UNIVERSITY OF WATERLOO

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Faculty of Mathematics

Centre for Education in
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# Grade 7/8 Math Circles
# Winter 2011
# Probability II

- **Binomial Distribution**

The binomial distribution gives the discrete probability function $P(n)$ of obtaining exactly $n$ successes out of $N$ Bernoulli trials.

**Example 1** In a coin toss, what is the probability that heads is flipped exactly two times out of three tosses? What if the coin is weighted so that the probability that heads occurs is 0.6?

In this case, heads is considered the success, and the number of tosses is the number of trials. So $n = 2$ and $N = 3$, and we want to find $P(n)$.

**Non weighted coin:**
All possible outcome are : {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}, so we can see there are 3 outcomes in which heads turns up twice, out of 8 total outcomes. Since the probability of heads occuring is equal to the probability of tails occuring, $P(n) = 3/8 = 0.375$

**Weighted coin:**
The probability of HHT is: $0.6 \times 0.6 \times 0.4 = 0.144$. But since there are 3 orderings in which two heads and one tail can occur, we get a total probability of:

$$\begin{aligned} 0.144 + 0.144 + 0.144 &= 3 \times 0.144 \\ &= 0.432 \end{aligned}$$

To get the total number of ways the coin can be flipped to get the desired outcome, we have $3 \times 2 \times 1 = 6$, however we can order the number of successes (i.e. heads) in $2 \times 1 = 2$ ways. so we get there are $6 \div 2 = 3$ unique ways the coins can be flipped to get the desired outcome. Which as we determined in the example above, this is just $\binom{3}{2}$. Then, we multiply this by the probabily of getting a success exponentiated to the number of successes we want, mulitplied by the probability of getting a failure exponentiated to the number of failures we want (i.e. the total number of trials subtracted by the number of successes:

$$\begin{aligned} P(2) &= \binom{3}{2} \times 0.6^2 \times 0.4^{3-2} \\ &= 3 \times 0.6^2 \times 0.4^1 \\ &= 0.432 \end{aligned}$$

In general the binomial distribution is given by:

$$\begin{aligned} P(n) &= \begin{pmatrix} N \\ n \end{pmatrix} p^n q^{N-n} \\ &= \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n} \end{aligned}$$

**Note:** $P(0) + P(1) + P(2) + ... + P(N-1) + P(N) = 1$

**Example 2** In a (non weighted) coin toss what is the probability that heads is flipped 18 times out of 25 tosses?

$$\begin{aligned} P(18) &= \binom{25}{18} \times 0.5^{18} \times 0.5^{7} \\ &= 0.014325976 \end{aligned}$$

**Exercise 1** A student randomly guesses the correct answer in a mulitple choice quiz of 10 question. If each question has 5 choices, what is the probability the student gets exactly 7 correct? What is the probability the students gets more than 7 correct?

**Exercise 2** A company owns 300 xbox consoles. Each console has an 8% probability of not working. You randomly select 25 consoles.

a) What is the probability that all xbox consoles are working?

b) What is the probability that exactly 3 consoles are not working?

c) What is the probability that 1 or 2 consoles are not working?

d) What is the probability that more than 3 consoles are not working?

For part c of exercise 4, we have to find the probability that either 1 or 2 consoles are broken. In this case we find the probability that 1 console is broken, and the probability that 2 consoles are broken, then we sum the two.

**Rule of Thumb:** When you see "or" you need to add

For part d, we need to find the probability that **more than** 3 consoles are not working. So we want when 4, or 5, or 6, or 7, ..., or 24, or 25 consoles are broken. The probability that 0, or 1, or 2, or 3, ..., or 24, or 25 consoles are broken is 1 since it is for sure that either none is broken ($n = 0$) or some are broken ($n \geq 1$). So instead of finding $P(4) + P(5) + P(6) + ... + P(25)$, we can find $P(0) + P(1) + P(2) + P(3)$, and then subtract it from 1. (We have already found these probability from parts a, b and c, so all that's left to do is subtract it them from 1.)

- **Hypergeometric Distribution**

In a hypergeometric distribution there are $n$ ways to make a successful selection and $m$ ways to make a failed selection, for a total of $m+n$ possibilities. You have N trails to make $i$ successful selections, without replacement. This is the probability that we wish to find in a hypergeometric probability distribution.

⋆ The only difference between binomial distribution and hypergeometric distribution is that binomial is **with** replacement and hypergeometric is **without** replacement

$$P(i \text{ successes}) = \frac{(\text{number of ways of getting } i \text{ successes}) \times (\text{number of ways of getting } N-i \text{ failures})}{\text{total number of ways of selection}}$$

◇ number of ways of getting $i$ successes $= \binom{n}{i}$

◇ number of ways of getting $N-i$ failures $= \binom{m}{N-i}$

◇ number of way of selection $= \binom{m+n}{N}$

$$P(x=i) = \frac{\binom{n}{i} \times \binom{m}{N-i}}{\binom{m+n}{N}}$$

**Example 3** You have a bag of 6 blue marbles and 4 red marbles. What is the probability that you choose 3 blue marble in 4 tries, without replacing the marbles?

$$\begin{aligned} P(x=3) &= \frac{\binom{6}{3} \times \binom{4}{1}}{\binom{10}{4}} \\ &= 0.380952 \end{aligned}$$

**Example 4** For halloween, your parents buy a box of candy that has 36 boxes of smarties, 24 kit-kat bars, and 24 packs of m&m's. You are the first person to choose from the box of candies, and you choose five pieces of candy. What is the probability that at least 2 are m&m's?

$$\begin{aligned} P(x \geq 2) &= 1 - P(x \leq 1) \\ &= 1 - [P(x=0) + P(x=1)] \\ &= 1 - \left[ \frac{\binom{24}{0} \times \binom{60}{5}}{\binom{84}{5}} + \frac{\binom{24}{1} \times \binom{84}{4}}{\binom{84}{5}} \right] \end{aligned}$$

**Exercise 3** A candy dish contains 100 jelly beans and 80 gumdrops. Fifty candies are picked at random. What is the probability that 35 of the 50 are gumdrops?

**Problem Set**

1. Seven Americans are randomly selected. Determine the probability that at least 3 of them are smokers if 27% of Americans smoke

2. Dave can make free throw 86% of the time. If he throws 8 free throws, what is the probability that he will make more than 5 of them?

3. On an assembly line of Acme Radio, the probability that a radio is defective is 1/15. If an inspector randomly checks 10 items, what is the probability of finding:

   a) exactly 2 defective

   b) no more than 2 defective

   c) at least 3 defective

4. A basketball player has a free throw percentage of 75%. In a single game, the player shoots 12 free throws. Compute the following probabilities.

   a) What is the probability that she makes exactly 6?

   b) What is the probability that she makes exactly 75% or 9 of the free throws?

   c) What is the probability that she makes all 12?

5. Suppose that a student is taking a multiple-choice exam in which each question has four choices. Assuming that she has no knowledge of the correct answer to any of the questions, she has decided on a strategy in which she will place four balls (marked A, B, C, and D) into a box. She randomly selects one ball for each question and replaces the ball in the box. The marking on the ball will determine her answer to the question.

   a) If there are five question on the exam, what is the probability that se will get five questions correct?

   b) What is the probability that she will get at least four questions correct?

   c) What is the probability that sh will get no question correct?

   d) What is the probability that she will get no more than two questions correct?

6. Suppose a shipment of 100 VCRs is known to have 10 defective VCRs. An inspector chooses 12 for inspection. He is interested in determining the probability that, among the 12, at most 2 are defective.

7. A company owns 300 xbox consoles. It is known that 8% of these consoles are not working. You randomly select 25 consoles.

   a) What is the probability that all xbox consoles are working?

b) What is the probability that exactly 3 consoles are not working?

c) What is the probability that 1 or 2 consoles are not working?

d) What is the probability that more than 3 consoles are not working?

8. You are president of an on-campus special events organization. You need a committee of 7 to plan a special birthday party for the president of the college. Your organization consists of 18 women and 15 men. You are interested in the number of men on your committee. What is the probability that your committee has more than 4 men?

9. In a package of 10 fireworks, 3 are defective. If you select 4 at random, what is the probability that

   a. All four fire?

   b. At most 2 will not fire?

10. You have a jar of 10 new batteries, an need to replace 4 batteries in your remote. You accidentally mix the dead batteries with the new ones. What is the probability that if you choose 4 batteries at random and put them in your remote, the remote will work?

11. An urn contains 10 marbles of which 5 are green, 2 are blue, and 3 are red. Three marbles are drawn from the urn, one at a time, without replacememt. What is the probability that all 3 marbles drawn are green?

12. Your school gym has 10 treadmills, 4 or which are out of order, but the signs fell off, so we do not know which ones are out of order. You and 4 friends randomly choose 5 tredmills. What is the probability that everyone got a working treadmill?

13. Twenty students tried out for a dance team, where 6 students were to be chosen. If 8 of the students were blond, what is the probability that the team has less than 2 blonds.