Intermediate Math Circles - Analytic Geometry I

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1. Cartesian Plane

We use a coordinate system to allow us to translate a geometric problem into an algebraic problem.

We bring a lot to the table: angle properties and theorems, similar and congruent triangles, etc. In the fall you examined Euclidean Geometry learning 10 facts about angles, 7 facts about side lengths, and 6 facts about circles.

Credit for developments in the area of Analytic Geometry go to Rene Descartes. Descartes is also credited with the phrase: "*I think*, *therefore I am*."


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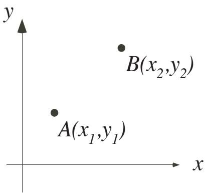
I thought, therefore I was.
search ID: rmcn90
Descartes in the afterlife

(a) Coordinate System and Points

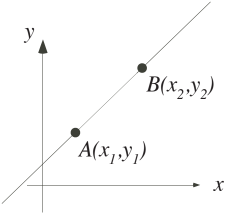
The coordinate system requires an origin, an x-axis, and a y-axis with which you should be familiar. Later in math we will extend to a third dimension.

A point is a specific location on the Cartesian Plane. The diagram shows two points A and B.

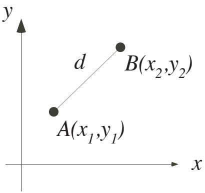


(b) Lines

A straight line can be drawn through two points. It has no beginning and no end.



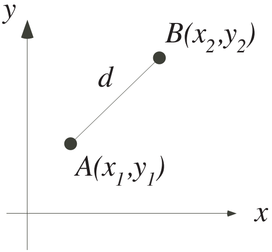
(c) Line Segments



AB is a line segment with endpoints A and B. A line segment is different from a line because it has a fixed length.

We are often interested in finding distances.

2. Distance Between Two Points

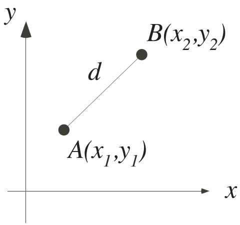


If d is the distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ then

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Can we prove this result? *Proof is shown in the video online.*

2. Distance Between Two Points

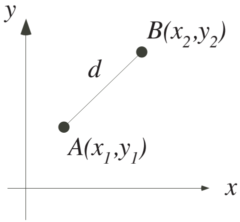


$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Does our formula allow for horizontal and vertical distances?

This result is shown in the video online.

2. Distance Between Two Points



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Does the order of the points matter when we use the formula? *Result is shown in the video online.*

Problem (i)

The line segment joining $A(2, 6)$ to $B(8, -2)$ forms the base of isosceles $\triangle ABP$. The x-coordinate of the third vertex P is -5 and $AP = BP$. Determine the y-coordinate of point P.

Solution is shown in the video online.

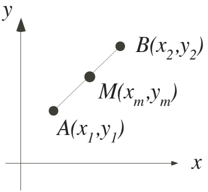
2. Distance Between Two Points

Problem (ii)

Determine the centre of a circle which passes through points $P(0, 3)$, $Q(2, -1)$, and $R(9, 0)$.

Solution is shown in the video online.

3. Midpoint Between Two Points



If $M(x_m, y_m)$ is the midpoint of AB, then

$$M = midpoint(AB) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Can we prove this? *Proof is shown in the video online.*

3. Midpoint Between Two Points

Problem (iii)

Points P, Q, and R divide the line segment from $A(0, 2)$ to $C(6, 0)$, in that order, into four equal parts. Determine the coordinates of the three points, P, Q, and R. *Solution is shown in the video online.*

4. Slope of a Line / Line Segment

(a) Definition and Formula

Slope is a measure of the steepness of a line (or line segment).

Slope is generally represented by the letter m.

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

If you have completed the grade 9 mathematics course you should be very familiar with the concept of slope!

(b) Special Cases

(i) Slope of Horizontal Lines

For horizontal lines, the y-coordinate is constant meaning $y_2 = y_1$ or $y_2 - y_1 = 0$.

$$\therefore m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0$$

(b) Special Cases

(i) Slope of Horizontal Lines: $m = 0$

(ii) Slope of Vertical Lines

For vertical lines, the x-coordinate is constant meaning $x_2 = x_1$ or $x_2 - x_1 = 0$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{0}$$

Since division by zero is undefined, the slope of a vertical line is undefined.

(b) Special Cases

(i) Slope of Horizontal Lines: $m = 0$

(ii) Slope of Vertical Lines: m is undefined

(iii) Slope of Parallel Lines

Parallel Lines have the same slope. For most students this is intuitively obvious. We will not spend much time here.

$$\text{If } l_1 || l_2 \text{ then } m_1 = m_2.$$

(b) Special Cases

(i) Slope of Horizontal Lines: $m = 0$

(ii) Slope of Vertical Lines: m is undefined

(iii) Slope of Parallel Lines: $m_1 = m_2$

(iv) Slope of Perpendicular Lines

Perpendicular Lines have negative reciprocal slopes.

$$\text{If } l_1 \perp l_2 \text{ then } m_1 = \frac{-1}{m_2} \text{ or } m_1 \times m_2 = -1.$$

(b) Special Cases Summary

(i) Slope of Horizontal Lines: $m = 0$

(ii) Slope of Vertical Lines: m is undefined

(iii) Slope of Parallel Lines: $m_1 = m_2$

(iv) Slope of Perpendicular Lines: $m_1 = \frac{-1}{m_2}$ or $m_1 \times m_2 = -1$

Problem (iv)

$\triangle BAH$ has vertices at $B(-1, 9)$, $A(a, 1)$ and $H(-7, -4)$ such that $\angle BAH = 90^\circ$. Determine the value of a.

Present two different solutions. *Solutions are shown in the video online.*

Problem (1)

Three points are *collinear* if they all lie on a straight line. Show that $P(-12, 1)$, $Q(-4, -3)$ and $R(6, -8)$ are collinear.

(a) Use a slope argument to show collinearity.

(b) Use a distance argument to show collinearity.

Problem (2)

The point $A(-2, y)$ is on a line that passes through the points $T(0, -2)$ and $W(4, 0)$. Determine the value of y.

Problem (3)

$\triangle ABC$ has vertex A on the x-axis at -2 and vertex C on the x-axis at 8. The third vertex B is on the y-axis at b such that $\angle ABC = 90^\circ$. Determine all possible values of b.

Problem (4)

A point W is located on the x-axis so that it is 13 units from the point $R(7, 5)$. Find the coordinates of point W.

Problem (5)

The points A and B are located in the first quadrant, equidistant from the origin, O. If the slope of OA is 7 and the slope of OB is 1, determine the slope of AB.

Problem (6)

The vertices of $\triangle ABC$ are $A(-2, -11)$, $B(10, 5)$ and $C(12, 3)$.

(a) Determine the midpoint M of line segment AB.

(b) Show that $AM = MB = MC$. This will prove that M is the centre of a circle containing points A, B and C on the circumference.

(c) Show that $\angle ACB = 90^\circ$.

Problem (7)

The line segment AB, where A is $(2, -4)$ and B is $(10, 8)$, is divided at Q in the ratio $3 : 5$. Find the coordinates of Q.