

Solutions

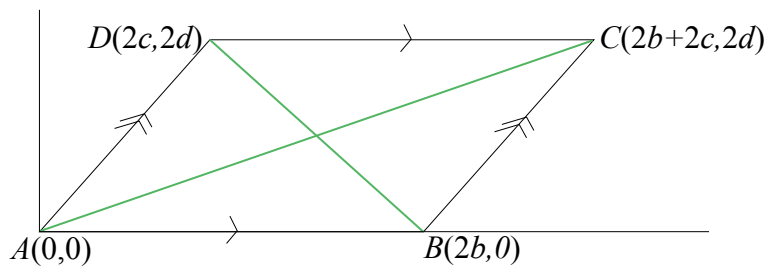
Problem 1

Find the midpoint of line AC and line BD:

$$M_{AC} = \left(\frac{2b+2c}{2}, \frac{2d}{2} \right) \quad \text{and} \quad M_{BD} = \left(\frac{2b+2c}{2}, \frac{2d}{2} \right)$$

$$= (b+c, d) \quad \quad \quad = (b+c, d)$$

Therefore, since the midpoints of each line are the same point, it is the point of intersection, and so the diagonals of a parallelogram bisect each other.



Problem 2

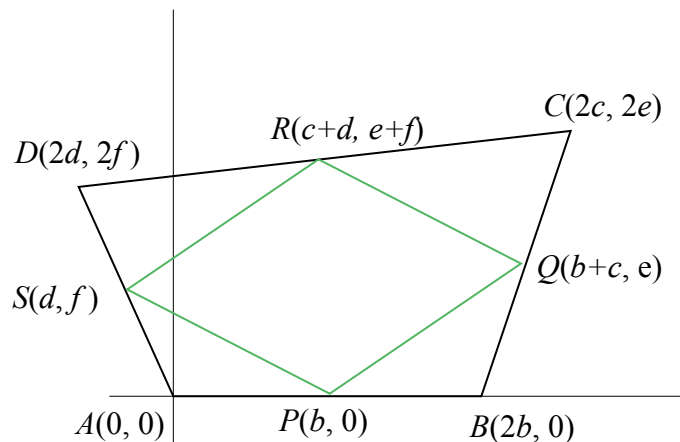
$$m_{PQ} = \frac{e}{(b+c)-b} \quad \text{and} \quad m_{SR} = \frac{(e+f)-f}{(c+d)-d}$$

$$= \frac{e}{c} \quad \quad \quad = \frac{e}{c}$$

$$m_{QR} = \frac{(e+f)-e}{(c+d)-(b+c)} \quad \text{and} \quad m_{PS} = \frac{f}{d-b}$$

$$= \frac{f}{d-b}$$

$\therefore m_{PQ} = m_{SR}$ and $m_{QR} = m_{PS}$, so $PQRS$ is a parallelogram.



Problem 3

$$m_{BC} = \frac{2d}{2c-2b} = \frac{d}{c-b}$$

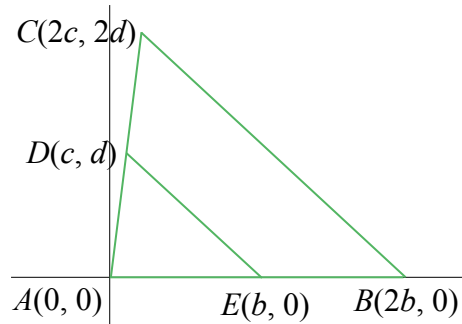
$$m_{ED} = \frac{d}{c-b}$$

and

$$BC = \sqrt{(2c-2b)^2 + (2d)^2} = 2\sqrt{(c-b)^2 + d^2}$$

$$ED = \sqrt{(c-b)^2 + d^2} = \frac{1}{2}BC$$

Therefore $m_{BC} = m_{ED}$ and $ED = \frac{1}{2}BC$ so the line segment joining the midpoints of two sides of the triangle is parallel to the third side and one-half the length of the third side.



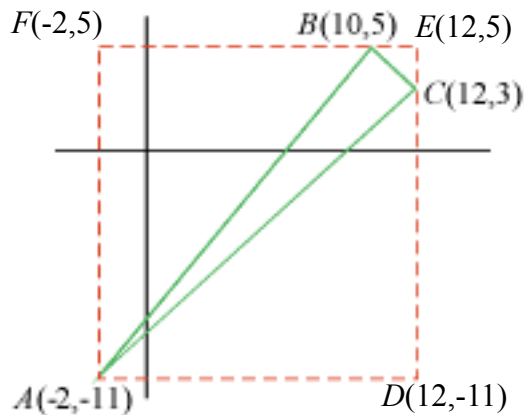
Problem 4

$$A_{\triangle ABC} = A_{\text{rectangle}} - A_{\text{outer triangles}}$$

$$A_{\text{rectangle}} = A_{ADEF} = (12 - (-2)) \times (5 - (-11)) = 14 \times 16 = 224$$

$$A_{\text{outer triangles}} = A_{\triangle ADC} + A_{\triangle CEB} + A_{\triangle ABF} = \frac{1}{2}[14(14) + 2(2) + 16(12)] = 196$$

$$\therefore A_{\triangle ABC} = 224 - 196 = 28$$



Verification using formula for area of a triangle:

$$\begin{array}{r}
 10 \quad 5 \\
 -2 \quad \times \quad -11 \\
 12 \quad \times \quad 3 \\
 10 \quad \times \quad 5
 \end{array}
 \quad
 A_{\triangle ABC} = \frac{1}{2} \left| 10(-11) + (-2)(3) + 12(5) - 5(-2) - (-11)(12) - 3(10) \right|$$

$$= \frac{1}{2} \left| -110 - 6 + 60 + 10 + 132 - 30 \right|$$

$$= 28$$

Problem 5

$$\begin{aligned}
 \text{Area} &= \frac{1}{2} |x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_3y_2 - x_2y_1| \\
 &= \frac{1}{2} |(-12)(-3) + (-4)(-8) + (6)(1) - (-12)(-8) - (6)(-3) - (-4)(1)| \\
 &= \frac{1}{2} |36 + 32 + 6 - 96 + 18 + 4| \\
 &= 0
 \end{aligned}$$

The triangle does not have an area because they all lie on the same line.

Problem 6

$$\begin{aligned}
 \text{Area} &= \frac{1}{2} |(4)(p) + (6)(6) + (0)(3) - (4)(6) - (0)(p) - (6)(3)| \\
 7 &= \frac{1}{2} |4p + 36 + 0 - 24 - 0 - 18| \\
 14 &= |4p - 6| \\
 7 &= |2p - 3|
 \end{aligned}$$

Since we have an absolute value, we have two answers.

$$\begin{array}{ll}
 7 = 2p - 3 & -7 = 2p - 3 \\
 p = 5 & p = -2
 \end{array}$$

Problem 7

Find point of intersection of BD and AC :

Equation of line BD :

$$y = -\frac{8}{p}x + 8$$

$$x = (8 - y)\frac{p}{8}$$

Equation of line AC :

$$y = \frac{10}{p}x$$

$$x = \frac{p}{10}y$$

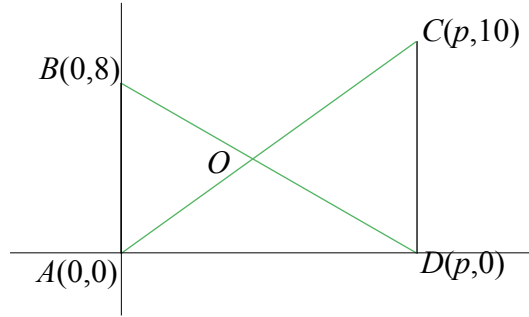
Equating lines BD and AC :

$$(8 - y)\frac{p}{8} = \frac{p}{10}y$$

$$10(8 - y) = 8y$$

$$80 = 18y$$

$$y = \frac{40}{9}$$



Problem 8

$$\begin{aligned}
 d &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \\
 &= \frac{|(2)(-3) + (-7)(5) + 1|}{\sqrt{2^2 + (-7)^2}} \\
 &= \frac{|-6 - 35 + 1|}{\sqrt{4 + 49}} \\
 &= \frac{40}{\sqrt{53}}
 \end{aligned}$$

Problem 9

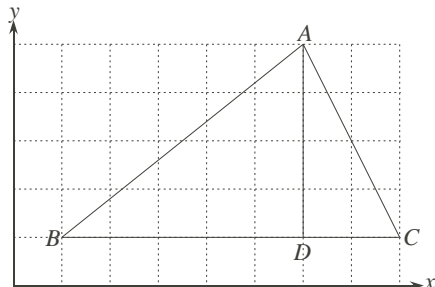
The slope the line is $\frac{1}{3}$, so the slope of the perpendicular is -3 .

$$\begin{aligned}
 -3 &= \frac{\frac{1}{3}x - \frac{2}{3} - (-6)}{x - 2} \\
 -9(x - 2) &= x - 2 + 18 \\
 -9x + 18 &= x + 16 \\
 10x &= 2 \\
 x &= \frac{1}{5} \Rightarrow y = -\frac{3}{5}
 \end{aligned}$$

Problem 10

Method 1:

By plotting the triangle, we can locate D.



After that, it is clear that the length of AD is 4.

Method 2:

The slope of line BC is

$$m_{BC} = \frac{1 - 1}{8 - 1} = 0$$

Since the slope of BC is 0, we know that any line perpendicular to BC is vertical. Thus $D = (6, 1)$, and the length of AD is

$$\sqrt{(6 - 6)^2 + (5 - 1)^2} = 4$$

Method 3:

Using the formula for distance from a point $(6, 5)$ to a line $y - 1 = 0$, we have

$$\begin{aligned} d &= \frac{|(0)(6) + (1)(5) + (-1)|}{\sqrt{0^2 + 1^2}} \\ &= \frac{|5 - 1|}{1} \\ &= 4 \end{aligned}$$

Method 4:

Using the length of BC is 7. The area of the triangle is

$$\begin{aligned} \text{Area} &= \frac{1}{2}|(6)(1) + (1)(1) + (8)(5) - (6)(1) - (8)(1) - (1)(5)| \\ &= \frac{1}{2}|6 + 1 + 40 - 6 - 8 - 5| \\ \frac{1}{2}bh &= \frac{1}{2}(28) \\ 7h &= 28 \\ h &= 4 \end{aligned}$$

Problem 11

Let $P = (p, 2p+3)$ and $Q = (q, -q+2)$, The midpoint would be $M = \left(\frac{p+q}{2}, \frac{(2p+3)+(-q+2)}{2} \right)$.

$$\frac{p+q}{2} = 2$$

$$p+q = 4$$

$$\frac{(2p+3)+(-q+2)}{2} = 5$$

$$2p - q + 5 = 10$$

$$2p - q = 5$$

Solving these two equations, we have

$$(p+q) + (2p-q) = 4+5$$

$$3p = 9$$

$$p = 3 \Rightarrow q = 1$$

$\therefore P = (3, 9)$ and $Q = (1, 1)$