Answers to Lecture 1 Problems:

1. The primes from 197 to 250 are: 197, 199, 211, 223, 227, 229, 233, 239, 241

2. (a)
$$\begin{array}{ccccc} \Omega(2)=1 & \Omega(3)=1 & \Omega(4)=2 & \Omega(5)=1 & \Omega(6)=2 \\ \Omega(7)=1 & \Omega(8)=3 & \Omega(9)=2 & \Omega(10)=2 & \Omega(11)=1 \\ \Omega(12)=3 & \Omega(13)=1 & \Omega(14)=2 & \Omega(15)=2 & \Omega(16)=4 \\ \Omega(17)=1 & \Omega(18)=3 & \Omega(19)=1 & \Omega(20)=3 & \end{array}$$

   (b) $\Omega(100)=4$, $\Omega(400)=6$, $\Omega(800)=7$

   (c) Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ and $n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$. Then $\Omega(m) = \alpha_1 + \cdots + \alpha_n$ and $\Omega(n) = \beta_1 + \cdots + \beta_n$. Also,
$$mn = p_1^{\alpha_1+\beta_1} p_2^{\alpha_2+\beta_2} \cdots p_n^{\alpha_n+\beta_n}$$
   so
$$\begin{aligned} \Omega(mn) &= (\alpha_1+\beta_1) + (\alpha_2+\beta_2) + \cdots + (\alpha_n+\beta_n) \\ &= \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n \\ &= \Omega(m) + \Omega(n) \end{aligned}$$
   A function with this property is said to be **completely additive**.

   (d)
$$\begin{aligned} \Omega(210000) &= \Omega(210 \times 1000) = \Omega(210) + \Omega(1000) \\ &= \big[\Omega(21) + \Omega(10)\big] + \big[\Omega(10) + \Omega(100)\big] = [2+2] + [2+4] = 10 \end{aligned}$$

   (e) $\Omega(20!) = \sum_{i=2}^{20} \Omega(i) = 1+1+2+1+2+1+3+2+2+1+3+1+2+2+4+1+3+1+3 = 36$

3. The n consecutive numbers $(n+1)!+2, (n+1)!+3, \ldots, (n+1)!+(n+1)$ are all composite. Hence, these form a prime gap of at least n numbers.

4. $22 = 11+11 = 5+17 = 3+19$ is the smallest even number that can be written as a sum of two primes in three ways.

   $34 = 17+17 = 23+11 = 29+5 = 31+3$ is the smallest even number that can be written as a sum of two primes in four ways.

5. Pick any odd number x greater than 5. Then $x = y+3$ where y is an even number greater than 2, so if Goldbach's Conjecture is true, then y can be written as a sum of two primes, so x is written as a sum of three primes. Similarly, for any even number greater x than 5, we can write $x = y+2$.