University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

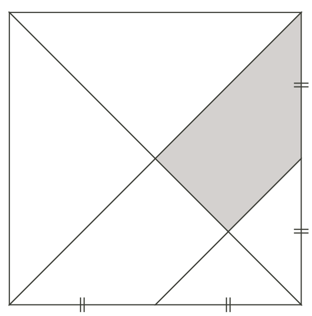
Grade 6 Math Circles
March 30, 2011
Jeopardy

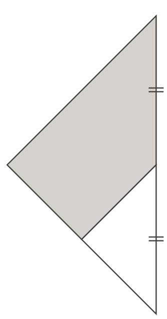
Gauss Questions

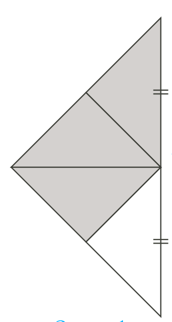
1. The month of April, 2000, had five Sundays. Three of them fall on even numbered days. What day is the eighth day of that month?

   Since April has 30 days, and five of them a Sundays, the Sundays must fall on the set {1, 8, 15, 22, and 29} or {2, 9, 16, 23, and 30}. Since three of them fall on even numbered days, Sundays are on {2, 9, 16, 23, and 30}, which means the eighth day of the month is a Saturday.

2. A square is divided, as shown. What fraction of the area of the square is shaded?



We know that  is $\frac{1}{4}$ of the area of the square.

We can split the triangle into congruent parts.  We can see that the shaded area is $\frac{3}{4}$ of the area of the triangle. So the shade area is $\frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$ of the area of the square.

3. In the square shown, the number in each row, column and diagonal multiply to give the same result. What is the sum of the missing numbers?

12	1	18
		4

$12 \times 1 \times 18 = 216$, so every row, column and diagonal must multiply to 216. To get a missing number, we can divide 216 by the other numbers in the group. For example, to find out the bottom right cell, we proceed with $216 \div 18 \div 4 = 3$. Continuing in this manner, we have

12	1	18
9	6	4
2	36	3

Adding the missing numbers up gives $9 + 6 + 2 + 36 + 3 = 56$

4. Each of the integers 226 and 318 have digits whose product is 24. How many three-digit positive integers have digits whose product is 24?

Let's list all groups of three 1-digit numbers that multiply to 24: $\{1, 3, 8\}$ $\{1, 6, 4\}$ $\{2, 3, 4\}$ and $\{2, 6, 2\}$.

For the first three groups, all the numbers are distinct, and so there are $3! = 3 \times 2 \times 1 = 6$ different arrangements for each group.

For the last group, since there are two identical numbers, there are $\frac{3!}{2!} = \frac{3 \times 2 \times 1}{2 \times 1} = 3$ different arrangements.

Therefore, there are $6 + 6 + 6 + 3 = 21$ three-digit positive integers whose digits multiply to 24.

5. In a softball league, after each team has played every other team 4 times, the total accumulated points are: Lions 22, Tigers 19, Mounties 14, and Royals 12. If each team received 3 points for a win, 1 point for a tie and no points for a loss, how many games ended in a tie?

Instead of worrying about each team's wins or ties, let's look at this problem as an accumulation of points in total.

Let w be the number of games that ended in a win/loss situation, and let t be the number of games that ended in tie.

In total, there are 6 ways to choose two teams out of 4: {Lions vs Tigers}, {Lions vs Mounties}, {Lions vs Royals}, {Tigers vs Mounties}, {Tigers vs Royals}, and {Mounties vs Royals}. Each team plays each other 4 times, so in total, there were $6 \times 4 = 24$ matches. Each match must end in either a win/loss or a tie, so the sum of w and t must be 24.

$$\begin{aligned} w + t &= 24 \\ w &= 24 - t \qquad (1) \end{aligned}$$

For every game that ends in a win/loss, one team gets 3 points and the other team gets 0, so 3 points are gained as a whole group. If the game ends in a tie, each team gets 1 point, so 2 points are gained as a whole group. In total, there are $22 + 19 + 14 + 12 = 67$ points. The 67 points are the sum of the 3 points for each win and 2 points for every match that ties.

$$3w + 2t = 67 \qquad (2)$$

Using substitution, we can substitute (1) into (2) to get

$$\begin{aligned} 3(24 - t) + 2t &= 67 \\ 72 - 3t + 2t &= 67 \\ 72 - t &= 67 \\ 72 - 67 &= t \\ t &= 5 \end{aligned}$$

Therefore, 5 games ended in ties.

Logic

1. Imagine that you have three crates, one containing oranges, one containing apples, and the third containing apples and oranges.

   The crates were originally labelled for their contents but someone has inadvertently switched the labels so that now every crate is incorrectly labelled!

   Without looking inside, you are allowed to take fruit at a time out of any hat that you wish, and by this process of sampling, you are to determine the contents of all three crates.

   What is the smallest number of drawings needed to do this?

   We only need 1 drawing! If we pick a fruit from the crate labelled "Apples and Oranges", then we know that it cannot contain a mixture of both, and therefore, that crate will only contain one type of fruit. Say the fruit was an apple. Then we know that the crate labelled "Apples and Oranges" contains only apples. We are left with the oranges and the mixture of both. Since the crate labelled "Oranges" cannot contain only oranges, it must contain the mixture, which only leaves the crate labelled "Apples" to contain the oranges. The logic is the same if we picked the orange instead of the apple from the crate labelled "mixture"

2. In your basement are three light switches, all of them currently in the OFF position. Each switch controls one of three different lamps on the floor above. You would like to find out which light switch corresponds to which lamp.

   You may move turn on any of the switches any number of times, but you may only go upstairs to inspect the lamps just once.

   How can you determine the switch for each lamp with just one trip upstairs?

   Turn Switch 1 on and leave it on for a little while... about five minutes or more... and then turn it off. Turn Switch 2 on and go upstairs to inspect the lamps.

   * The lamp with the bulb that is off but warm is controlled by Switch 1.
   * The lamp that is currently on is controlled by Switch 2.
   * The lamp that is off and cold is controlled by Switch 3.

3. Kyle, Neal, and Grant were rounded up by their mother yesterday, because one of them was suspected of having grabbed a few too many cookies from the cookie jar. The three brothers made the following statements under very intensive questioning:

   * Kyle: I'm innocent.
   * Neal: I'm innocent.
   * Grant: Neal is the guilty one.

   If only one of these statements was true, who took the cookies?

Assume Neal took the cookies. If so, then both Kyle's and Grant's statements would be true. Hence it was not Neal.

Now assume Grant took the cookies. If so, then both Kyle's and Neal's statements would be true. Hence it was not Grant.

Now assume it was Kyle. Then only Neal's statement was true, so Kyle took the cookie.

4. You have two strings whose only known property is that when you light one end of either string it takes exactly one hour to burn. The rate at which the strings will burn is completely random and each string is different.

   How do you measure 45 minutes?

   Light both the ends of the first string and one end of the second string. 30 minutes will have passed when the first string is fully burned, which means 30 minutes have burned off the second string. Light the end of the second string and when it is fully burned, 45 minutes will have passed.

5. A king wants his daughter to marry the smartest of 3 extremely intelligent young princes. The princes are gathered into a room and seated, facing one another, and are shown 2 black hats and 3 white hats.

   They are blindfolded, and 1 hat is placed on each of their heads, with the remaining hats hidden in a different room. The king tells them that the first prince to deduce the color of his hat without removing it or looking at it will marry his daughter. The blindfolds are then removed.

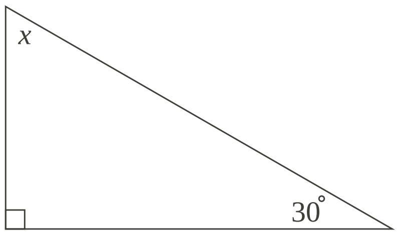
   You are one of the princes. You see 2 white hats on the other prince's heads. After some time you realize that the other prince's are unable to deduce the color of their hat, or are unwilling to guess. What color is your hat?

   If you were wearing the black hat, it would not take long for one of the other princes to deduce he was wearing a white hat.

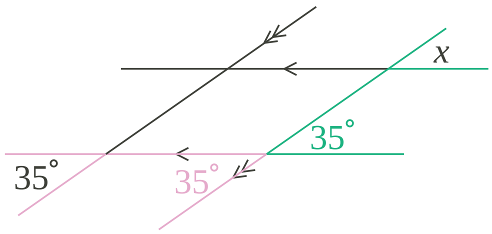
   If an intelligent prince saw a white hat and a black hat, he would eventually realize that the king would never select two black hats and one white hat. Any prince seeing two black hats would instantly know he was wearing a white hat. Therefore if a prince can see one black hat, he can work out he is wearing white.

   Therefore, you are wearing a white hat.

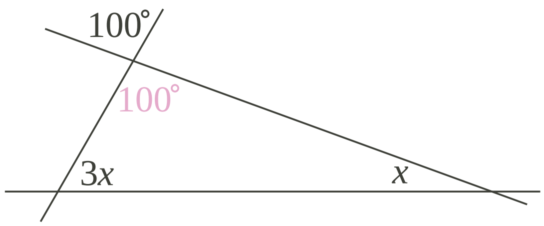
Angle Chasing

1. 

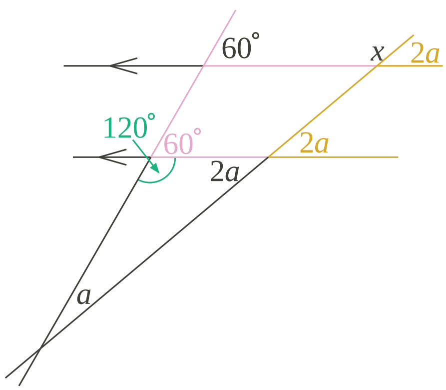
$$180 = 90 + 30 + x$$
$$180 - 90 - 30 = x$$
$$x = 60°$$

2. 

Using opposite angles once and two corresponding angles (“F” patterns), we can see that $x = 35°$

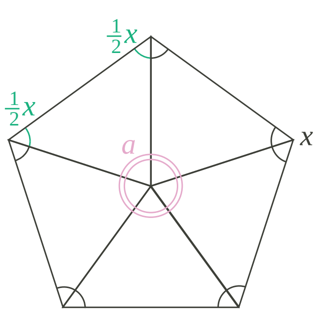
3. 

$$180 = 100 + 3x + x$$
$$80 = 4x$$
$$x = 20°$$

4. 

$$180 = 120 + a + 2a$$
$$60 = 3a$$
$$a = 20°$$

$$180 = 2a + x$$
$$180 = 40 + x$$
$$x = 140°$$

5. 

$$360 = 5a$$
$$a = 72$$

$$180 = a + \frac{1}{2}x + \frac{1}{2}x$$
$$180 = 72 + x$$
$$x = 108$$

Arithmetic

1. $158 + 348 - 90$

$$\begin{aligned} 158 + 348 - 90 &= 506 - 90 \\ &= 416 \end{aligned}$$

2. $25 - 15 \times 20$

$$\begin{aligned} 25 - 15 \times 20 &= 25 - 300 \\ &= -275 \end{aligned}$$

3. $45 + 58 \div [5 \div (27 - 12) \times 3]$

$$\begin{aligned} 45 + 58 \div [5 \div (27 - 12) \times 3] &= 45 + 58 \div [5 \div 15 \times 3] \\ &= 45 + 58 \div [5 \times 3 \div 15] \\ &= 45 + 58 \div [15 \div 15] \\ &= 45 + 58 \div 1 \\ &= 103 \end{aligned}$$

4. $\frac{7}{4} - \frac{2}{3} \times [3 \times 15 \div (17 - 14) + 6]$

$$\begin{aligned} 15 - \frac{2}{3} \times [3 \times 15 \div (17 - 14) + 6] &= \frac{7}{4} - \frac{2}{3} \times [3 \times 15 \div 3 + 6] \\ &= 15 - \frac{2}{3} \times [3 \div 3 \times 15 + 6] \\ &= 15 - \frac{2}{3} \times [15 + 6] \\ &= 15 - \frac{2}{3} \times 21 \\ &= 15 - \frac{2}{1} \times 7 \\ &= 15 - 14 \\ &= 1 \end{aligned}$$

5. $\{[3 \times 5 \times (3 + 6 \times 2) - 125] + 20\} \div \frac{35}{14}$

$$\begin{aligned} \{[3 \times 5 \times (3 + 6 \times 2) - 125] + 20\} \div \frac{35}{14} &= \{[15 \times (3 + 12) - 125] + 20\} \div \frac{5}{2} \\ &= \{[15 \times (15) - 125] + 20\} \times \frac{2}{5} \\ &= \{[225 - 125] + 20\} \times \frac{2}{5} \\ &= \{100 + 20\} \times \frac{2}{5} \\ &= \{120\} \times \frac{2}{5} \\ &= 24 \times \frac{2}{1} \\ &= 48 \end{aligned}$$

Spot the Pattern

1. $2, 5, 14, 41, \ldots$; what is the next term?

   The pattern is that to get to the next term, we triple the previous term and minus 1. The next term is $3 \times 41 - 1 = 123 - 1 = 122$

2. $1, 1, 2, 3, 5, 8, 13, 21, \ldots$; what is the 10th term?

   To get the next term, we add the two previous terms. This is known as the Fibonacci sequence.
   The 9th term is $13 + 21 = 34$
   The 10th term is $21 + 34 = 55$

3. $2, -4, 8, -16, \ldots$; what is the 7th term?

   To get the next term, we multiply the previous term by -2
   The 5th term is $-16 \times (-2) = 32$
   The 6th term is $32 \times (-2) = -64$
   The 7th term is $-64 \times (-2) = 128$

4. $1, 3, 6, 10, 15, \ldots$; what is the 100th term?

   The first term 1
   The second term is $1 + 2$
   The third term is $1 + 2 + 3$
   The fourth term is $1 + 2 + 3 + 4$

   Continuing in this pattern, the 100th term will be $1 + 2 + 3 + ... + 98 + 99 + 100$
   Let's rearrange is expression into

   $100 + (1 + 99) + (2 + 98) + (3 + 97) + ... + (47 + 53) + (48 + 52) + (49 + 51) + 50.$

   Notice that the sums in the brackets are all 100. In fact, there are 49 of these sums which makes 4900.
   $100 + 4900 + 50 = 5050$. The 100th term is 5050.

5. $0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, \ldots$;
   which term in the pattern is 1101?

   The pattern is counting in binary starting from 1. The binary number 1101 converts to 13 in base-10. Therefore, 1101 is the 13th term in the pattern.

Final Jeopardy

1. Let x be the three-digit number with digits ABC and y be the three-digit number with digits CBA. The digits A and C are not 0. If $x - y = 495$, how many possibilities are there for x? [**Gauss8 2004 Q24**]
   Let $x = 100A + 10B + C$ and $y = 100C + 10B + C$. Then

   $$\begin{aligned} 495 &= x - y \\ &= (100A + 10B + C) - (100C + 10B + A) \\ &= 100A + 10B + C - 100C - 10B - A \\ &= 99A - 99C \\ &= 99(A - C) \\ \frac{495}{99} &= A - C \\ 5 &= A - C \\ A &= 5 + C \end{aligned}$$

   Since A nor C can be 0, the pair (A, C) are restricted to only 4 choices: $(6, 1), (7, 2), (8, 3),$ and$(9, 4)$. B is not restricted, so it can be any digit; hence, 10 choices for B.
   $\therefore$ There are $4 \times 10 = 40$ possibilities.

2. A large block, which has dimensions n by 11 by 10, is made up of a number of unit cubes and one 2 by 1 by 1 block. There are exactly 2362 positions in which the 2 by 1 by 1 block can be placed. What is the value of n? [**Gauss8 2004 Q25**]
   Consider positions of the 2 by 1 by 1 block which are entirely contained in one layer. In each layer, there are 9 possible positions for the 2 by 1 by 1 block in each column (crossing rows 1 and 2, 2 and 3, 3 and 4, and so on, up to 9 and 10), and there are 10 possible positions in each row (crossing columns 1 and 2, 2 and 3, 3 and 4, and so on, up to 10 and 11). Therefore, within each layer, there are $11(9) + 10(10) = 199$ positions for the 2 by 1 by 1 block. In the large block, there are thus $199n$ positions of this type for the 2 by 1 by 1 block, since there are n layers.

   Next, consider positions of the 2 by 1 by 1 block which cross between two layers. Since each layer has 110 blocks (in 11 columns and 10 rows) then there are 110 positions for the 2 by 1 by 1 block between each pair of touching layers. Since there are $n?1$ pairs of touching layers, then there are $110(n?1)$ positions of this type.

   Thus, we have

   $$\begin{aligned} 199n + 110(n - 1) &= 2362 \\ 309n - 110 &= 2362 \\ 309n &= 2472 \\ n &= 8 \end{aligned}$$