each of the 5 regular polyhedra, find the number of vertices V, the number of edges E and the number of faces F, and verify that $V - E + F = 2$. Do the same for the solid described in question 4.

6: A regular tetrahedron is inscribed in a cube with sides of length 1. Every second vertex of the cube is a vertex of the tetrahedron, and each edge of the tetrahedron is a diagonal of the cube. Find the volume of the tetrahedron.

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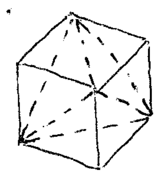

7: A regular octahedron is inscribed in a cube with sides of length 1. The centre of each face of the cube is a vertex of the octahedron. Find the volume of the octahedron.

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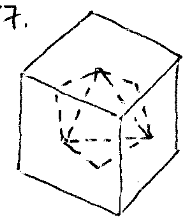

8: A regular octahedron is inscribed in a regular tetrahedron with sides of length 1. The midpoint of each edge of the tetrahedron is a vertex of the octahedron. Find the volume of the octahedron.

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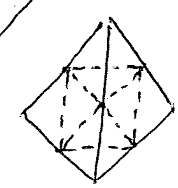

9: A regular icosahedron is inscribed in a cube with sides of length 1. The icosahedron has 2 vertices on each face of the cube as shown. Find the volume of the icosahedron.

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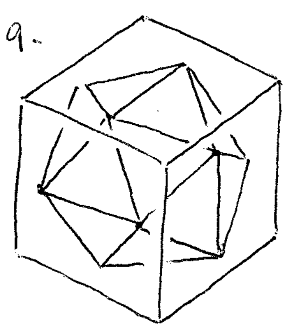

10: (a) Explain whybricks shaped like regular tetrahedra with sides of length 1 can be fit together snugly with bricks shaped like regular octahedra with sides of length 1 to fill space.
(b) The angle between two faces of a regular tetrahedron is almost equal to 72°. Show that it is not exactly equal to 72°, and so bricks shaped like regular tetrahedra with sides of length 1 cannot be fit together snugly to fill space.
(c) Determine whether bricks shaped like regular dodecahedra with sides of length 1 can be fit together snugly to fill space.

Problems Involving Areas and Volumes of Solids

1: A rectangular-based prism has sides of length 1, 2 and 3. Find the volume and the surface area of the set of all points whose distance from the box is at most 1.

2: A square-based prism with base sides of length 2 and hight 1 is inscribed in a sphere. Find the volume and surface area of the sphere.

3: A light is at a distance of 5 units from the center of a sphere of radius 3 units. Find the proportion of the surface of the sphere which is lit.

4: Some people live on a spherical planet. They carefully survey a large triangle and find that its interior angles are $90.0000°$, $45.0000°$ and $45.0126°$, and that its area is $22,000\ \text{m}^2$. Using a calculator, determine the radius of the planet.

5: Let R be the radius of the Earth ($R \cong 6000$ km).

(a) Find the area of that part of the Earth's surface which lies between 30° and 60° longitude.

(b) Find the area of that part of the Earth's surface which lies between 30° and 60° latitude.

(c) Find the area of that part of the Earth's surface which lies between 30° and 60° longitude and also between 30° and 60° latitude.

6: A hemispherical bowl of radius 5 cm contains water which is 1 cm deep. Find the volume of the water.

7: A cone-shaped hole is bored into a solid sphere with the tip of the hole at the centre of the sphere. The radius of the sphere is 2 and the radius of the circular rim of the hole is $\sqrt{3}$. Find the volume of the portion of the sphere that remains.

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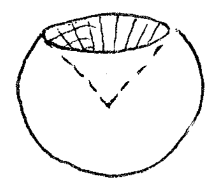

8: A spherical-shaped hole is bored into a solid sphere, with the bottom of the hole at the centre of the sphere. The radius of both the solid sphere and the spherical hole are 2 units. Find the volume of the portion of the sphere that remains.

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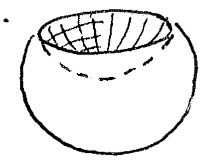

9: A cylindrical hole is bored through the centre of a solid sphere. The radius of the sphere is 5 units and the sides of the cylindrical hole are 6 units long. What is the volume of the portion of the sphere that remains.

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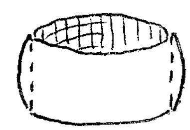

10: (a) Find the surface area and the volume of the doughnut with inner radius 1 cm and outer radius 5 cm.

(b) A parabolic dish is shaped so that when it contains water which is y meters deep, the surface of the water is a circle of radius $r = \sqrt{y}$ meters. Find the volume of the water when it is 1 meter deep.