University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles
March 9, 2011
Combinations

Review

1. Evaluate 6! $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

2. Evaluate $\frac{7!}{5!}$ $\frac{7 \times 6 \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = 7 \times 6 = 42$

3. Assuming a "word" is just an arrangement of letters. How many 3-letter "words" are there?

   $26 \times 26 \times 26 = 26^3 = 17576$

4. How many 3-letter "words" are there with no repeating letters?

   $26 \times 25 \times 24 = \frac{26!}{23!} = 15600$

5. How many 3-letter "words" are there with at least one repeating letter?

   We can take all the 3-letter words, and take away all the 3-letter words with no repeating letters
   $17576 - 15600 = 1976$

6. 10 swimmers compete in a race. How many arrangements of first, second and third place are there assuming there is no tie?

   $10 \times 9 \times 8 = \frac{10!}{7!} = 720$

7. You are judging a talent show with 9 contestants. For the first half of the show, you need to pick the order of the first four acts. How many possible ways are there?

   $9 \times 8 \times 7 \times 6 = \frac{9!}{5!} = 3024$

General Rule: The number of permutations of n taken k items at a time is

$$\frac{n!}{(n-k)!}$$

The short form of this notation is ${}_n\mathrm{P}_k$

Permutations with Identical Elements

So far, we have seen permutations of items that are distinct. What happens when there are identical elements in our set?

Problem: You are making a bracelet and need to put 4 beads in a line. You have 2 red beads, 1 green beads, and 1 blue bead. How many different ways can you make the line?

Let's pretend that the two red beads are different (say, R and r). We then know that there are $4! = 24$ permutations or arrangements of the beads. However, we have double counted, because in reality, the two red beads can switch places and we would still have the same arrangement.

BGRr	BRGr	BRrG	GBRr	GRBr	GRrB
BGrR	BrGR	BrRG	GBrR	GrBR	GrRB
RBGr	RGrB	RrBG	RGBr	RBrG	RrGB
rBGR	rGRB	rRBG	rGBR	rBRG	rRGB

For any one arrangement, there are 2! copies because, we can switch the red beads around. $\therefore$ We have $\frac{24}{2!} = 12$ arrangements.

Problem: How many ways can you arrange three A's and two B's?

Once again, let's assume that the A's and B's are all different, so that we can have $5! = 120$ arrangements. Now, let's take a look at one of those arrangements.

$$A_1A_2A_3B_1B_2$$

Now, if we take away the assumption that the A's and B's are different, then $A_1A_2A_3B_1B_2$ is the same as $A_2A_1A_3B_1B_2$ or $A_2A_1A_3B_2B_1$. In fact, there are 3! permutations of A's an 2! permutations of B's, so in total, for every arrangement, there are $3! \times 2! = 12$ permutations associated with it. We can prevent the double counting by dividing the total number of permutations by the number of "double counts".

$$\frac{5!}{3!2!} = \frac{120}{12} = 10$$

There are only 10 possible arrangements with three A's and two B's.

Exercises:

1. Your weekly chores include washing the dishes 3 times, taking out the garbage once and cleaning your room twice. How many ways can you order these tasks?

   In total, you have 6 tasks to do.

$$\frac{6!}{2!3!} = 60$$

2. Instead of having 4 beads, you now have 10 beads: 3 yellow, 3 green, 1 blue, 1 orange, 1 red and 1 pink. How many ways can you line all the beads up for the bracelet?

   There are 10! ways to line up the beads if all of them are different, but we double counted the ones of the same colour. There are 3!=6 ways to rearrange the green beads and still get the same order, and 3!=6 ways to rearrange the yellow beads and still get the same order.

$$10! \div 3! \div 3! = 100800$$

   $\therefore$ there are 100800 ways to line up the beads

General Rule: The number of permutations of n items with a identical items of one kind, b identical items of another kind, etc. is

$$\frac{n!}{!a!b!\ldots}$$

<u>Combinations: Order Doesn't Matter!</u>

Problem: In a singing competition, there are 5 finalists (A, B, C, D and E). As a judge, you only pick the *top three* and do not specify their placing. How many possible outcomes could there be?

Let's list the outcomes.

ABC ABD ABE ACD ACE ADE BCD BCE BDE CDE

$\therefore$ There are 10 possible outcomes

In the above problem we had to choose 3 people out of a group of 5 and did not care about the order. If order mattered, then we know the answer is just $\frac{5!}{2!}$. But when order matters, we are double counting occurrences (we counted ABC as well as ACB, BAC, BCA, CAB and CBA). For any group of 3, there are $3! = 6$ groups that contain the same elements, but just in a different order. In a the case where order does not matter, we have to divide the permutation by this so that we don't double count.

Example: You just bought 7 paintings and want to put 3 of them on a wall. How many ways can you do this?

We know that there are $\frac{7!}{(7-3)!} = 210$ ways of putting the paintings up **if order mattered**. Now all we have to do is take out the double counted arrangements. For any three paintings on the wall, there are $3! = 6$ ways to order them (say, from left to right). So in our 210 permutations, we actually counted the same painting arrangements 6 times!

$$\frac{210}{6} = 35$$

$\therefore$ There are 35 ways to pick 3 paintings out of 7.

Notation: In general, if we want to choose k elements from a group of n, we know that there are $\frac{n!}{k!(n-k)!}$ ways of doing so. We can make up a short hand to represent the number of ways to choose k elements from a group of n.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We say $\binom{n}{k}$ as "n choose k".

Problem Set

1. On an announcement board, you take out the letters to the word "CAUTION". How many ways can you arrange the letters and put them back on the board?

2. On an announcement board, you take out the letters to the word "WARNING". How many ways can you arrange the letters and put them back on the board?

3. How many ways can you order the letters of "MISSISSIPPI"?

4. From the top 40's chart, you want to pick 3 songs to move to your playlist. How many different combinations are there?

5. You have a deck of cards. How many possible five card hands can you draw?

6. With a deck of cards, a full house is a 5 card combination that has 3 of a kind and a a pair. How many possible full house combinations are there?

Answers:

1. $7! = 5040$
2. $\dfrac{7!}{2!} = 2520$
3. $\dfrac{11!}{4!4!2!} = 34650$
4. $\dbinom{40}{3} = \dfrac{40!}{37!3!} = 9880$
5. $\dbinom{52}{5} = \dfrac{52!}{47!5!} = 2598960$
6. There are 13 suits to choose from for the triplet. After that, there are 12 suits remaining for the pair. There are $\binom{4}{3}$ ways to choose 3 cards out of 4 cards of the same suit, and $\binom{4}{2}$ ways to choose 2 cards out of 4 of the same suit. Putting all this information together, we have $13 \times \dbinom{4}{3} \times 12 \times \dbinom{4}{2} = 3744$ different full house combinations.