Series

Brian Forrest

October 20, 2010

A Problem:

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Problem: Can one perform infinitely many tasks in a finite amount of time?

The Paradox of Achilles:

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- Achilles races a tortoise

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- Achilles races a tortoise who is given a head start.

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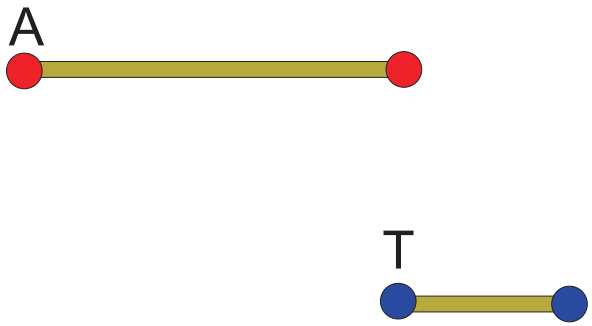

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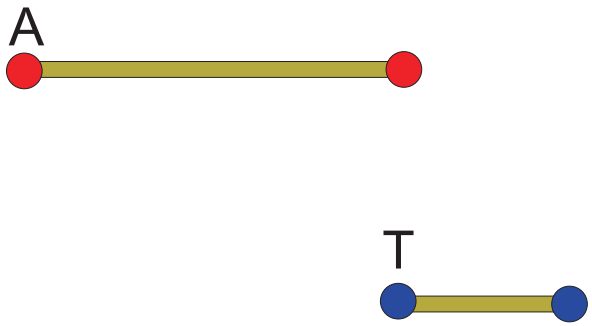

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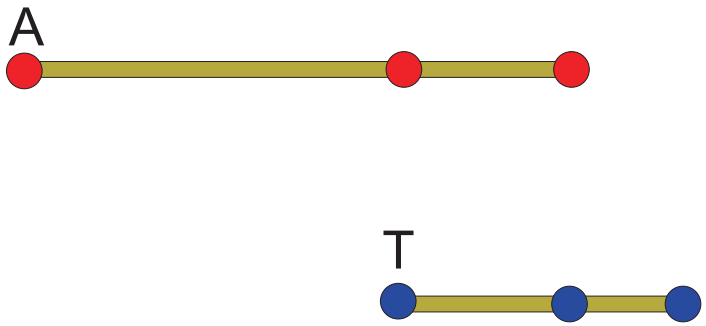

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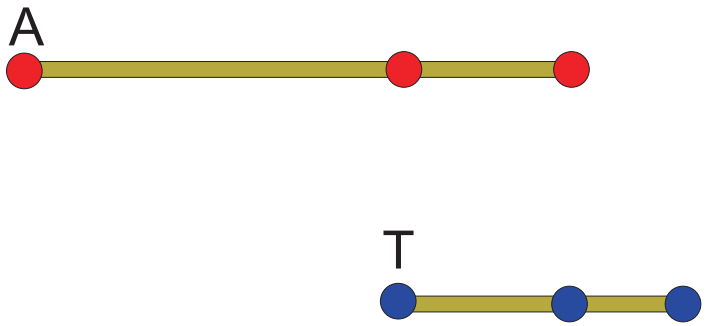

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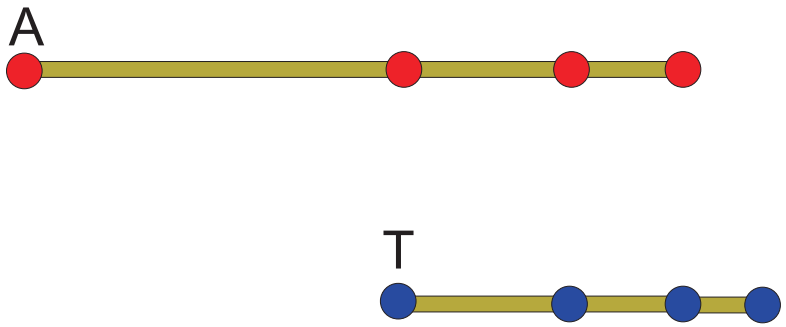

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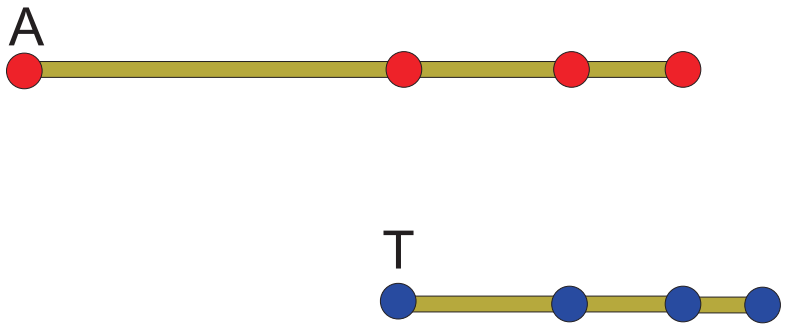

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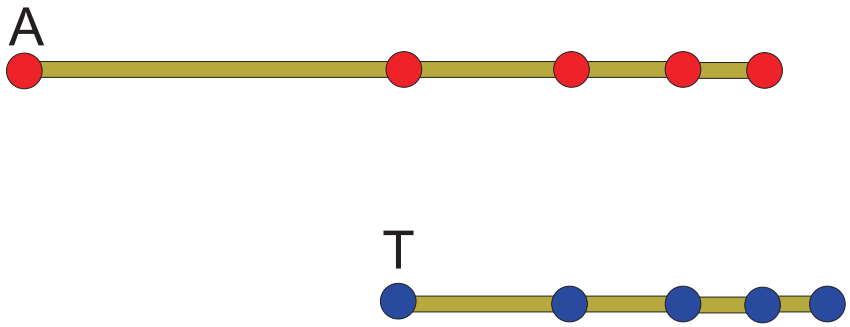

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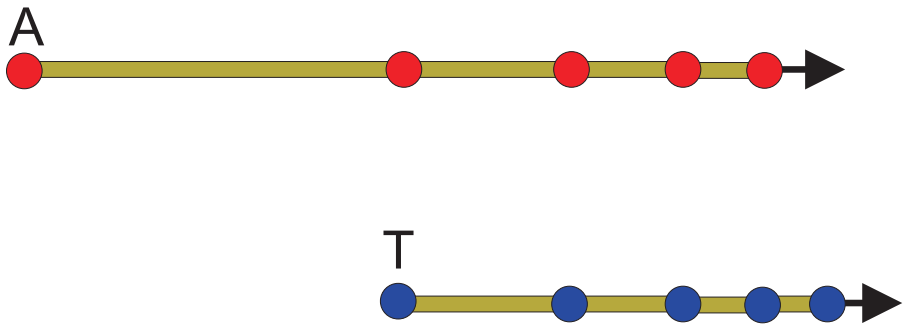

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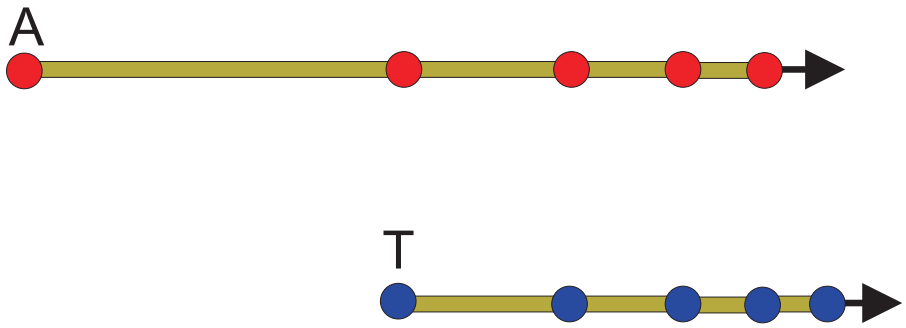

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- **Conclusion:**

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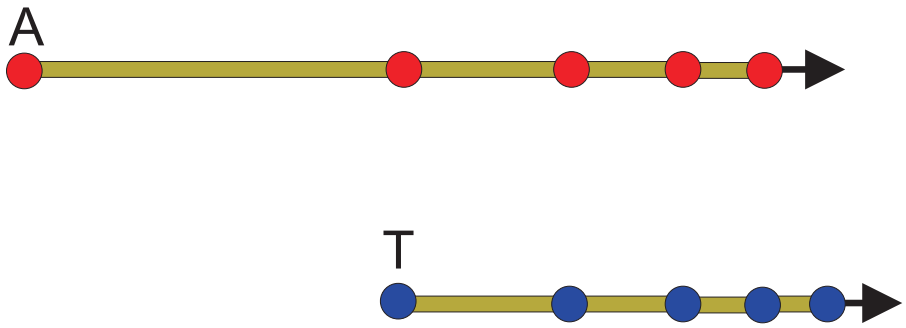

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- And so on Ad infinitum.
- **Conclusion:** Achilles can never catch the tortoise!!!.

Resolving The Paradox of Achilles:

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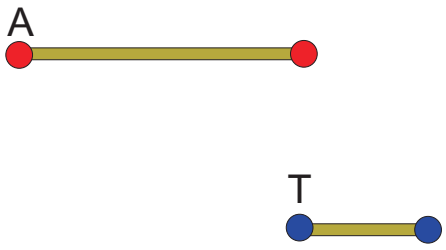

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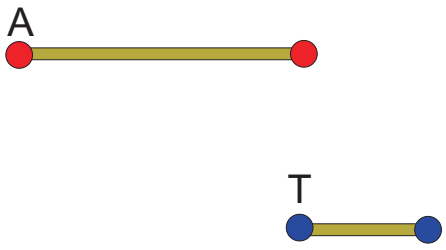

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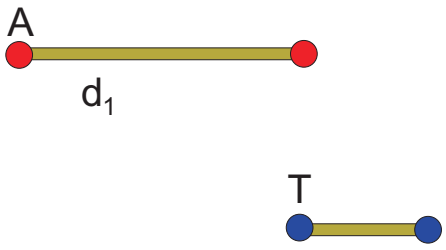

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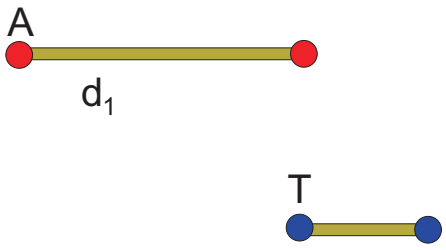

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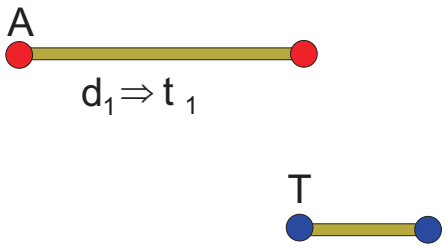

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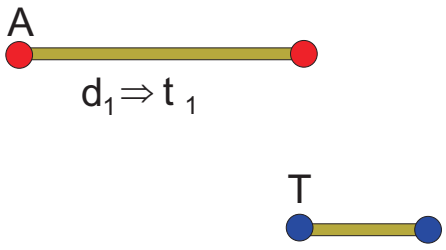

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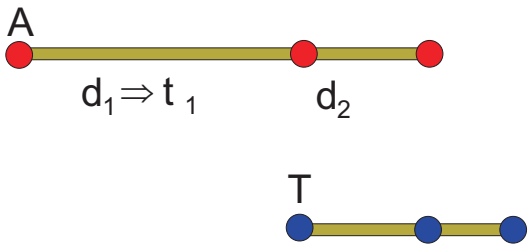

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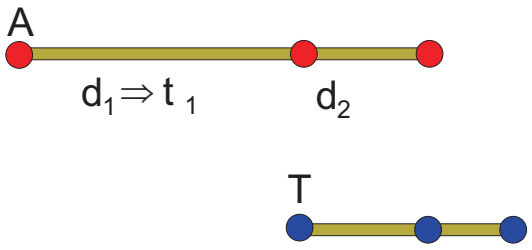

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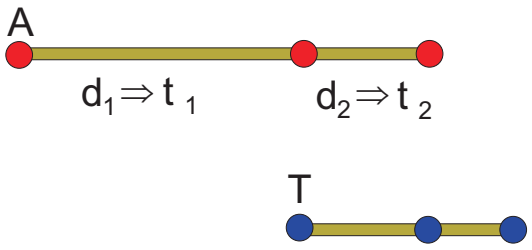

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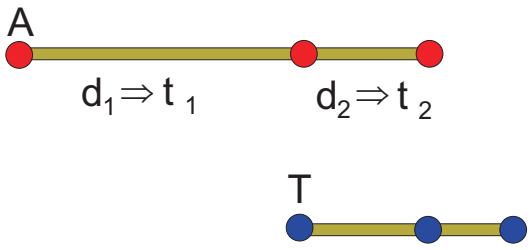

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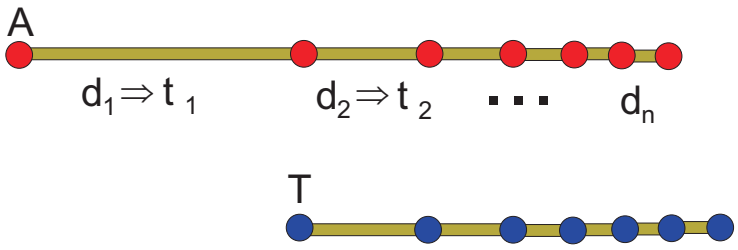

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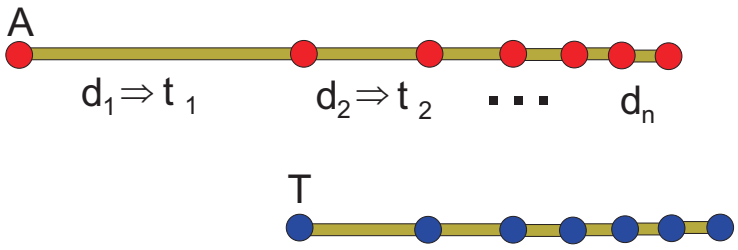

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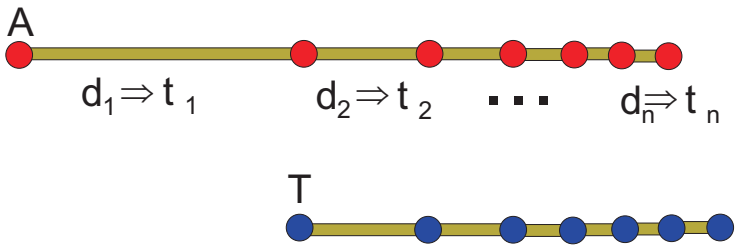

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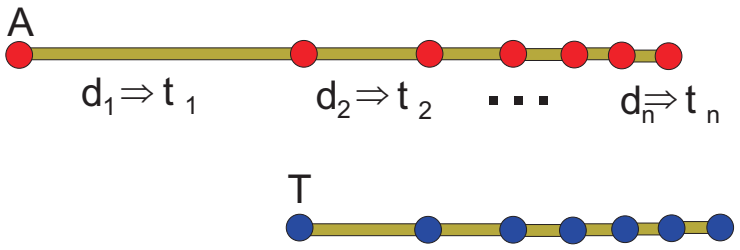

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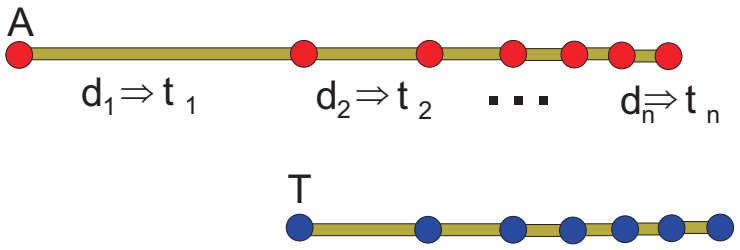

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Time to catch the Tortoise

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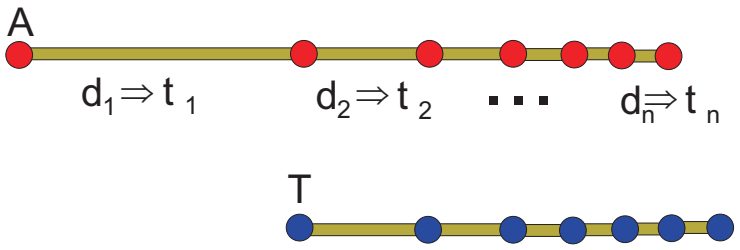

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Time to catch the Tortoise $= t_1$

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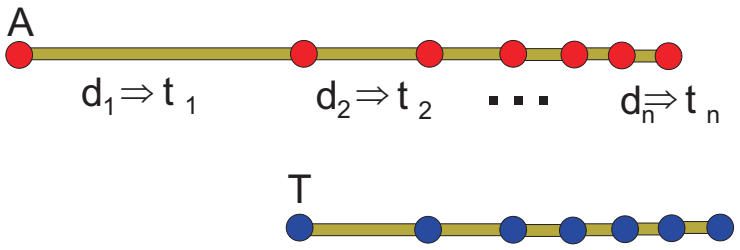

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Time to catch the Tortoise $= t_1 + t_2$

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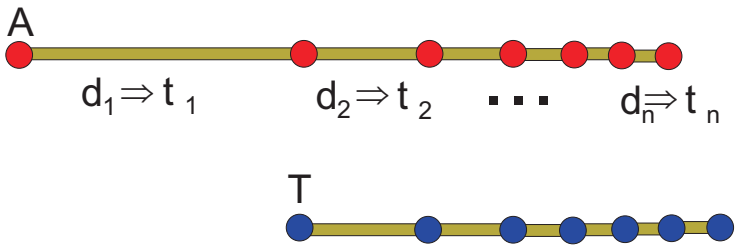

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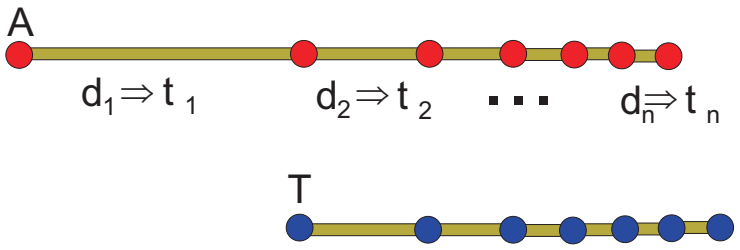

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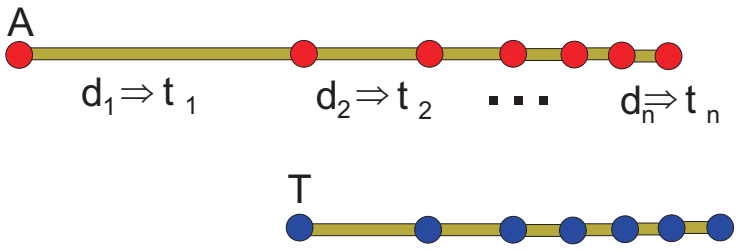

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$$\begin{aligned} \text{Time to catch the Tortoise} &= t_1 + t_2 + \cdots + t_n + \cdots \\ &= \infty \ ? \end{aligned}$$

Introduction to Series

Problem:

Introduction to Series

Problem: Can we add infinitely many numbers at the same time?

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Question:

What does this formal sum represent? Does it have a value?

Introduction to Series

Example:

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Example: What is

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$$\frac{1}{2} +$$

Introduction to Series

Example: What is

$$\frac{1}{2} + \frac{1}{4} +$$

Introduction to Series

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$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$$

Introduction to Series

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Introduction to Series

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Introduction to Series

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Introduction to Series

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Introduction to Series

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Introduction to Series

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Introduction to Series

Example: What is

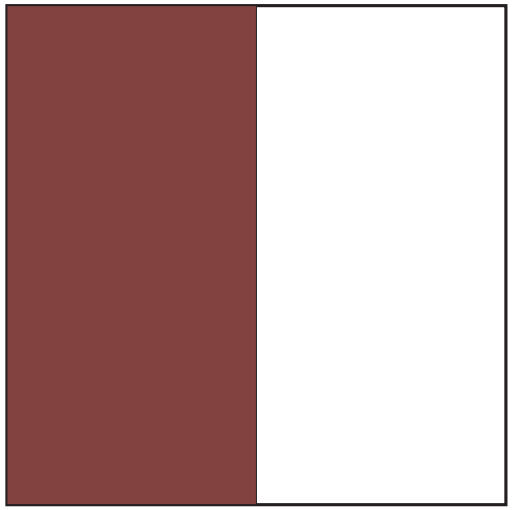
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\cdots$$

$$=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}+\cdots$$

Introduction to Series

Geometric Interpretation:

Introduction to Series



$$\frac{1}{2} +$$

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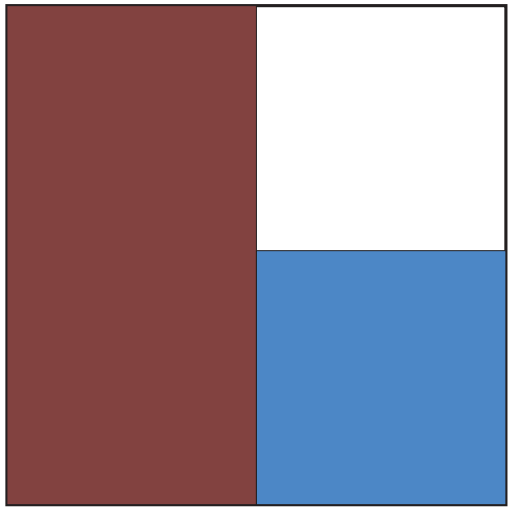

$$\frac{1}{2} + \frac{1}{4} +$$

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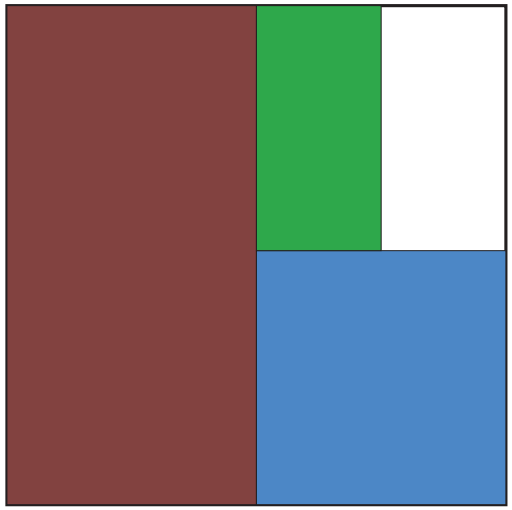

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$$

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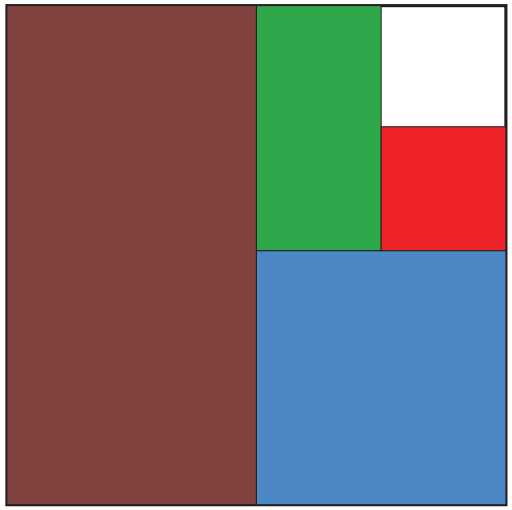

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$$

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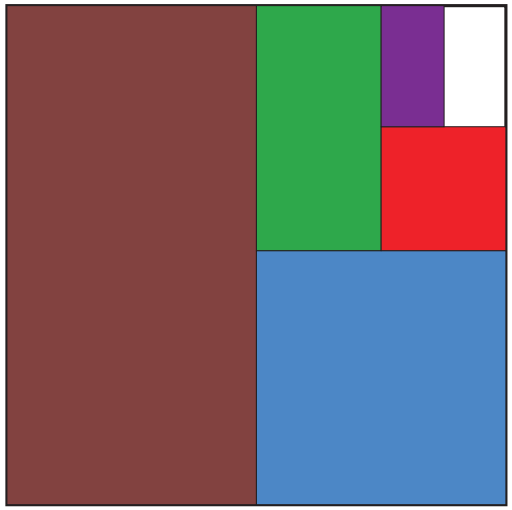

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} +$$

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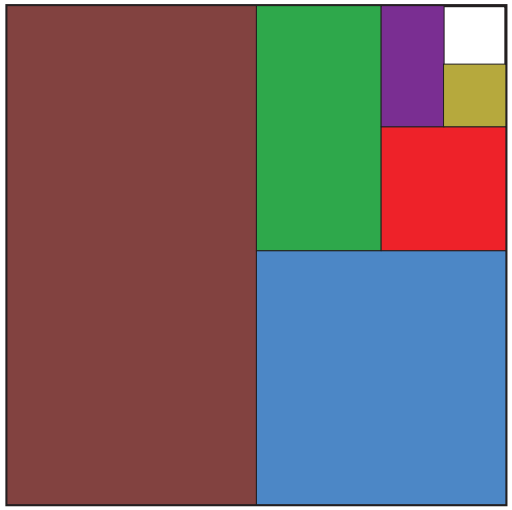

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} +$$

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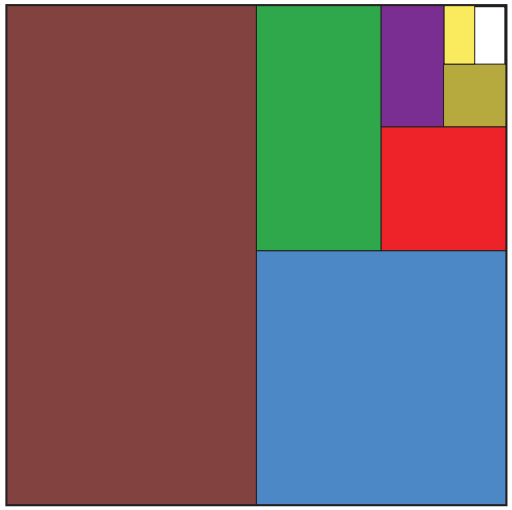
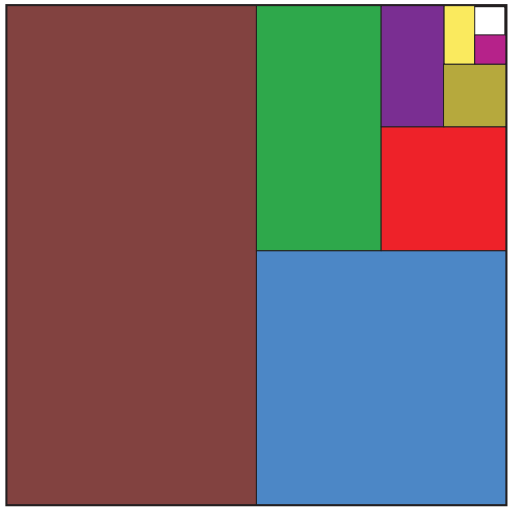


Geometric Interpretation:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+$$

Introduction to Series



Geometric Interpretation:

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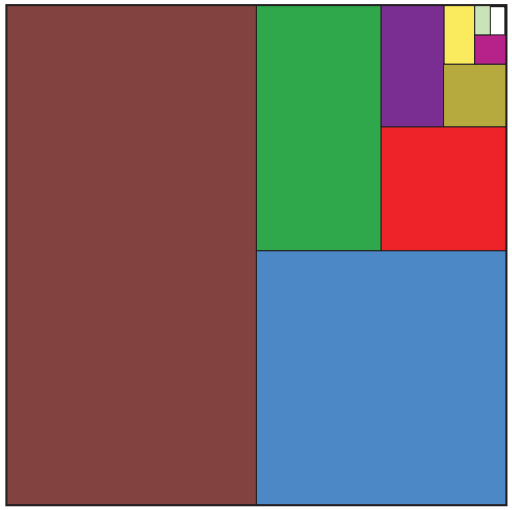
Introduction to Series



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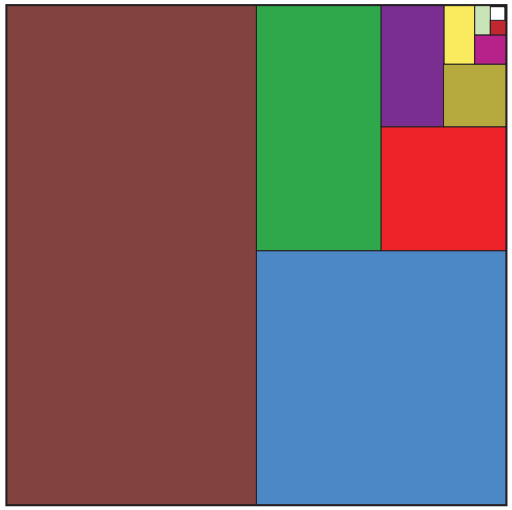

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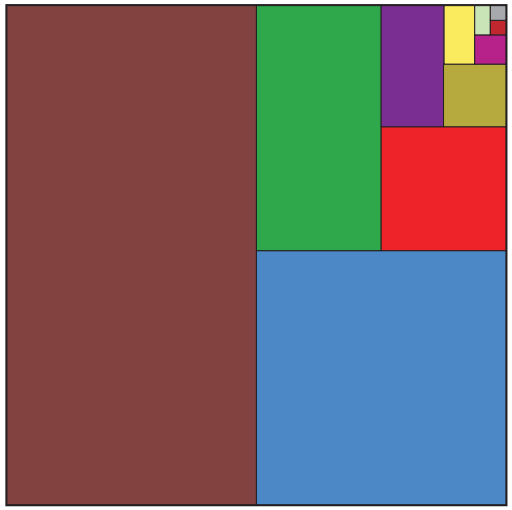

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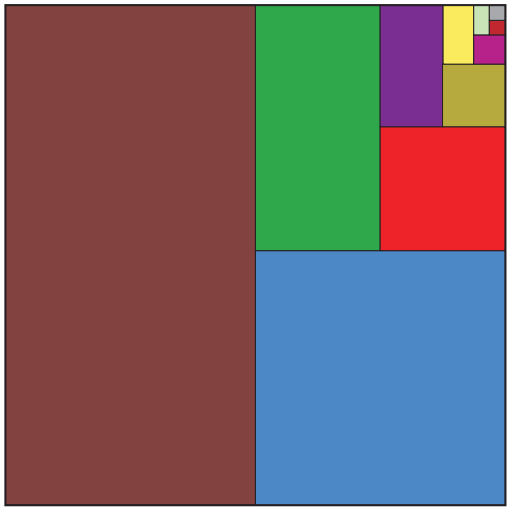

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$$=1?$$

Introduction to Series

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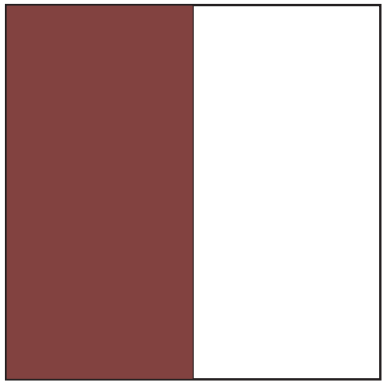


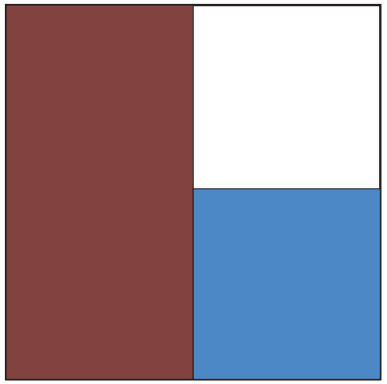
Sum

$\frac{1}{2}$

Area

$1 - \frac{1}{2}$

Introduction to Series



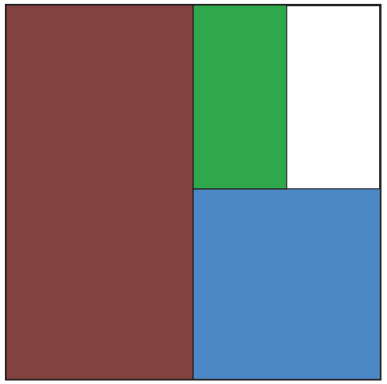
Sum

$\frac{1}{2} + \frac{1}{4}$

Area

$1 - \frac{1}{4}$

Introduction to Series



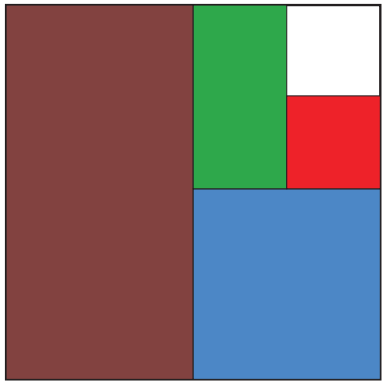
Sum

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

Area

$1 - \frac{1}{8}$

Introduction to Series



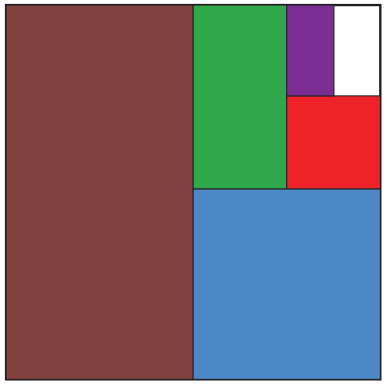
Sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

Area

$$1 - \frac{1}{16}$$

Introduction to Series



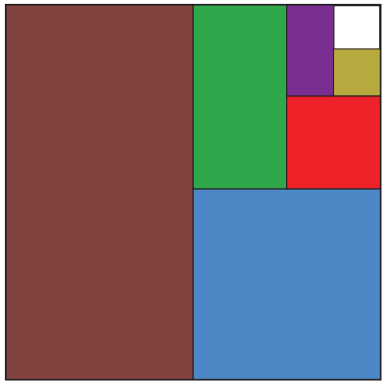
Sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$$

Area

$$1 - \frac{1}{32}$$

Introduction to Series



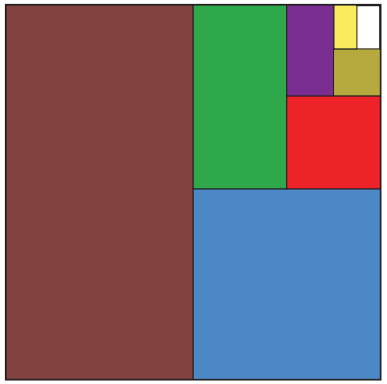
Sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$$

Area

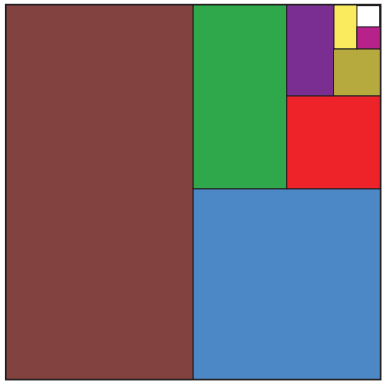
$$1 - \frac{1}{64}$$

Introduction to Series



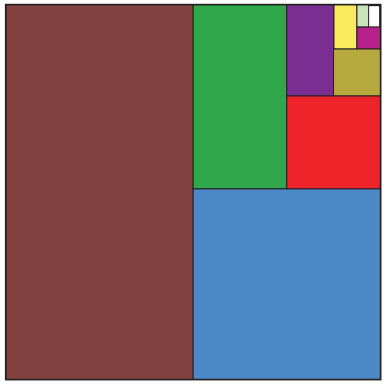
Sum	Area
$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$	$1 - \frac{1}{128}$

Introduction to Series



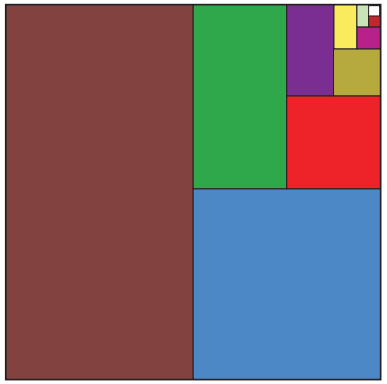
Sum	Area
$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256}$	$1 - \frac{1}{256}$

Introduction to Series



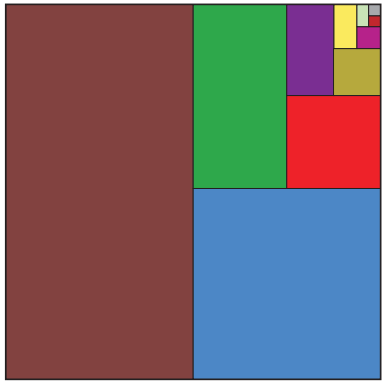
Sum	Area
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}$	$1-\frac{1}{512}$

Introduction to Series



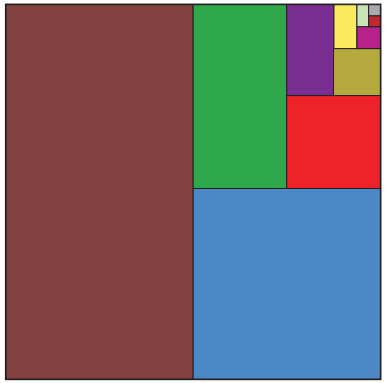
Sum	Area
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}+\frac{1}{1024}$	$1-\frac{1}{1024}$

Introduction to Series



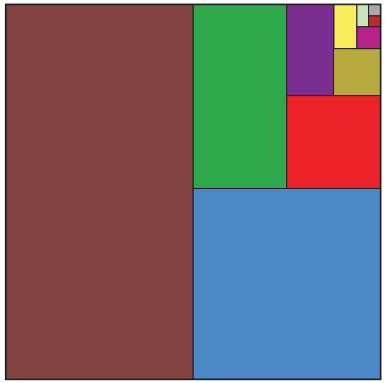
Sum	Area
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}+\frac{1}{1024}+\cdots+\frac{1}{2^k}$	$1-\frac{1}{2^k}$

Introduction to Series



Note:

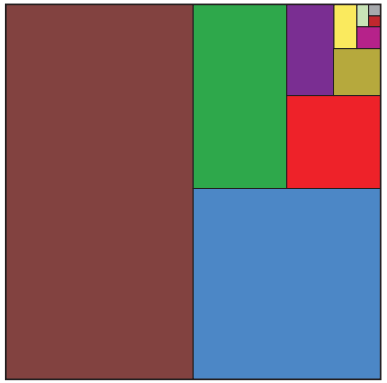
Introduction to Series



Note:

$$\lim_{k\to\infty} \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + + \cdots \frac{1}{2^k}$$

Introduction to Series



Note:

$$\lim_{k\to\infty} \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + + \cdots \frac{1}{2^k} \quad = \quad \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{2^k}$$

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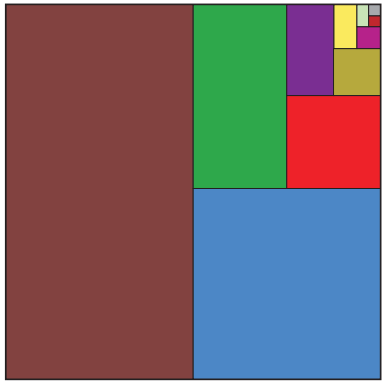
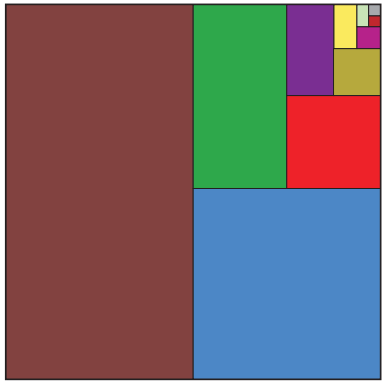


Note:

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + + \cdots \frac{1}{2^k} &= \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{2^k} \\ &= \lim_{k\to\infty} 1 - \frac{1}{2^k}\end{aligned}$$

Introduction to Series



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&= \lim_{k\to\infty} 1 - \frac{1}{2^k} \\
&= 1
\end{aligned}$$

Convergence for Series

Definition: [Convergent Series]
Given a sequence $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$,

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Example:

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Hence, $\sum_{n=1}^{k} \frac{1}{2^n}$ converges with

$$\sum_{n=1}^{k} \frac{1}{2^n} = 1.$$

Why use limits?

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Our result is ambiguous;

Why use limits?

Question Why use limits? Suppose $a_n = (-1)^{n+1}$. Then our formal sum looks like

$$a_1 + a_2 + a_3 + \cdots = 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots$$

We could parenthesize our formal sum as follows:

$$[1 + (-1)] + [1 + (-1)] + [1 + (-1)] + \cdots = 0 + 0 + 0 + \cdots = 0.$$

Alternatively , we could parenthesize our formal sum in this way:

$$1 + [(-1) + 1] + [(-1) + 1] + [(-1) + 1] + \cdots = 1 + 0 + 0 + 0 + \cdots = 1.$$

Our result is ambiguous; the "sum" changes if we change the way we parenthesize the terms.

Why use limits?

Observe:

$$S_1$$

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$$S_1 = 1$$

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$$\begin{aligned} S_1 &= 1 \\ S_2 & \end{aligned}$$

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So $\{S_k\} = \{1, 0, 1, 0, 1, 0, \cdots\}$ diverges.

Geometric Series

Geometric Series

Geometric Series

Geometric Series Let $r \in \mathbb{R}$.

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$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$$

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To answer this question, we must look at its sequence partial sums;

$$S_k = \sum_{n=0}^{k} r^n = 1 + r + r^2 + \cdots + r^k.$$

Geometric Series

Problem: For which r does $\sum_{n=0}^{\infty} r^n$ series converge?

Case 1:

Geometric Series

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Since $\{S_k\} = \{k + 1\}$ diverges,

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Case 1: $r = 1$

$$S_k = 1 + 1 + 1 + \cdots + 1 = k + 1.$$

Since $\{S_k\} = \{k + 1\}$ diverges, the series $\sum_{n=0}^{\infty} 1^n$ diverges.

Geometric Series

Problem: For which r does $\sum_{n=0}^{\infty} r^n$ series converge?

Case 2:

Geometric Series

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$$S_k = 1 + (-1) + 1 + \cdots + (-1)^k =$$

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Geometric Series

Problem: For which r does $\sum_{n=0}^{\infty} r^n$ series converge?

Case 3:

Geometric Series

Problem: For which r does $\sum_{n=0}^{\infty} r^n$ series converge?

Case 3: $r \neq 1$

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Case 3: $r \neq 1$

$$S_k = 1 + r + r^2 + \cdots + r^k$$

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Problem: For which r does $\sum_{n=0}^{\infty} r^n$ series converge?

Case 3: $r \neq 1$

$$\begin{aligned} S_k &= 1 + r + r^2 + \cdots + r^k \\ rS_k & \end{aligned}$$

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$$S_k$$

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$$\begin{aligned}
S_k &= 1 + r + r^2 + \cdots + r^k \\
rS_k &= \quad\quad r + r^2 + \cdots + r^k + r^{k+1} \\
\Rightarrow (1-r)S_k &= 1 - r^{k+1} \\
S_k &= \frac{1 - r^{k+1}}{1 - r}
\end{aligned}$$

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Problem: For which r does $\sum_{n=0}^{\infty} r^n$ series converge?

Now

$$| \ r^{k+1} \ | \longrightarrow$$

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Theorem: [Geometric Series Test]

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Example:

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Example:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$$

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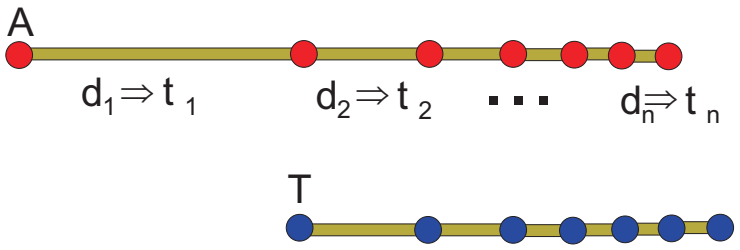
Resolving The Paradox of Achilles:



Resolving the Paradox:

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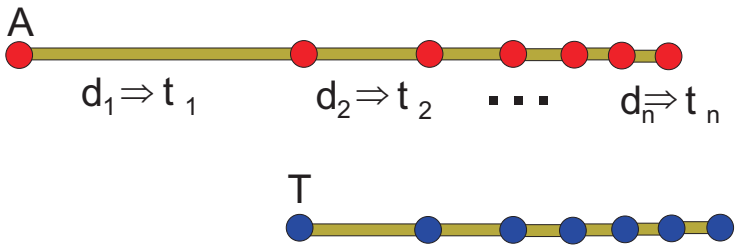
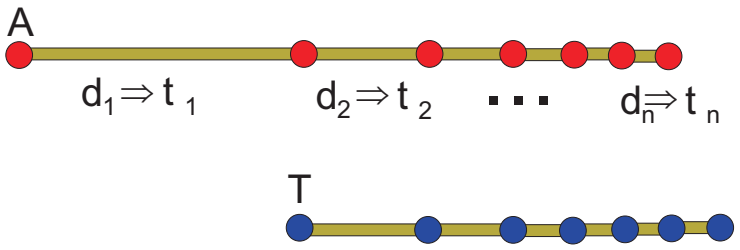


Resolving the Paradox:
We will call each time Achilles moves from a point

Resolving The Paradox of Achilles:



Resolving the Paradox:
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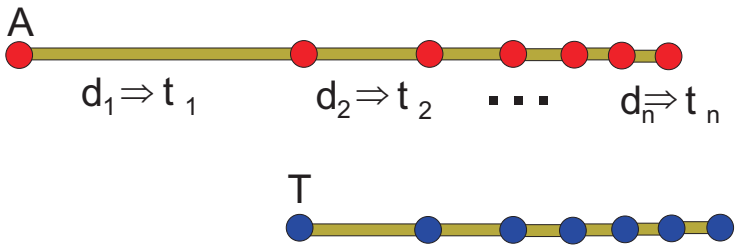
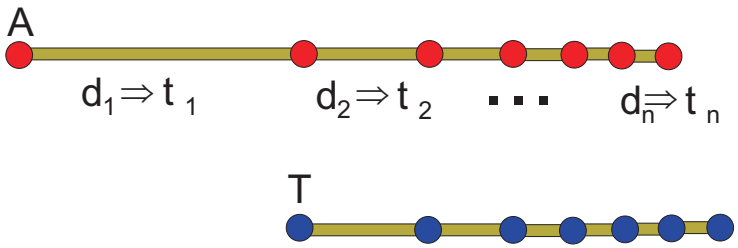
Resolving The Paradox of Achilles:



Resolving the Paradox:
We will call each time Achilles moves from a point to where the tortoise was *a stage*.

Resolving The Paradox of Achilles:



- $d_n =$

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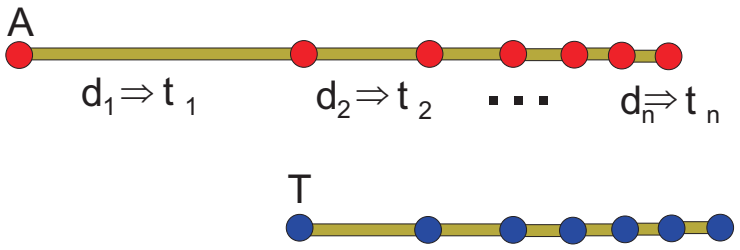
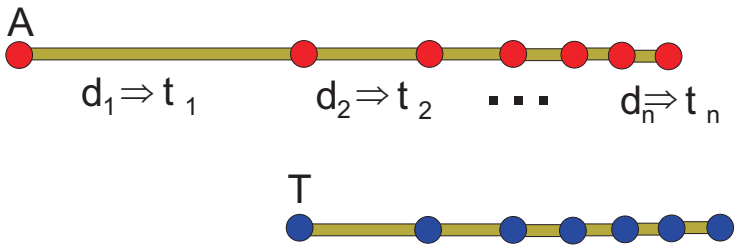


Resolving the Paradox:
We will call each time Achilles moves from a point to where the tortoise was *a stage*.

- d_n = distance traveled in stage n

Resolving The Paradox of Achilles:



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- $d_n =$ distance traveled in stage n $\Rightarrow t_n =$

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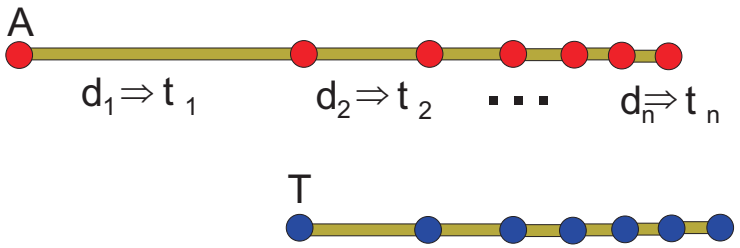

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- d_n = distance traveled in stage n $\Rightarrow$ t_n = time to complete stage n

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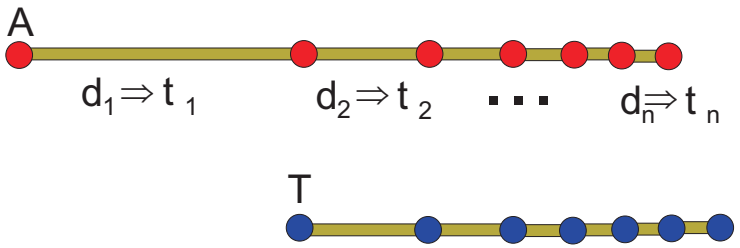

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Time to catch the Tortoise

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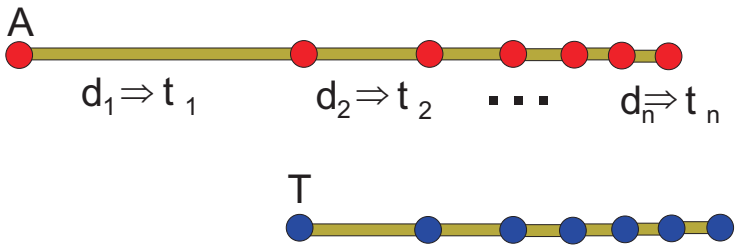

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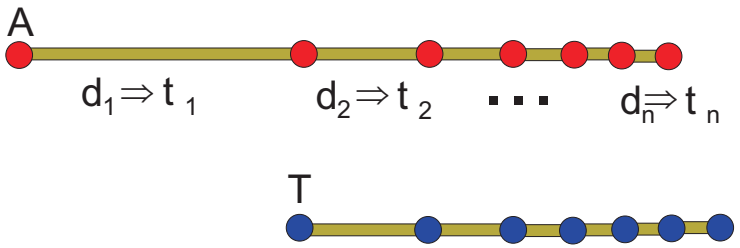
Time to catch the Tortoise =

Resolving The Paradox of Achilles:



Resolving the Paradox:
We will call each time Achilles moves from a point to where the tortoise was *a stage*.

- d_n = distance traveled in stage n $\Rightarrow$ t_n = time to complete stage n

Time to catch the Tortoise $= t_1$

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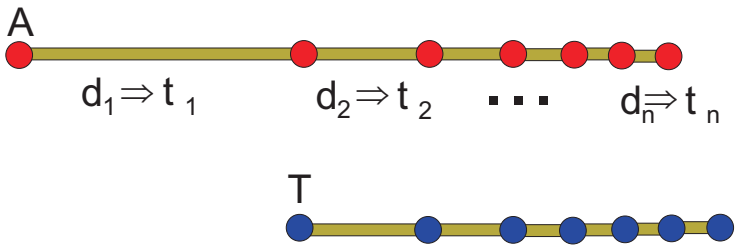

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Time to catch the Tortoise $= t_1 + t_2$

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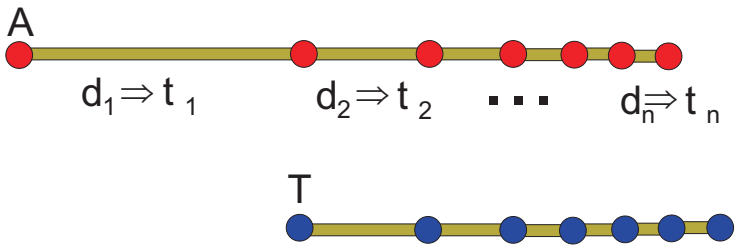

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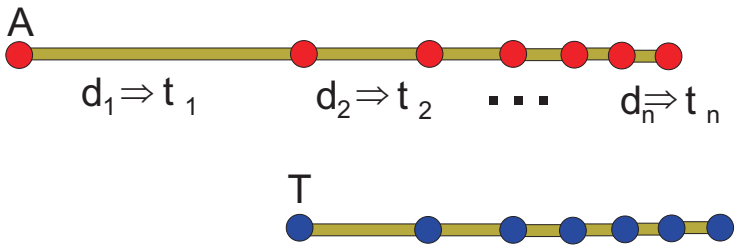

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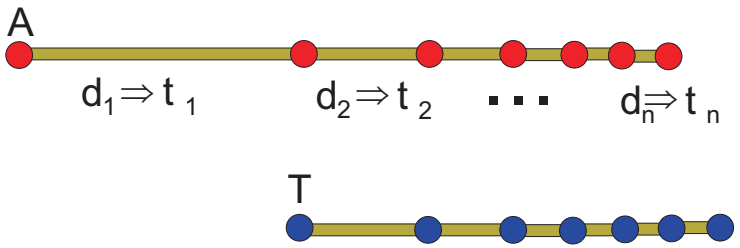

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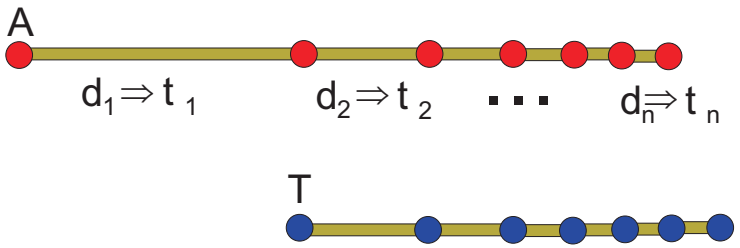

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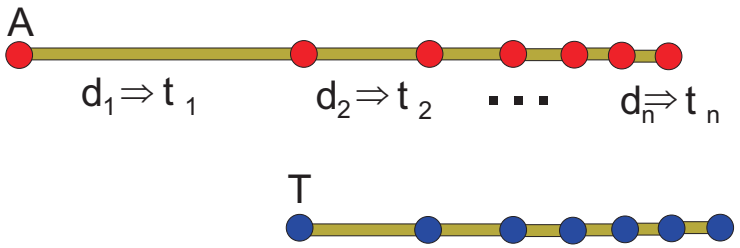

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Conclusion:

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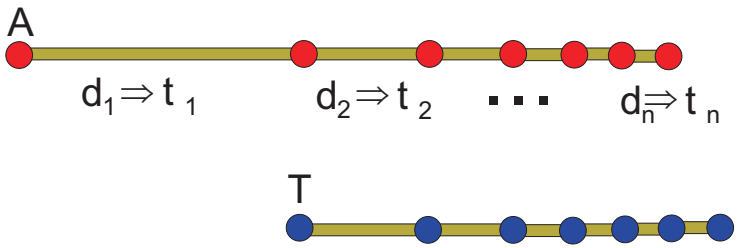

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Conclusion: Achilles catches the tortoise at time T_0!