MATH Circles: October, 2010

Problem Set: Day 1

1) Use Induction to prove that:

a) $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for each $n \in \mathbb{N}$.

b) $2^n + 3^n$ is divisible by 5 for each odd $n \in \mathbb{N}$ (Hint: Odd n's are of the form $2k-1$ for $k \in \mathbb{N}$. Use induction on k §).

2) Let $a_1 = 1$ and for each $n \geq 1$ let

$$a_{n+1} = \sqrt{3 + 2a_n}$$

(This is called a recursively defined sequence.)

Prove that for every $n \in \mathbb{N}$, we have

$$0 \leq a_n \leq a_{n+1} \leq 3.$$

3) Tower of Hanoi:

You are given three pegs. On one of the pegs is a tower made up of n rings placed on top of one another so that as you move down the tower each successive ring has a larger diameter than the previous ring. The object of this puzzle is to reconstruct the tower on one of the other pegs by moving one ring at a time, from one peg to another, in such a manner that you never have a ring above any smaller ring on any of the three pegs.

Prove that for any $n \in \mathbb{N}$, if you begin with n rings, then the puzzle can be completed in $2^n - 1$ moves. Moreover, prove that for each n, this is the minimum number of moves necessary to complete the task.

Note: The key to this question is coming up with an appropriate description of the statement $P(n)$.

4 a) The Well Ordering Principle for $\mathbb{N}$ states that every nonempty subset S of $\mathbb{N}$ has a least element. Prove this using the Principle of Induction.

(Hint: Let S be a subset of $\mathbb{N}$ that does not have a least element. Let

$$T = \mathbb{N} \backslash S = \{n \in \mathbb{N} \mid n \notin S\}$$

and show that $T = \mathbb{N}$, and hence that $S = \emptyset$. To do this, let $P(n)$ be the statement that $\{1, 2, \ldots, n\} \subset T$.)

5) Let n be a natural number greater than or equal to 2. We say that n is prime if whenever n can be written as a product of the form $n = m_1 m_2$ where m_1 and m_2 are natural numbers, then either $m_1 = 1$ or $m_2 = 1$. Use the Well Ordering Principle to show that if $n \in \mathbb{N}$ and $n \geq 2$, then n is a product of primes.

Note: The factorization above is unique in the sense that if

$$n = p_1^{n_1} p_2^{n_2} \cdots p_j^{n_j}$$

and

$$n = q_1^{k_1} q_2^{k_2} \cdots q_l^{k_l}$$

where $p_1 < p_2 < \cdots < p_j$ and $q_1 < q_2 < \cdots < q_l$ are distinct primes, then $j = l$, $p_i = q_i$ and $n_i = k_i$ for each $1 \leq i \leq j$.

6 a) Use the Least Upper Bound Property to prove the Archimedian Property I:

Archimedian Property I: $\mathbb{N}$ is unbounded.

(Hint: Assume that $\mathbb{N}$ is bounded with $lub(\mathbb{N}) = \alpha$. What can you say about $\alpha - \frac{1}{2}$.)

b) Prove that $\mathbb{R}$ has the *Archimedean Property II:*

Let $\epsilon > 0$. Then there exists an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \epsilon$.

c) Let $x > 1$. Assume that $x \notin \mathbb{N}$. Use the Archimedian Property and the Well Ordering Property to show that there exists an $n \in \mathbb{N}$ such that $n < x < n + 1$.

d) Let $0 \leq a < b$. Show that there exists a rational number r with $a < r < b$ and an irrational number γ with $a < \gamma < b$. (Hint: You may assume that $\sqrt{2}$ is irrational.)