

Series

Brian Forrest

October 20, 2010

A Problem:

Problem:

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Problem: Can one perform infinitely many tasks in a finite amount of time?

The Paradox of Achilles:

A


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Paradox of Achilles

- ▶ Achilles races a tortoise

The Paradox of Achilles:

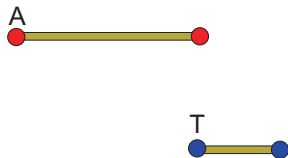
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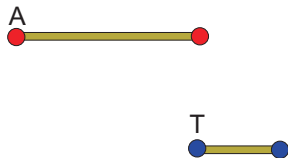
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- ▶ Achilles races a tortoise who is given a head start.
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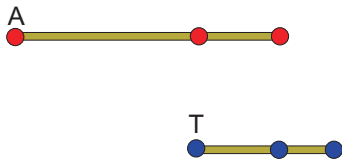
The Paradox of Achilles:



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- ▶ Achilles races a tortoise who is given a head start.
- ▶ Achilles **reaches where the tortoise began** and **the tortoise has moved ahead.**

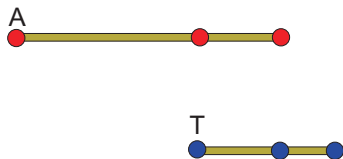
The Paradox of Achilles:



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- ▶ Achilles races a tortoise who is given a head start.
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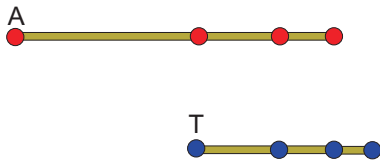
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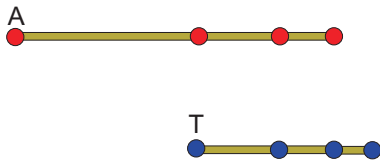


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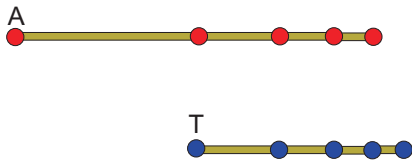
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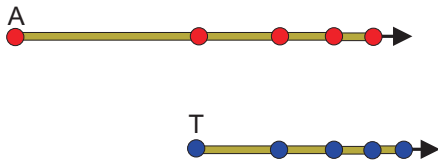
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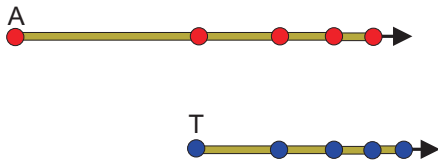
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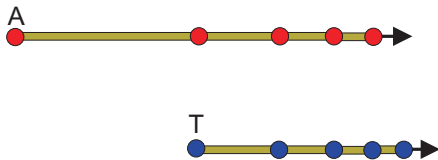
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- ▶ **Conclusion:**

The Paradox of Achilles:



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- ▶ And so on Ad infinitum.
- ▶ **Conclusion:** Achilles can never catch the tortoise!!!.

Resolving The Paradox of Achilles:

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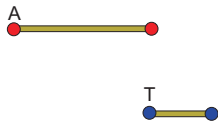
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Resolving the Paradox:

We will call each time Achilles moves from a point

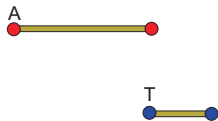
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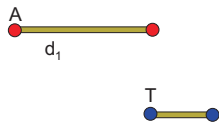
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We will call each time Achilles moves from a point to where the tortoise was a *stage*.

Resolving The Paradox of Achilles:

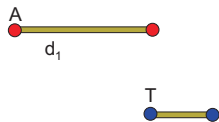


► $d_1 =$

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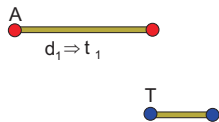


- ▶ d_1 = distance traveled in stage 1

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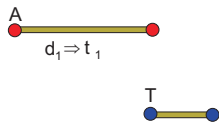


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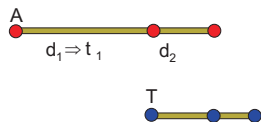


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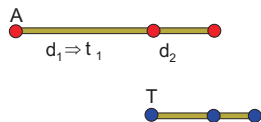


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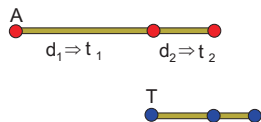


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- ▶ d_2 = distance traveled in stage 2

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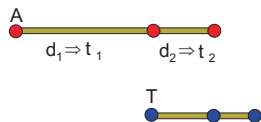


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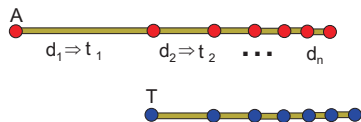


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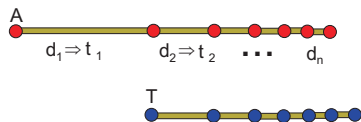


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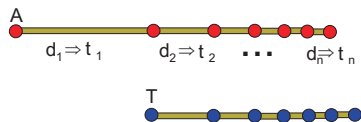


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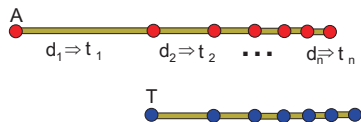


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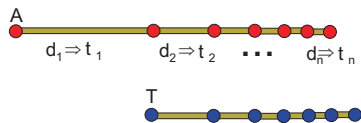


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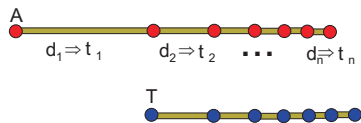
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Time to catch the Tortoise

Resolving The Paradox of Achilles:



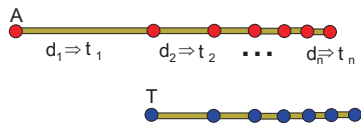
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Time to catch the Tortoise = t_1

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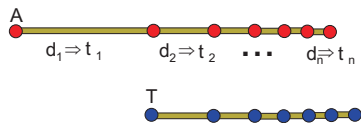
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$$\text{Time to catch the Tortoise} = t_1 + t_2$$

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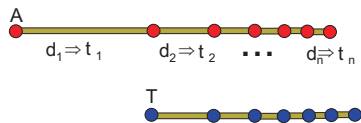
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$$\text{Time to catch the Tortoise} = t_1 + t_2 + \cdots + t_n$$

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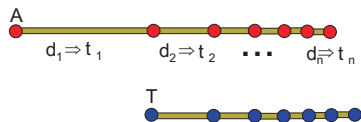
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$$\begin{aligned} \text{Time to catch the Tortoise} &= t_1 + t_2 + \dots + t_n + \dots \\ &= \infty ? \end{aligned}$$

Introduction to Series

Problem:

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Question:

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which we call a *series*.

Question:

What does this formal sum represent? Does it have a value?

Introduction to Series

Example:

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Example: What is

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Example: What is

$$\frac{1}{2} +$$

Introduction to Series

Example: What is

$$\frac{1}{2} + \frac{1}{4} +$$

Introduction to Series

Example: What is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$$

Introduction to Series

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Introduction to Series

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Introduction to Series

Example: What is

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Introduction to Series

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$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} +$$

Introduction to Series

Example: What is

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Introduction to Series

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$$= \frac{1}{2} +$$

Introduction to Series

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$$= \frac{1}{2} + \frac{1}{2^2} +$$

Introduction to Series

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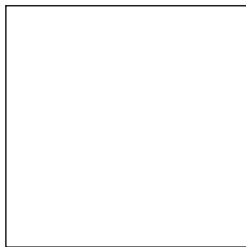
Introduction to Series

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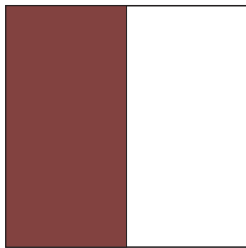
$$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} + \dots$$

Introduction to Series



Geometric Interpretation:

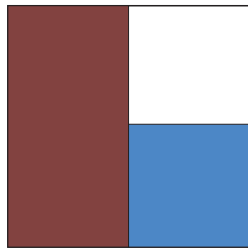
Introduction to Series



$$\frac{1}{2} +$$

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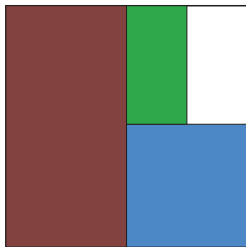
Introduction to Series



$$\frac{1}{2} + \frac{1}{4} +$$

Geometric Interpretation:

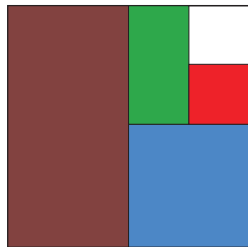
Introduction to Series



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$$

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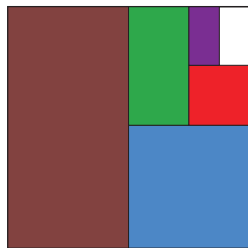
Introduction to Series



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$$

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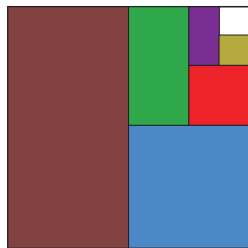
Introduction to Series



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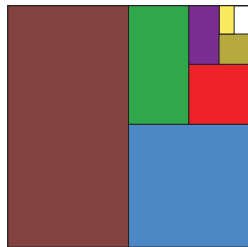
Introduction to Series



Geometric Interpretation:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} +$$

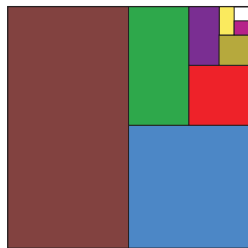
Introduction to Series



Geometric Interpretation:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} +$$

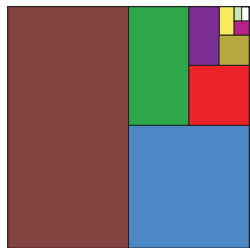
Introduction to Series



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$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} +$$

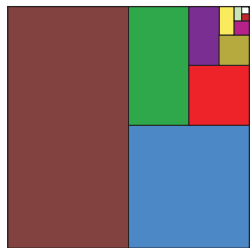
Introduction to Series



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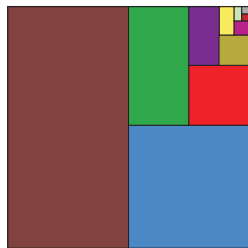
Introduction to Series



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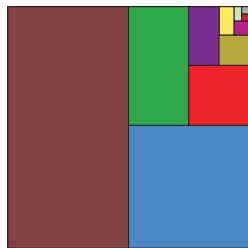
Introduction to Series



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Introduction to Series

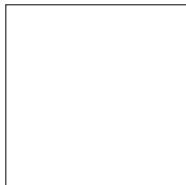


Geometric Interpretation:

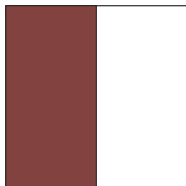
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \dots$$

$$= 1?$$

Introduction to Series



Introduction to Series



$$\frac{1}{2}$$

Sum

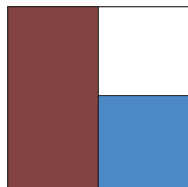
Area

$$1 - \frac{1}{2}$$

Introduction to Series

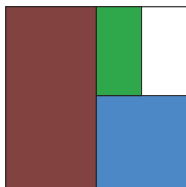
$$\frac{1}{2} + \frac{1}{4}$$

Sum



Area
 $1 - \frac{1}{4}$

Introduction to Series



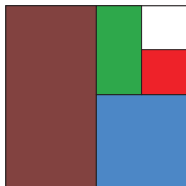
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

Sum

Area

$$1 - \frac{1}{8}$$

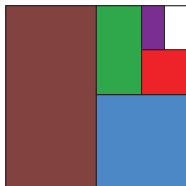
Introduction to Series



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \quad \text{Sum}$$

$$1 - \frac{1}{16} \quad \text{Area}$$

Introduction to Series



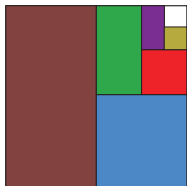
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$$

Sum

$$1 - \frac{1}{32}$$

Area

Introduction to Series



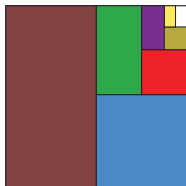
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$$

Sum

$$1 - \frac{1}{64}$$

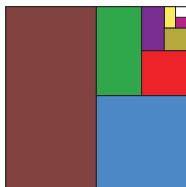
Area

Introduction to Series



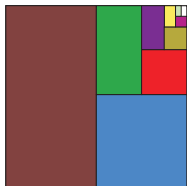
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} \quad \begin{array}{l} \text{Sum} \\ \text{Area} \end{array} \quad 1 - \frac{1}{128}$$

Introduction to Series



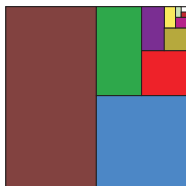
$$\begin{array}{r}
 \text{Sum} \\
 \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} \\
 + \frac{1}{256}
 \end{array}
 \quad
 \begin{array}{r}
 \text{Area} \\
 1 - \frac{1}{256}
 \end{array}$$

Introduction to Series



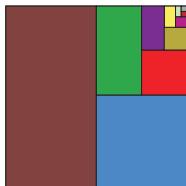
$$\begin{array}{r}
 \text{Sum} \\
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 + \frac{1}{256} + \frac{1}{512}
 \end{array}
 \qquad
 \begin{array}{r}
 \text{Area} \\
 1 - \frac{1}{512}
 \end{array}$$

Introduction to Series



Sum	Area
$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$ $+ \frac{1}{256} + \frac{1}{512} + \frac{1}{1024}$	$1 - \frac{1}{1024}$

Introduction to Series



Sum	Area
$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$ $+ \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \cdots + \frac{1}{2^k}$	$1 - \frac{1}{2^k}$

Introduction to Series



Note:

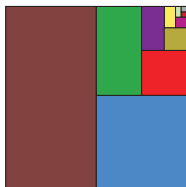
Introduction to Series



Note:

$$\lim_{k \rightarrow \infty} \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k}$$

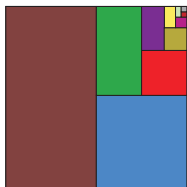
Introduction to Series



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Introduction to Series



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Introduction to Series



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Convergence for Series

Definition: [Convergent Series]

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Example:

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Why use limits?

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Our result is ambiguous; the "sum" changes if we change the way we parenthesize the terms.

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So $\{S_k\} = \{1, 0, 1, 0, 1, 0, \dots\}$ diverges.

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To answer this question, we must look at its sequence partial sums;

$$S_k = \sum_{n=0}^k r^n = 1 + r + r^2 + \dots + r^k.$$

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$$S_k = 1 + 1 + 1 + \cdots + 1$$

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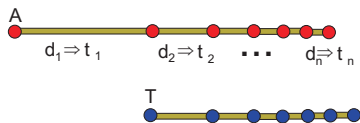
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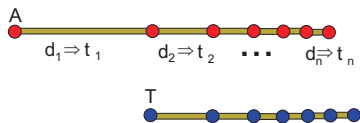
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Resolving The Paradox of Achilles:



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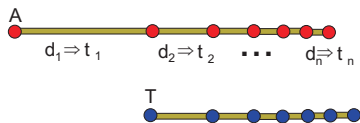
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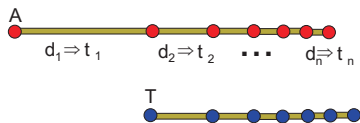
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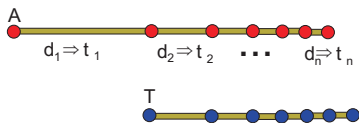
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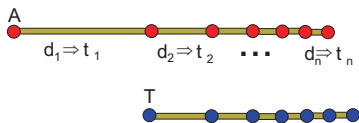


► $d_n =$

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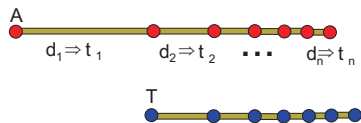


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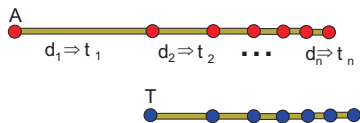


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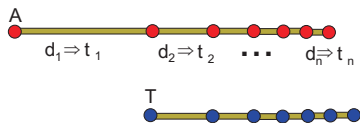


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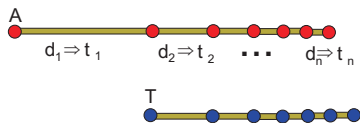
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Time to catch the Tortoise

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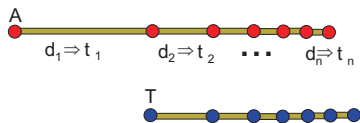
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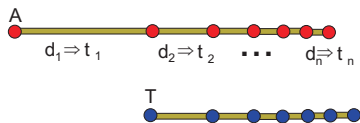
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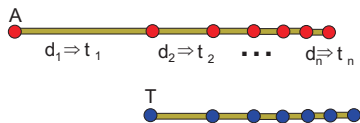
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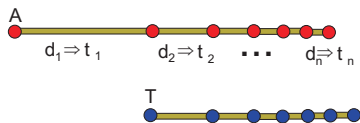
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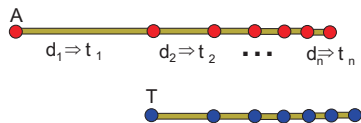
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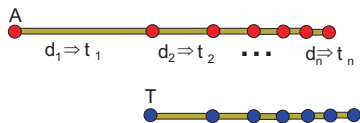
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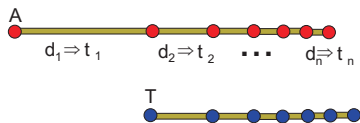
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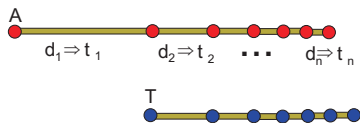
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Conclusion: Achilles catches the tortoise at time T_0 !